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Research Article

Toughness and Maximum Extension of Certain t-Tough Sets of the Bloom Graph $B_{m,n}$, $m \geq 3$, $n \geq 3$

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Abstract. Data broadcasting is the process of distributing data sets from one or more nodes to other nodes in the network. The fault tolerance of the data broadcasting network plays key importance in its efficient performance. The toughness of graphs is a measure for the fault tolerance of a graph. In this paper, we investigate the toughness and maximum extension of certain t-tough sets of the bloom graph $B_{m,n}$, $m \geq 3$, $n \geq 3$.

Keywords. Toughness, Maximum extension, Bloom graph

Mathematics Subject Classification (2020). 05C42; 05C40; 05C10

1. Introduction

Data analytics is the process of analysing data to derive useful insights and make decisions from the information they contain. This is the era where a soaring population has an easier access to electronic gadgets. Thus data is increasing in volume with a great magnitude making it an asset to obtain information. Due to this reason, many organizations use data analytics to make better business solutions thus increasing their revenue. Moreover, data has become an expensive asset because of the fact that organizations require highly sophisticated tools to collect data from various sources and process the same to be ready for analyses.

Data broadcasting is the process of distributing data from one or more source nodes to other nodes. The efficiency of the data broadcasting network relies on its performance under fault

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tolerant conditions. We propose that graph theoritical tools, namely, toughness and maximum extension of a t-tough set can identify the nodes prone to faults and provide a fault propagation warning.

Toughness is a measure to estimate the closeness of the vertices of a graph. Chvátal [4] introduced and coined the definition of toughness of a graph as follows:

Definition 1. *Toughness* [4] of a graph G is defined as the real number $\tau > 0$ such that it is the minimum of the ratio of the number of vertices in the cutset S to the number of components in $G \setminus S$ taken over all possible cutsets S of G.

$$\tau = \min \frac{|S|}{\omega(G \setminus S)}, \quad \text{for all } S \subset V. \tag{1.1}$$

For a connected graph G, the upper bound and lower bound of $\tau(G)$ are governed by the following theorems:

Theorem 1 ([4])**.** *If G is not complete, then $\tau \le \frac{\kappa}{2}$.*

Theorem 2 ([8])**.** *For a connected graph G, $\tau \ge \frac{\kappa}{\Delta}$.*

Whereas, for a connected planar graph the following theorem defines the upper bound and lower bound of $\tau(G)$:

Theorem 3 ([9])**.** *If G is a connected planar graph of connectivity κ, then*

$$\frac{\kappa}{2} - 1 < \tau(G) \le \frac{\kappa}{2}.$$

We have introduced and characterized the extension of a t-tough set of a graph and hence the maximum extension of the same.

Definition 2. A *t-tough set* [6] of a connected graph G, denoted as S_t, is defined as a cutset $S \subset V(G)$ which satisfies the following equation:

$$t = \frac{|S|}{\omega(G \setminus S)}, \quad t \ge \tau.$$

Definition 3. For a connected graph G, a t'-tough set $S_{t'}$ is called an *extension* [6] of a t-tough set S_t if whenever $t' \le t$, $S \subseteq S'$.

(i) If $t' = t$, then $S_{t'}$ is called a weak extension of S_t.

(ii) If $t' < t$, then $S_{t'}$ is called a strong extension of S_t.

Definition 4. A t_m-tough set S_{t_m} is called a *maximum extension* of a t-tough set S_t if there does not exist a t_0-tough set S_{t_0} such that $t_0 \le t_m \le t$ and $S_{t_0} \supset S_{t_m} \supset S_t$.

1.1 Literature Survey

An extensive study on the toughness of various graphs in available in literature. Chvátal [4] investigated the toughness of complete graphs, product of complete graphs and complete bipartite graphs. Kevin [7] derived the toughness of generalised Petersen graphs and established all of its tough sets. The toughness of split graphs and a polynomial time algorithm to generate the same were determined by Woeginger [12,14]. The toughness of cubic graphs was investigated by Goddard [10]. Cynthia *et al.* [5] investigated the toughness of cyclic split graphs and generalised prism graphs. The toughness and extension of certain t-tough sets of the mesh graphs were established by Cynthia *et al.* [6].

