Communications in Mathematics and Applications
Vol. 12, No. 4, pp. 919–930, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i4.1645

Research Article

Application of Mahgoub Integral Transform to Bessel's Differential Equations

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Received: 17 July, 2021 **Accepted:** 8 October, 2021

Abstract. In this paper, we apply Mahgoub integral transform method to solve various types Bessel's differential equations with initial conditions. Also, we provide some numerical examples to illustrate the Application of Mahgoub transform for some particular Bessel's differential equations.

Keywords. Mahgoub transform, Bessel's differential equations

Mathematics Subject Classification (2020). 35G15; 44A85; 44A35; 44A05; 34A02

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1. Introduction

There are many numerous integral transforms and which are widely used in the field of science and engineering. It plays a vital role for solving many advance problems like radioactive decay problems, population growth problems, vibration problems of beam, electric circuit problems and motion of a particle under gravity which appear in many branches of engineering and sciences [6, 8, 9, 16, 23, 27].

Many researchers use different kind of the integral transforms such as Laplace transform [2, 19], Fourier transform [13, 20, 21], Sumudu transform [7], Elzaki transform [25, 26] and etc.

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Integral transforms are very efficient methodology to solve the linear and nonlinear differential and integral equations [15, 24]. Many problems of physical interest are described by ordinary or partial differential equations with appropriate initial or boundary conditions, these problems are usually formulated as initial value problems or boundary value problems (see also, [6, 8, 9, 15, 16, 23, 24]. Generally, we use integral transforms, namely, Laplace, Fourier, Sumudu, Elzaki transform method and etc. (see [2, 3, 7, 12, 13, 25, 26]).

Mahgoub integral transform method [1, 4, 5, 14, 22] is one of the new integral transform, which is particularly very useful for finding solutions for these kind of problems. This means that we can use both Mahgoub transform and other integral transforms are very fruitful methods to investigate many linear and nonlinear ordinary and partial differential equations and integral equations.

In the modern time, Bessel's functions appear in solving many problems of sciences and engineering together with many equations such as heat equation, wave equation, Laplace equation, Schrodinger equation, Helmholtz equation in cylindrical or spherical coordinates. Recently, Aggarwal *et al*. [5] established the Mahgoub transform of Bessel's functions and also investigated some applications of Mahgoub transform of Bessel's functions for evaluating the integral, which contain Bessel's functions. In 2021, Selvan *et al*. [17] obtain the solutions of various forms of Bessel's equations by using the Elzaki integral transform (see also [18]).

Motivated by the above facts, in this paper, our main aim is to obtain the various forms of solutions of the various types of Bessel's differential equations of order zero of the form

$$tx'' + x' + tx = 0, \tag{1.1}$$

$$tx'' + x' + k^2\, tx = 0 \tag{1.2}$$

by using Mahgoub differential transform method. Also, we introduce some relationship between Laplace, Sumudu, Elzaki and Mahgoub transforms, further; for the comparison purpose, we apply these transforms to solve differential equations to establish the differences and similarities. Finally, we provide some examples regarding to second order Bessel's differential equations with non constant coefficients as special case.

2. Preliminaries

In this section, we provide some basic notations, definitions, theorems which are very useful to prove our main results.

Throughout this paper, $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f:[0,\infty)\to\mathbb{K}$ is of exponential order if there exist constants $A,B\in\mathbb{R}$ such that $|f(t)|\le Ae^{Bt}$ for all $t\ge 0$. Similarly, a function $g:(-\infty,0]\to\mathbb{K}$ is of exponential order if there exist constants $A,B\in\mathbb{R}$ such that $|g(t)|\le Ae^{Bt}$ for all $t\le 0$.

A new differential transform called the Mahgoub transform defined for function of exponential order. We consider the function in the set A is defined by

$$A=\left\{f(t):\ \exists\ M,k_1,k_2>0,\ |f(t)|<Me^{\frac{|t|}{k_j}}\right\}.$$

For a given function in the set A, the constant M must be finite number, k_1,k_2 may be finite or infinite.