In this paper, we investigate the toughness and maximum extension of certain t-tough sets of the bloom graph $B_{m,n}$, $m \geq 3$, $n \geq 3$.

Definition 5. The bloom graph [15] $B_{m,n}$, $m \geq 3$, $n \geq 3$ is defined as follows:

$$V(B_{m,n}) = \{v_{ij} \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}.$$

Two distinct vertices $v_{i_1 j_1}$ and $v_{i_2 j_2}$ are adjacent if and only if

(i) $i_2 = i_1 + 1$ and $j_1 = j_2$,

(ii) $i_1 = i_2 = 0$ and $j_1 + 1 \equiv j_2 (\text{mod}\, n)$,

(iii) $i_1 = i_2 = m-1$ and $j_1 + 1 \equiv j_2 (\text{mod}\, n)$,

(iv) $i_2 = i_1 + 1$ and $j_1 + 1 \equiv j_2 (\text{mod}\, n)$.

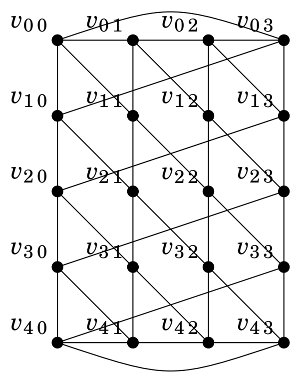


Figure 1. Bloom graph $B_{5,4}$

2. Toughness and τ-Tough Sets of $B_{3,n}$, $n \geq 3$

The toughness of bloom graph $B_{3,3}$ is 1.5. The cutset and components of $B_{3,3}$ are illustrated as follows:

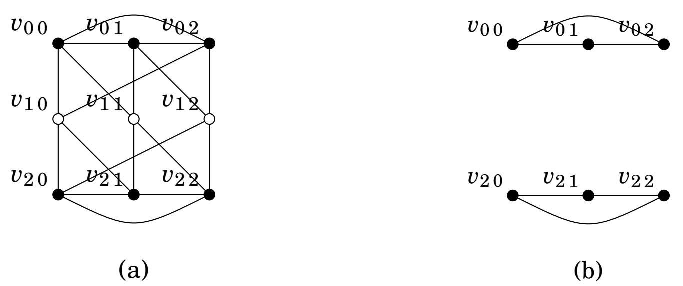


Figure 2. (a) Cutset S of the Bloom graph $B_{3,3}$; (b) Components of $B_{3,3} \setminus S$

Theorem 4. *Let $B_{m,n}$, $m \geq 3$, $n > 3$ be the bloom graph on mn vertices. Then, the minimum toughness of the bloom graph $B_{3,n}$, $n > 3$ is given by*

$$\tau(B_{3,n}) = 2.$$

Proof. Consider the bloom graph $B_{3,n}$, $n > 3$. The bloom graph is planar [15] and it is easy to verify that $\kappa(B_{m,3}) = 4$. Therefore, theorem 1 and theorem 3 imply that

$$1 < \tau(B_{3,n}) \leq 2. \tag{2.1}$$

We claim that the bound for $\tau(B(3,n))$ is sharp at 2. Equation (2.1) imply that

$$\tau(B_{3,n}) > 1.$$

Therefore, there exists a τ-tough set S of $B_{3,n}$, such that

$$|S| \geq n\lfloor \frac{3}{2} \rfloor$$

which implies

$$\tau(B_{3,n}) > \frac{n\lfloor\frac{3}{2}\rfloor}{3n - n\lfloor\frac{3}{2}\rfloor - n}.$$

By the adjacency of vertices in $B_{3,n}$, for every

$$v_{1j} \in V(B_{3,n}).$$

We have

$$(v_{0j}, v_{1j}), (v_{0j-1}, v_{1j}), (v_{2j}, v_{1j}), (v_{2j+1}, v_{1j}) \in E(B_{3,n}).$$