Definition 2.1 ([1])**.** The Mahgoub transform of the function $f:[0,\infty)\to\mathbb{F}$ is defined by

$$\mathcal{M}\{f(t)\}=u\int_0^\infty f(s)e^{-us}ds=F(u), \tag{2.1}$$

for $t>0$ and $k_1\le u\le k_2$, where $\mathcal{M}$ is the Mahgoub integral transform operator, where the variable u in this transform is used to factor the variable t in the argument of the function f.

Remark 2.2. For any function $f(t)$, we assume that the integral equation (2.1) exist. The Mahgoub integral transform for the function $f:[0,\infty)\to\mathbb{F}$ exists if $f(t)$ is piecewise continuous and of exponential order. These conditions are the only sufficient conditions for the existence of Mahgoub transform of the function $f(t)$ for $t\ge 0$. Otherwise Mahgoub transform may or may not exist.

Theorem 2.3 ([1], Convolution theorem for Mahgoub transform)**.** *Assume that $f(t)$ and $g(t)$ are given functions defined for $t\ge 0$. If $\mathcal{M}\{f(t)\}=F(u)$ and $\mathcal{M}\{g(t)\}=G(u)$, then*

$$\mathcal{M}\{f(t)*g(t)\}=\frac{1}{u}F(u)G(u).$$

Definition 2.4 ([1], Inverse Mahgoub transform)**.** If $\mathcal{M}\{f(t)\}=F(u)$, then $f(t)$ is called the inverse Mahgoub transform of $F(u)$ and is denoted as $f(t)=\mathcal{M}^{-1}\{F(u)\}$, where $\mathcal{M}^{-1}$ is the inverse Mahgoub transform operator.

Theorem 2.5 ([1])**.** *Let $F(u)$ is the Mahgoub transform of the function $f(t)$, then*

(i) $\mathcal{M}\{f'(t)\}=uF(u)-uf(0)$;

(ii) $\mathcal{M}\{f''(t)\}=u^2F(u)-uf'(0)-u^2f(0)$;

(iii) $\mathcal{M}\{f^{(n)}(t)\}=u^nF(u)-\sum\limits_{k=0}^{n-1}u^{n-k}f^{(k)}(0)$.

Theorem 2.6 ([4])**.** *If $\mathcal{M}$ is the Mahgoub integral transform operator and $\mathcal{M}\{f(t)\}=F(u)$ then*

(i) $\mathcal{M}\{tf(t)\}=\frac{1}{u}F(u)-\frac{d}{du}F(u)$,

(ii) $\mathcal{M}\{tf'(t)\}=\frac{d}{du}\{uf(0)\}-f(0)-u\frac{d}{du}F(u)$,

(iii) $\mathcal{M}\{t^2f'(t)\}=\frac{1}{u}\frac{d}{du}\{u\ f(0)\}-\frac{f(0)}{u}-\frac{d}{du}f(0)-\frac{d^2}{du^2}\{uf(0)\}+u\frac{d^2}{du^2}F(u)$,

(iv) $\mathcal{M}\{tf''(t)\}=\frac{d}{du}\{u^2f(0)+uf'(0)\}-u^2\frac{d}{du}F(u)-uF(u)-f'(0)-uf(0)$,

(v) $\mathcal{M}\{t^2f''(t)\}=\frac{1}{u}\frac{d}{du}\{u^2f(0)+uf'(0)\}+\frac{d}{du}\{f'(0)+uf(0)\}$

$$-\frac{d^2}{du^2}\{u^2f(0)+uf'(0)\}+2u\frac{d}{du}F(u)+u^2\frac{d^2}{du^2}F(u)-\frac{1}{u}f'(0)-f(0).$$

Definition 2.7. The linear second order ordinary differential equation of type

$$t^2x''+tx'+(t^2-v^2)x=0 \tag{2.2}$$

is called Bessel equation, where v is a non-negative constant. The number v is called the order of the Bessel equation.

Here the most important cases are when ν is an integer or half-integer. Bessel functions for integer ν are also known as cylinder functions or the cylindrical harmonics because they appear in the solution to Laplace's equation in cylindrical coordinates. Spherical Bessel functions with half-integer ν are obtained when the Helmholtz equation is solved in spherical coordinates.