Therefore, let

$$S = \{v_{ij} \mid i = 0, 2, 0 \leq j \leq n-1\}.$$

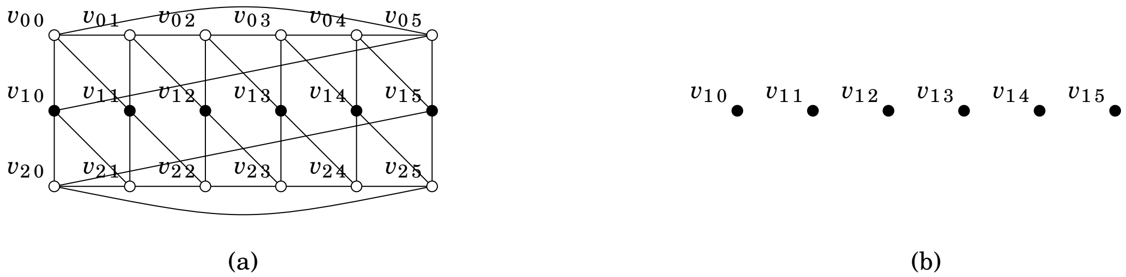


Figure 3. (a) Cutset S of the Bloom graph $B_{3,6}$; (b) Components of $B_{3,6} \setminus S$

Clearly, $|S| = 2n$ and $B_{3,n} \setminus S$ yields n components. Hence,

$$\frac{|S|}{\omega(B_{3,n} \setminus S)} = 2. \tag{2.2}$$

We claim that $\tau(B_{3,n}) = 2$. Suppose $\tau(B_{3,n}) \neq 2$, then it is possible to find a cutset of $B_{3,n}$, say S^0, such that $|S^0| < |S|$ and $\omega(B_{3,n} \setminus S^0) \leq \omega(B_{3,n} \setminus S)$ and

$$\frac{|S^0|}{\omega(B_{3,n} \setminus S^0)} < 2.$$

Since, S is the largest cutset of $B_{3,n}$ satisfying eq. (2.2), we have the following two cases:

***Case* (i).** $S^0 \subset S$

Then, there exists atleast one vertex $v_{i\,j} \in S$ such that $v_{i\,j} \notin S^0$. Without loss of generality, let $v_{i\,j} \simeq v_{0\,j}$. Then, by the adjacency of vertices in $B_{3,n}$, $P_3 \simeq v_{1\,j} - v_{0;j} - v_{0,j+1}$ and $n-2$ trivial vertices are the component of $B_{3,n} \setminus S^0$. Therefore,

$$\frac{|S^0|}{\omega(B_{3,n} \setminus S^0)} = \frac{2n-1}{n-2} > 2.$$

The components of $B_{3,n} \setminus S$ are trivial and the vertices of $B_{3,n} \setminus S$ are adjacent to exactly 2 vertices in $\{v_{i\,j} \mid i = 2, 0 \le j \le n-1\}$ and $\{v_{i\,j} \mid i = 0, 0 \le j \le n-1\}$, respectively. Hence, it is not possible to construct S^0. The minimal cutset $S^0 \subset S$ is of the form

$$S^0 = \{v_{0\,j-1}, v_{0\,j}, v_{2\,j}, v_{2\,j+1}\}$$

and

$$\frac{|S^0|}{\omega(B_{3,n} \setminus S^0)} = 2$$

***Case* (ii).** $S^0 \not\subset S$

Then, there exists atleast one vertex $v_{i\,j} \in S^0$ such that $v_{i\,j} \notin S$. Since, S is the larget cutset, $|S^0| < |S|$ and

$$\frac{|S^0|}{\omega(B_{3,n} \setminus S^0)} < \frac{|S|}{\omega(B_{3,n} \setminus S)}.$$

We have $\omega(B_{3,n} \setminus S^0) < \omega(B_{3,n} \setminus S)$. Therefore,

$$2 \le \omega(B_{3,n} \setminus S^0) < n.$$

Then, $n < |S^0| < 2n$. Then, the argument that such an S^0 does not exist is similiar to the previous case. □

2.1 τ-Tough Sets of $B_{3,n}$, $n \ge 3$

In this subsection, we exhibit the 2-tough sets of $B_{3,n}$. Since, $B_{3,n}$ is 4-regular, $\kappa = 4$ and $\tau = 2$, there exists 2-tough sets S_2 such that $|S_2| = 4$.

Denote the 2-tough set on 4 vertices using $S_2^{4_l}$, $1 \le l \le 5$. Then,

$$S_2^{4_1} = \{v_{0\,j}, v_{0\,j+k}, v_{2\,j+1}, v_{2\,j+k+1} \mid 1 \le j \le n-1,\ 1 \le k \le n-1\},$$
$$S_2^{4_2} = \{v_{0\,j}, v_{1\,j+2}, v_{2\,j+1}, v_{2\,j+3} \mid 1 \le j \le n-1,\ 1 \le k \le n-1\},$$
$$S_2^{4_3} = \{v_{0\,j}, v_{1\,j-1}, v_{2\,j+1}, v_{2\,j-1} \mid 1 \le j \le n-1,\ 1 \le k \le n-1\},$$
$$S_2^{4_4} = \{v_{0\,j}, v_{1\,j+1}, v_{0\,j+2}, v_{2\,j+3} \mid 1 \le j \le n-1,\ 1 \le k \le n-1\},$$
$$S_2^{4_5} = \{v_{0\,j}, v_{1\,j+2}, v_{0\,j+2}, v_{2\,j+1} \mid 1 \le j \le n-1,\ 1 \le k \le n-1\},$$

generates all possible 2-tough sets of cardinality 4.

For each graph $B_{3,n} \setminus S_2^{4_l}$, $1 \le l \le 5$ and $d(v_{i\,j}) = 2$ if and only if $v_{i\,j}$ is adjacent to atmost one pair of vertices from $S_2^{4_l}$ in $B_{3,n}$ where $v_{i\,j} \in V(B_{3,n} \setminus S_2^{4_l})$. Then, 2-tough sets of cardinality 6 can be generated including the two vertices adjacent to $v_{i;j}$ in the corresponding $S_2^{4_l}$. Using the same argument for the graph $B_{3,n}$, 2-tough sets of cardinality $8, 10, 12, \dots, 2n$ can be generated.

3. Toughness and τ-Tough Sets of $B_{m,n}$, $m>3$, $n=3,4$

Theorem 5. *Let $B_{m,n}$, $m\geq 3$, $n\geq 3$ be the bloom graph on mn vertices. Then, the minimum toughness of the bloom graph $B_{m,n}$, $m>3$, $n=3,4$ is given by*

$$\tau(B_{m,n})=\begin{cases}\frac{n(m-1)}{n(m-3)+4} & m\ odd,\\ \frac{n(m-2)}{n(m-1)+2} & m\ even.\end{cases}$$

Proof. Consider the bloom graph $B_{m,n}$, $m>3$, $n=3,4$. The bloom graph is planar [15] and it is easy to verify that $\kappa(B_{m,3})=4$. Therefore, Theorem 1 and Theorem 3 imply that

$$1<\tau(B_{m,n})\leq 2. \tag{3.1}$$

***Case* 1:** When m odd

We claim that the bound for $\tau(B_{m,n})$ is sharp at $\frac{n(m-1)}{n(m-3)+4}$.

Equation (3.1) imply that

$$\tau(B_{m,n})>1$$

Therefore, there exists a τ-tough set S of $B_{m,n}$, such that

$$|S|\geq n\lfloor\frac{m}{2}\rfloor$$

which implies

$$\tau(B_{m,n})>\frac{n\lfloor\frac{m}{2}\rfloor}{mn-n\lfloor\frac{m}{2}\rfloor-n}. \tag{3.2}$$

Consider,

$$S=\{v_{i\,j}\mid i=1,3,5,\dots,m-2,0\leq j\leq n-1\}. \tag{3.3}$$

Clearly, $|S|=n\lfloor\frac{m}{2}\rfloor$ and $B_{m,n}\setminus S$ yields $mn-n\lfloor\frac{m}{2}\rfloor-2n+2$ components.