Definition 2.8. If $\nu = 0$, then the Bessel equation (2.3) becomes

$$tx''(t) + x'(t) + tx(t) = 0 \tag{2.3}$$

is called Bessel equation of order zero.

The Mahgoub transform and other integral transforms are very useful methods to investigate many linear and nonlinear ordinary and partial differential equations and integral equations. The variable u in this transform is used to factor the variable t in the argument of the function f. This transform has deeper connection with the Laplace, Elzaki and Sumudu transform. The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear ordinary differential equations with constant and non-constant coefficients. In the next we compare two solutions by using the two different methods. The following definition was given in [10, 11].

Definition 2.9 ([10, 11])**.** The space W of test functions of exponential decay is the space of complex valued functions $\phi(t)$ satisfying the following properties:

(1) $\phi(t)$ is infinitely differentiable; that is, $\phi(t) \in C^{\infty}(R^n)$,

(2) $\phi(t)$ and its derivatives of all orders vanish at infinity faster than the reciprocal of the exponential of order $\frac{1}{w}$ that is $|e^{\frac{t}{w}} D^k \phi(t)| < M$, for all $\frac{1}{w}$, k.

Then the function $f(t)$ is said to be of exponential growth if and only if $f(t)$ together with all its derivatives grow more slowly than the exponential function of order $\frac{1}{w}$. That is, there exists a real constant $\frac{1}{w}$ and M such that $|D^k \phi(t)| < Me^{\frac{t}{w}}$. A linear continuous functional over the space W of test functions is called a distribution of exponential growth and this dual space of W is denoted by W'.

3. Main Results

In this section, first we apply the Mahgoub transform to the Bessel function $J_0(t)$ in the following equation $tJ_0''(t) + J_0'(t) + tJ_0(t) = 0$, for $t > 0$. Let us consider the Bessel equation of order zero is $t^2 x''(t) + tx'(t) + t^2 x(t) = 0$. Let $x = J_0 H$. Then, except $t = 0$, we have (1.1), for $t > 0$. Applying Mahgoub transform to equation (1.1), we obtain that

$$\mathcal{M}\{tx'' + x' + tx\} = 0 \quad \Rightarrow \quad \mathcal{M}\{tx''\} + \mathcal{M}\{x'\} + \mathcal{M}\{tx\} = 0,$$

then we arrive that

$$\begin{aligned} &\frac{d}{du}\{u^2 x(0) + ux'(0)\} - u^2 \frac{d}{du} X(u) - uX(u) \\ &\quad - x'(0) - ux(0) + uX(u) - ux(0) + \frac{1}{u} X(u) - \frac{d}{du} X(u) = 0 \\ \Rightarrow \quad & -u^2 X'(u) - X'(u) + \frac{1}{u} X(u) = 0. \end{aligned}$$

On simplifying, we get

$$X'(u) = \frac{1}{u(u^2+1)} X(u).$$

Then integrating on both sides, we get

$$\log[X(u)] = \log u - \frac{1}{2}\log(1+u^2) + \log L_1 = \log\left[\frac{L_1 u}{\sqrt{1+u^2}}\right].$$

Therefore

$$X(u) = \frac{L_1 u}{\sqrt{1+u^2}}.$$

It can be written as,

$$\mathcal{M}\{x(t)\} = \frac{L_1 u}{\sqrt{1+u^2}}.$$

Applying inverse Mahgoub transform on both sides, we obtain

$$x(t) = \mathcal{M}^{-1}\left[\frac{L_1 u}{\sqrt{1+u^2}}\right] = L_1 J_0(t)$$

which is the required solution of the Bessel's differential equation (1.1), where L_1 is the integration constant, we can find the value of L_1 with the help of initial conditions.

Next, we are going to use the Mahgoub transform Bessel function $J_0(t)$ in the following Bessel's differential equation

$$t^2 J_0''(t) + t J_0'(t) + k^2 t^2 J_0(t) = 0$$

for $t > 0$. Now, we will consider the Bessel equation of order zero of the form

$$t^2 x''(t) + t x'(t) + k^2 t^2 x(t) = 0.$$

Let $x = J_0 H$. Then, except $t = 0$, we have (1.2), for $t > 0$. Applying Mahgoub transform to equation (1.2), we obtain that

$$\mathcal{M}\{tx'' + x' + k^2\, tx\} = 0 \quad \Rightarrow \quad \mathcal{M}\{tx''\} + \mathcal{M}\{x'\} + k^2 \mathcal{M}\{tx\} = 0,$$

then we arrive that

$$\begin{aligned} &\frac{d}{du}\{u^2 x(0) + u x'(0)\} - u^2 \frac{d}{du} X(u) - u X(u) \\ &\quad - x'(0) - u x(0) + u X(u) - u x(0) + k^2\left[\frac{1}{u} X(u) - \frac{d}{du} X(u)\right] = 0 \end{aligned}$$

$$\Rightarrow \qquad (u^2 + k^2) X'(u) = \frac{k^2}{u} X(u).$$

Simplifying the above equation, we get

$$\frac{X'(u)}{X(u)} = \frac{k^2}{u(u^2+k^2)} = \frac{1}{u} - \frac{u}{u^2+k^2}.$$

Then integrating, we have

$$\log[X(u)] = \log u - \frac{1}{2}\log(k^2+u^2) + \log L_2 = \log\left[\frac{L_2 u}{\sqrt{k^2+u^2}}\right]$$

$$\Rightarrow \qquad X(u) = \frac{L_2 u}{\sqrt{k^2+u^2}}.$$

It can be written as,

$$\mathcal{M}\{x(t)\} = \frac{L_2 u}{\sqrt{k^2+u^2}}.$$

Applying inverse Mahgoub transform on both sides, we obtain

$$x(t) = L_2 \mathcal{M}^{-1}\left[\frac{(u/k)}{\sqrt{1+(u/k)^2}}\right] = L_2 J_0(kt)$$

is the required solution of the Bessel's differential equation (1.2), where L_2 is the integration constant and it can be evaluated by using the initial conditions.

Proposition 3.1. *Let f be n times continuously differentiable function on $(0,\infty)$ and let $f(t)=0$ for $t<0$. Suppose that $f^{(n)} \in L_{loc}$. Then $f^{(k)} \in L_{loc}$ for $0 \le k \le n-1$, $\mathrm{dom}(\mathcal{M}f) \subseteq \mathrm{dom}(\mathcal{M}f^{(n)})$ and for any polynomial P of degree n,*

$$P(u)\mathcal{M}(x)(u) = F(u) + M_P(u)\psi(x,n) \tag{3.1}$$

for $u \in \mathrm{dom}(\mathcal{M}f)$.

Proof. In particular, we have

$$\begin{aligned}\mathcal{M}\{f^{(n)}\} &= u^n F(u) - \sum_{k=0}^{n-1} u^{n-k} f^{(k)}(0)\\ &= u^n(\mathcal{M}f)(u) - (u^n, u^{n-1}, u^{n-2}, \dots, u^2, u)\psi(f;n),\end{aligned} \tag{3.2}$$

where $\psi(f;n)$ is written as a column vector. For $n=2$, we have

$$\mathcal{M}\{f''(t)\} = u^2 F(u) - uf'(0) - u^2 f(0).$$

In particular, if we will consider $x(t) = \frac{1}{\mu}\sin(\mu t)$. Then clearly $x''(t)+\mu^2 x(t) = 0$ and in the operator form we write $(D^2+\mu^2)f = 0$. Since $dom(\mathcal{M}f)$ contains $(0,\infty)$ then on using the equation (3.1) with $n=2$ and $P(m) = m^2+\mu^2$, for $u>0$, we have

$$0 = \left(u^2+\mu^2\right)\mathcal{M}(f) - \begin{bmatrix} u & u^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

since $\psi(x;2) = (f(0), f'(0)) = (0,1)$. Thus, we can obtain the same result without using the definitions or transforms table as

$$V(t) = \mathcal{M}\left\{\frac{1}{\mu}\sin(\mu t)\mathcal{H}(t)\right\} = \frac{u}{u^2+\mu^2}$$