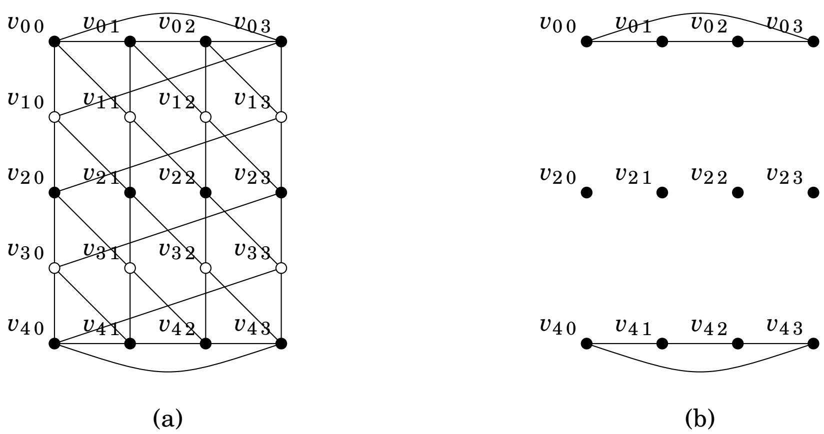


Figure 4. (a) Cutset S of the Bloom graph $B_{5,4}$; (b) Components of $B_{5,4}\setminus S$

Therefore,

$$\frac{|S|}{\omega(B_{m,n}\setminus S)}=\frac{n\lfloor\frac{m}{2}\rfloor}{mn-n\lfloor\frac{m}{2}\rfloor-2n+2}. \tag{3.4}$$

Moreover, S defined in eq. (3.3) is the only cutset of $B_{m,n}$ which satisfies eq. (3.4). By contrary, consider the following cutset with $n\lfloor\frac{m}{2}\rfloor$ vertices.

$$S^1 = \{v_{i\,j} \mid i = 0,2,4,\dots,m-3, 0 \le j \le n-1\}.$$

$B_{m,n}\setminus S^1$ yields $mn - n\lfloor\frac{m}{2}\rfloor - 2n + 1$ components. Therefore,

$$\omega(B_{m,n}\setminus S^1) < \omega(B_{m,n}\setminus S).$$

It can be similarly proved for other cutsets with cardinality $n\lfloor\frac{m}{2}\rfloor$ by showing that they are analogous to S^1.

Then, eqs. (3.1), (3.2) and (3.4) imply that

$$1 \le \frac{n\lfloor\frac{n}{2}\rfloor}{mn - n\lfloor\frac{m}{2}\rfloor - 2n + 2} < \tau(B_{m,n}) \le 2.$$

As m increases, eq. (3.4) tends to 1. Therefore,

$$\tau(B_{m,n}) = \frac{n\lfloor\frac{m}{2}\rfloor}{mn - n\lfloor\frac{m}{2}\rfloor - 2n + 2}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{n(m-1)}{n(m-3)+4}. \tag{3.5}$$

***Case* 2:** When m even

We claim that the bound for $\tau(B_{m,n})$ is sharp at $\frac{n(m-2)}{n(m-1)+2}$.

Equation (3.1) imply that

$$\tau(B_{m,n}) > 1.$$

Therefore, there exists a τ-tough set S of $B_{3,n}$, such that

$$|S| \ge \frac{mn}{2}. \tag{3.6}$$

which implies

$$\tau(B_{m,n}) > \frac{\frac{mn}{2}}{mn - \frac{mn}{2}}$$

Consider the following cutset:

$$S = \{v_{i\,j} \mid i = 1,3,5,\dots,m-1, 0 \le j \le n-1\}.$$

Clearly, $|S| = \frac{mn}{2}$ and $B_{m,n}\setminus S$ yields $\frac{mn}{2} - n + 1$ components.

Therefore,

$$\frac{|S|}{\omega(B_{m,4}\setminus S)} = \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}.$$

Also, cutsets of $B_{m,n}$ analogous to S satisfy the following equation:

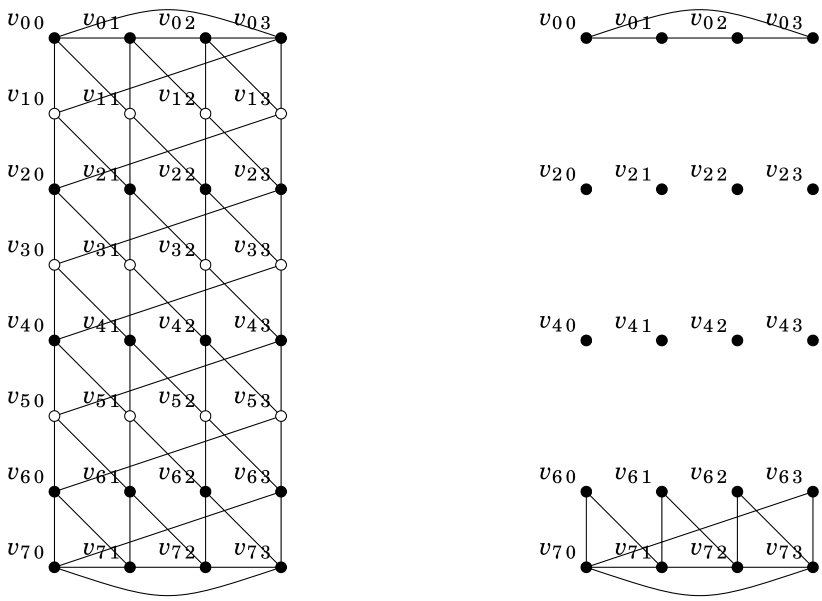
$$1 \le \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1} < \tau(B_{m,n}) \le 2.$$

We claim that it is possible to find a cutset S^1 such that $|S^1| < |S|$ and

$$\frac{|S^1|}{\omega(B_{m,n}\setminus S^1)} < \frac{|S|}{\omega(B_{m,n}\setminus S)}.$$

We attain S^1 by excluding some vertices from S. Let

$$S^1 = \{v_{ij} \mid i = 1,3,5,\ldots,m-3, 0 \le j \le n-1\}. \tag{3.7}$$



(a) (b)

Figure 5. (a) Cutset S^1 of the Bloom graph $B_{8,4}$; (b) Components of $B_{8,4} \setminus S^1$

Clearly, $|S^1| < |S|$. Also,

$$\frac{|S^1|}{\omega(B_{m,n} \setminus S^1)} = \frac{\frac{n(m-2)}{2}}{mn - \frac{n(m-2)}{2} - 3n + 2} < \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}$$

implies that S^1 is a contradiction to eq. (3.6). Therefore, if S is a τ - tough set of $B_{m,n}$, then

$$|S| \ge \frac{n(m-2)}{2}.$$

By contradiction, suppose it is possible to find a cutset S^0 such that $|S^0| < |S^1| < |S|$. Consider,

$$S^0 = \{v_{ij} \mid i = 1,3,5,7,\ldots,m-5, 0 \le j \le n-1\}.$$

It can be observed that

$$\frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1} > \frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{\frac{n(m-4)}{2}}{mn - \frac{n(m-4)}{2} - 4n + 1} > \frac{\frac{n(m-2)}{2}}{mn - \frac{n(m-2)}{2} - 3n + 2}.$$

Hence,

$$1 < \frac{\frac{n(m-2)}{2}}{mn - \frac{n(m-2)}{2} - 3n + 2} \le \tau(B_{m,n}).$$