then the inverse is given by

$$\frac{1}{\mu}\sin(\mu t)\mathcal{H}(t) = \mathcal{M}^{-1}\left\{\frac{\mu u}{u^2+\mu^2}\right\}$$

let $V = g\mathcal{H}$, where $g = \frac{1}{\mu}\sin(\mu t)$ and $\mathcal{H}$ is Heaviside function

$$DV = gD\mathcal{H} + g'\mathcal{H} = g\delta + g'\mathcal{H} = g\delta(0) + g'\mathcal{H} = g'\mathcal{H},$$

then

$$D^2V = g'D\mathcal{H} + g''\mathcal{H} = g'\delta + g''\mathcal{H} = g'(0)\delta + g''\mathcal{H} = \delta + g''\mathcal{H}.$$

Therefore,

$$P(D)V = D^2V + \mu^2 V = \delta + (g''+\mu^2 g)\mathcal{H} = \delta.$$

□

Proposition 3.2. *Let us consider the function f be Mahgoub transformable and its satisfies $f(t) = 0$ for $t < 0$. Then*

$$\lim_{u\to 0}\{\mathcal{M}[f](u)\} = 0.$$

Proof. Assume that $f_0(t) = f(t)[1-\mathcal{H}(t-1)]$ and $f_1(t) = f(t)\mathcal{H}(t-1)$. Since f_0 vanishes outside of $[0,1]$, we have

$$\mathcal{M}[f](u) = u\int_0^1 e^{-ut} f(t)dt$$

for any u. Moreover, $f = f_0 + f_1$, $\operatorname{dom}\{\mathcal{M}(f_1)\} = \operatorname{dom}\{\mathcal{M}(f)\}$ and

$$\mathcal{M}[f](u) = u\int_0^1 e^{-ut} f(t)dt + \mathcal{M}(f_1)(u)$$

for all $u \in \operatorname{dom}\{\mathcal{M}(f)\}$. Let $u_0 \in \operatorname{dom}\{\mathcal{M}(f)\}$ and also apply $|\mathcal{M}[f](u)| \le \mathcal{A}e^{-cu}$ to f_1 we conclude that there is a constant $\mathcal{A}$ such that

$$|\mathcal{M}[f](u)| \le u\int_0^1 e^{-ut}|f(t)|\, dt + \mathcal{A}e^{-cu}$$

for all $u \ge u_0$ as $u \to \infty$, the second term on the right clearly tends to zero. The same applies to the first term. □

4. Application

Finally, in this section, we illustrate the main results with some examples.

Example 4.1. Consider the following Bessel's differential equation

$$t^2 J_0''(t) + tJ_0'(t) + 36t^2 J_0(t) = 0$$

for $t > 0$. Now, we will consider the Bessel equation of order zero of the form

$$t^2 x''(t) + tx'(t) + 36t^2 x(t) = 0.$$

Let us choose $x = J_0(t)H$. Then, except $t = 0$, we have

$$tx''(t) + x'(t) + 36tx(t) = 0 \tag{4.1}$$

for $t > 0$. Applying Mahgoub transform to equation (4.1), we obtain that

$$\mathcal{M}\{tx'' + x' + 36tx\} = 0$$

then we arrive that $(u^2+36)X'(u) = \frac{36}{u}X(u)$, thus

$$\frac{X'(u)}{X(u)} = \frac{36}{u(u^2+36)} = \frac{1}{u} - \frac{u}{u^2+36}.$$

Then integrating on both sides, we have

$$\log[X(u)] = \log\left[\frac{K_1 u}{\sqrt{36+u^2}}\right]$$

$$\Rightarrow \quad X(u) = \frac{K_1 u}{\sqrt{36+u^2}}$$

thus we have,

$$\mathcal{M}\{x(t)\} = \frac{K_1\, u}{\sqrt{36+u^2}}.$$

Applying inverse Mahgoub transform on both sides, we obtain

$$x(t) = K_1 \mathcal{M}^{-1}\left[\frac{(u/6)}{\sqrt{1+(u/6)^2}}\right] = K_1 J_0(6t),$$

which is the required solution of the Bessel's differential equation (4.1), where K_1 is the integration constant and it can be evaluated by using initial conditions.