On simplifying,

$$1 < \frac{n(m-2)}{n(m-1) + 2} \le \tau(B_{m,n}).$$

As m increases, $\frac{n(m-2)}{n(m-1)+2}$ tends to 1. Therefore,

$$\tau(B_{m,n}) = \frac{n(m-2)}{n(m-1)+2}.$$ □

3.1 τ-Tough Sets of $B(m,n)$, $m > 3$, $n = 3,4$

***Case* (i).** When m odd

Since, S defined in eq. (3.3) is the only cutset satisfying eq. (3.5), the τ-tough set of $B(m,n)$, $m > 3$, $n = 3,4$ is given by

$$S_\tau = \{v_{ij} \mid i = 1,3,5,\dots,m-3,\ 0 \le j \le n-1\}.$$

***Case* (ii).** When m even

Since, S defined in eq. (3.7) is a cutset attaining $\tau(B_{m,n})$,

$$S_\tau^1 = \{v_{ij} \mid i = 1,3,5,\dots,m-3,\ 0 \le j \le n-1\}$$

is a τ-tough set of $B_{m,n}$, m even, $n = 3,4$. Since m is even, it is possible to find a cutset analogous to S such that it attains $\tau(B_{m,n})$, say S_τ^2. Then,

$$S_\tau^2 = \{v_{ij} \mid i = 2,4,6,\dots,m-2,\ 0 \le j \le n-1\}.$$

4. Toughness and τ-Tough Sets of $B_{m,n}$, $m \ge 4$, $n > 4$, m odd

Theorem 6. *Let $B_{m,n}$, $m \ge 3$, $n \ge 3$ be the bloom graph on mn vertices. Then, the minimum toughness of the bloom graph $B_{m,n}$, $m \ge 4$, $n > 4$, m odd is given by*

$$\tau(B_{m,n}) = \begin{cases} \frac{n(m-1)}{n(m-3)+4} & n \le m, \\ \frac{n(m+1)-2}{n(m-1)-2} & n > m,\ n \text{ odd}, \\ \frac{m+1}{m-1} & n > m,\ n \text{ even}. \end{cases}$$

Proof. Consider the bloom graph $B_{m,n}$, $m < n-1$, $n > 4$. The bloom graph is planar [15] and it is easy to verify that $\kappa(B_{m,3}) = 4$. Therefore, Theorem 1 and Theorem 3 imply that

$$1 < \tau(B_{m,n}) \le 2. \tag{4.1}$$

Therefore, there exists a τ-tough set S of $B_{m,n}$, such that

$$|S| \ge n\lfloor \frac{m}{2} \rfloor$$

which implies

$$\tau(B_{m,n}) > \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - n} \tag{4.2}$$

Consider,

$$S = \{v_{ij} \mid i = 1,3,5,\dots,m-2,\ 0 \le j \le n-1\}. \tag{4.3}$$

Clearly, $|S| = n\lfloor \frac{m}{2} \rfloor$ and $B_{m,n} \setminus S$ yields $mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2$ components. Therefore,

$$\frac{|S|}{\omega(B_{m,n} \setminus S)} = \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2}. \tag{4.4}$$

Moreover, S defined in eq. (4.3) is the only cutset of $B_{m,n}$ which satisfies eq. (4.4). By contrary, consider the following cutset with $n\lfloor\frac{m}{2}\rfloor$ vertices.

$$S^1 = \{v_{i\,j} \mid i = 0,2,4,\dots,m-3,\ 0 \le j \le n-1\}.$$

Then, $B_{m,n} \setminus S^1$ yields $mn - n\lfloor\frac{m}{2}\rfloor - 2n + 1$ components. Therefore,

$$\omega(B_{m,n} \setminus S^1) < \omega(B_{m,n} \setminus S).$$

It can be similarly proved for other cutsets with cardinality $n\lfloor\frac{m}{2}\rfloor$ by showing that they are analogous to S^1.