Example 4.2. Consider the following Bessel's differential equation

$$t^2 J_0''(t) + tJ_0'(t) + \frac{t^2}{4} J_0(t) = 0$$

for $t > 0$. Now, we will consider the Bessel equation of order zero of the form

$$t^2 x''(t) + tx'(t) + \frac{t^2}{4} x(t) = 0.$$

Let us choose $x = J_0(t)H$. Then, except $t = 0$, we have

$$tx''(t) + x'(t) + \frac{t}{4} x(t) = 0 \tag{4.2}$$

for $t > 0$. Applying Mahgoub transform to equation (4.1), we obtain that

$$\mathcal{M}\left\{tx''(t) + x'(t) + \frac{t}{4} x(t)\right\} = 0$$

then we arrive that $\left(u^2 + \frac{1}{4}\right) X'(u) = \frac{1}{4u} X(u)$, thus

$$\frac{X'(u)}{X(u)} = \frac{1}{4u\left(u^2 + \frac{1}{4}\right)} = \frac{1}{u} - \frac{u}{u^2 + \frac{1}{4}}.$$

Then integrating on both sides, we have

$$\log[X(u)] = \log\left[\frac{K_2 u}{\sqrt{\frac{1}{4} + u^2}}\right] \quad \Rightarrow \quad X(u) = \frac{K_2 u}{\sqrt{\frac{1}{4} + u^2}}$$

thus we have, $\mathcal{M}\{x(t)\} = K_2 \frac{u}{\sqrt{\frac{1}{4}+u^2}}$. Applying inverse Mahgoub transform on both sides, we obtain

$$x(t) = K_2 \mathcal{M}^{-1}\left[\frac{(2u)}{\sqrt{1+(2u)^2}}\right] = K_2 J_0\left(\frac{t}{2}\right)$$

is a solution of (4.2), where K_2 is the integration constant, with the help of initial conditions, we can obtain K_2.

Example 4.3. Let us consider the linear differential equation of second order with variable coefficients of the form

$$tf''(t) - tf'(t) + f(t) = 2 \tag{4.3}$$

with initial conditions

$$f(0) = 2, \quad f'(0) = -1. \tag{4.4}$$

First solution by Laplace Transform. By applying the Laplace transform method to the equation (4.3) and using the initial conditions, we have

$$F'(s) + \left(\frac{2}{s}\right) F(s) = \frac{-2}{s^2(s-1)}$$

then we obtain the solution

$$F(s) = -\frac{\ln(s-1)}{s^2} + \frac{c}{s^2},$$

where c is a constant, then by taking the inverse Laplace transform, we get

$$f(t) = 2 - 2\pi i t + ct. \tag{4.5}$$

Second solution by Sumudu Transforms. On using the Sumudu transform for the differential equation (4.3), we have

$$F'(u) - \frac{F(u)}{u} = \frac{-2}{u}.$$

By applying the same technique that was used with the same problem in the case of Laplace transform, then we have the following solution

$$F(u) = 2 + au, \tag{4.6}$$

where a is a constant, by using inverse Sumudu transform for equation (4.6) with respect to u we obtain the solution in the form of $f(t) = 2 + at$.

Third Solution by Elzaki Transform. By using the Elzaki transformation method to the equation (4.3) and by applying the initial conditions (4.4), we have

$$(1-v)F'(v) + 3\left(1 - \frac{1}{v}\right) F(v) = 2 - 2v.$$

On simplifying the above equation, we get

$$F'(v) - \frac{3}{v}F(v) = 2. \tag{4.7}$$

By applying the same technique that was used in the same problem, in case of Laplace transform and by using the inverse Elzaki transform for equation (4.7) with respect to v we obtain the solution for equation (4.3).

Fourth Solution by Mahgoub Transform. By using the Mahgoub transform method to (4.3) and by using the initial conditions (4.4), we have

$$u(u-1)F'(u) + (u-1)F(u) = 2 + 2u$$

$$\Rightarrow \quad F'(u) + \frac{F(u)}{u} = \frac{2u-2}{u(u-1)} = \frac{2}{u}$$

By applying the same technique that was used with the same problem in case of Laplace transform and by using inverse Mahgoub transform, we will reach the solution for the differential equation with respect to u, we can have the solution in the form of $f(t) = 2 + at$, where a is a constant.

Remark 4.4. We note that, if we compare the solutions obtained by the different transformations for the differential equations, we see that the solution which is given by Laplace transform in complex domain and given by Sumudu, Elzaki and Mahgoub transform in real domain. Thus this leads us to consider that if the solution exists by Mahgoub transform then the solution also exists by Laplace, Sumudu and Elzaki transform.

Remark 4.5. If Mahgoub transform exists then Laplace transform is also exists.

Remark 4.6. If Mahgoub transform exists then Sumudu transform is also exists.

Remark 4.7. If Mahgoub transform exists then Elzaki transform is also exists.

Remark 4.8. The converse of these above statements is always need not be true.

5. Conclusion

As stated in the introduction of this paper, the Mahgoub transform can be used as an effective tool for analyzing the basic characteristics of ordinary and partial differential equations. Here we analyzed the solution of various forms Bessel's differential equation using Mahgoub transform, proposed some properties of Mahgoub transform and illustrated the main results with an examples. That is, we provided an illustration to comparing the solution for a particular differential equation by using Laplace, Sumudu, Elzaki and Mahgoub transformations. The authors are also confident that the paper will inspire many young mathematicians to read more about Mahgoub transform and will also help to apply this transform to solve many problems related to various types of linear and nonlinear differential equations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. M. A. Mahgoub, The new integral transform mahgoub transform, *Advances in Theoretical and Applied Mathematics* **11**(4) (2016), 391 – 398.

[2] S. Aggarwal, A. R. Gupta, D. P. Singh, N. Asthana and N. Kumar, Application of Laplace transform for solving population growth and decay problems, *International Journal of Latest Technology in Engineering, Management & Applied Science* **7**(9) (2018), 141 – 145.

[3] S. Aggarwal, D. P. Singh, N. Asthana and A. R. Gupta, Application of Elzaki transform for solving population growth and decay problems, *Journal of Emerging Technologies and Innovative Research* **5**(9) (2018), 281 – 284, URL: `https://www.jetir.org/papers/JETIR1809545.pdf`.

[4] S. Aggarwal, N. Sharma, R. Chauhan, A. R. Gupta and A. Khandelwal, A new application of mahgoub transform for solving linear ordinary differential equations with variable coefficients, *Journal of Computer and Mathematical Sciences* **9**(6) (2018), 520 – 525, URL: `http://compmath-journal.org/dnload/Sudhanshu-Aggarwal-Nidhi-Sharma-Raman-Chauhan-Anjana-Rani-Gupta4and-Astha-Khandelwal5/CMJV09I06P0520.pdf`.

[5] S. Aggarwal, N. Sharma and R. Chauhan, Mahgoub transform of Bessel's functions, *International Journal of Latest Technology in Engineering, Management & Applied Science* **VII**(VIII) (2018) 32 – 36, URL: `https://www.ijltemas.in/DigitalLibrary/Vol.7Issue8/32-36.pdf`.

[6] Z. Ahsan, *Differential Equations and Their Applications*, Prentice Hall of India Pvt. Ltd, New Delhi (1999).

[7] F. B. M. Belgacem and A. A. Karaballi, Sumudu transform fundamental properties investigations and applications, *Journal of Applied Mathematics and Stochastic Analysis* **2006** (2006), Article ID 091083, 23 pages, DOI: 10.1155/JAMSA/2006/91083.

[8] W. E. Boyce and R. C. Di Prima, *Elementary Differential Equations and Boundary Value Problems*, John Willey & Sons, NY (2001).

[9] C. Chicone, *Ordinary Differential Equations With Applications*, Springer Verlag, NY (1999).

[10] H. Eltayab and A. Kilicman, On some application of a new integral transform, *International Journal of Mathematical Analysis* **4**(3) (2010), 123 – 132.

[11] H. Eltayab, A. Kilicman and B. Fisher, A new integral transform and associated distributions, *Integral Transforms and Special Functions* **21**(5) (2010), 367 – 379, DOI: 10.1080/10652460903335061.

[12] S. Islam, Y. Khan, N. Faraz and F. Austin, Numerical solution of logistic differential equations by using the laplace decomposition method, *World Applied Sciences Journal* **8**(9) (2010), 1100 – 1105, URL: `https://www.idosi.org/wasj/wasj8(9)/10.pdf`.

[13] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, 2nd edition, Chapman and Hall/CRC (2006).

[14] M. A. M. Mahgoub and A. A. Alshikh, An application of new transform Mahgoub transform to partial differential equations, *Mathematical Theory and Modeling* **7**(1) (2017), 7 – 9.

[15] N. Guzel and M. Nurulay, Solution of shiff systems by using differential transform method, *Dunlupinar Universities Fen Bilimleri Enstitusu Dergisi* **16** (2008), 49 – 59, URL: `https://birimler.dpu.edu.tr/app/views/panel/ckfinder/userfiles/16/files/Dergiler/16/07.pdf`.

[16] D. H. P. Gowda, *Pharmaceutical Mathematics With Application to Pharmacy*, Pharma Med Press, Hyderabad (2014).

[17] A. P. Selvan, K. Varadharajan and J. Jini, Application of Elzaki transform to Bessel's differential equations, Chapter 16, *Physics and Statistics* (16), ESN Publications (2021), 106 – 111.

[18] A. P. Selvan, P. Ashokkumar and P. Madhu, A general form of Bessel's ordinary differential equations and an application of Elzaki transform, Chapter 17, *Physics and Statistics* (16), ESN Publications (2021), 112 – 117.

[19] R. Murali and A. Ponmana Selvan, Mittag-Leffler-Hyers-Ulam stability of a linear differential equations of first order using Laplace transforms, *Canadian Journal of Applied Mathematics* **2**(2) (2020), 47 – 59, URL: `https://www.prior-sci-pub.com/cjam_2020_art13.pdf`.

[20] M. Ramdoss and P. S. Arumugam, Fourier transforms and Ulam stabilities of linear differential equations, Chapter 12, *Frontiers in Functional Equations and Analytic Inequalities* **AG 2019**, Springer Nature, Switzerland (2019), 195 – 217, DOI: 10.1007/978-3-030-28950-8_12.

[21] J. M. Rassias, R. Murali and A. P. Selvan, Mittag-Leffler-Hyers-Ulam stability of linear differential equations using fourier transforms, *Journal of Computational Analysis and Applications* **29**(1) (2021), 68 – 85, URL: `http://www.eudoxuspress.com/images/JOCAAA-VOL-29-2021-ISSUE-1.pdf`.

[22] S. M. Jung, P. S. Arumugam and M. Ramdoss, Mahgoub transform and Hyers-Ulam stability of first-order linear differential equations, *Journal of Mathematical Inequalities* **15**(3) (2021), 1201 – 1218, DOI: 10.7153/jmi-2021-15-80.

[23] P. D. Ritger and N. J. Rose, *Differential Equations With Applications*, McGraw Hill, NY (1968).

[24] S.-H. Chang and I.-L. Chang, A new algorithm for calculating one-dimensional differential transform of nonlinear functions, *Applied Mathematics and Computation* **195** (2008), 799 – 808, DOI: 10.1016/j.amc.2007.05.026.

[25] T. M. Elzaki, The new integral transform Elzaki transform, *Global Journal of Pure and Applied Mathematics* **7**(1) (2011), 57 – 64.

[26] T. M. Elzaki and S. M. Elzaki, On the Elzaki transform and ordinary differential equation with variable coefficients, *Advances in Theoretical and Applied Mathematics* **6**(1) (2011), 13 – 18.

[27] D. Zwillinger, *Handbook of Differential Equations*, Academic Press, Inc., NY (1989), DOI: 10.1016/C2013-0-07676-1.