Then, eq. (4.1), eq. (4.2) and eq. (4.4) imply that

$$1 \le \frac{n\lfloor\frac{m}{2}\rfloor}{mn - n\lfloor\frac{m}{2}\rfloor - 2n + 2} < \tau(B_{m,n}) \le 2.$$

For $B_{m,n} \setminus S$ it is clear that for $S \cup \{v_{i\,j}\}$ where, $v_{i\,j} \in \{v_{i\,j} \mid i = 2,4,6,\dots,m-3,\ 0 \le j \le n-1\}$, we have

$$\frac{|S \cup \{v_{i\,j}\}|}{\omega(B_{m,n} \setminus S \cup \{v_{i\,j}\})} = \frac{n\lfloor\frac{m}{2}\rfloor + 1}{mn - n\lfloor\frac{m}{2}\rfloor - 2n + 1} > \frac{|S|}{\omega(B_{m,n} \setminus S)}.$$

Then, depending on the vertices that can be included in S, we have the following cases:

***Case* (i):** When, $n \le m$

***Subcase* (i):** n odd

Without loss of generality, let

$$S^0 = S \cup \{v_{i\,j} \mid i = 0, m-1, j = 0,2,4,\dots,n-3\}.$$

Then, $n \le m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{n\lfloor\frac{m}{2}\rfloor + n - 1}{mn - n\lfloor\frac{m}{2}\rfloor - n + 1} > \frac{n\lfloor\frac{m}{2}\rfloor}{mn - n\lfloor\frac{m}{2}\rfloor - 2n + 2}$$

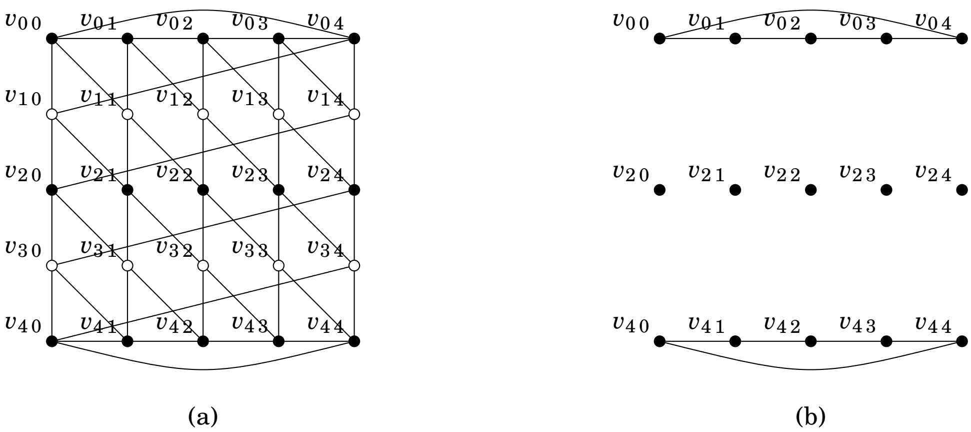


Figure 6. (a) Cutset S of the Bloom graph $B_{5,5}$; (b) Components of $B_{5,5} \setminus S$

Also, including every pair of independent vertices from $\{v_{i\,j} \mid i = 0, j = 0,2,4,\dots,n-3\}$ and $\{v_{i\,j} \mid i = m-1, j = 0,2,4,\dots,n-3\}$ respectively increases the ratio of number of vertices in S to the number of components in $B_{m,n} \setminus S$ in the ratio $2:1$. Hence,

$$\tau(B_{m,n}) = \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{n(m-1)}{n(m-3)+4}. \tag{4.5}$$

***Subcase* (ii):** n even

Without loss of generality, let

$$S^0 = S \cup \{v_{ij} \mid i = 0, m-1, j = 0, 2, 4, \dots, n-2\}.$$

Then, $n \le m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{n\lfloor \frac{m}{2} \rfloor + n}{mn - n\lfloor \frac{m}{2} \rfloor - n} > \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2}.$$

Also, including every pair of independent vertices from $\{v_{ij} \mid i = 0, j = 0, 2, 4, \dots, n-2\}$ and $\{v_{ij} \mid i = m-1, j = 0, 2, 4, \dots, n-2\}$ respectively increases the ratio of number of vertices in S to the number of components in $B_{m,n} \setminus S$ in the ratio $2:1$. Hence,

$$\tau(B_{m,n}) = \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{n(m-1)}{n(m-3)+4}. \tag{4.6}$$

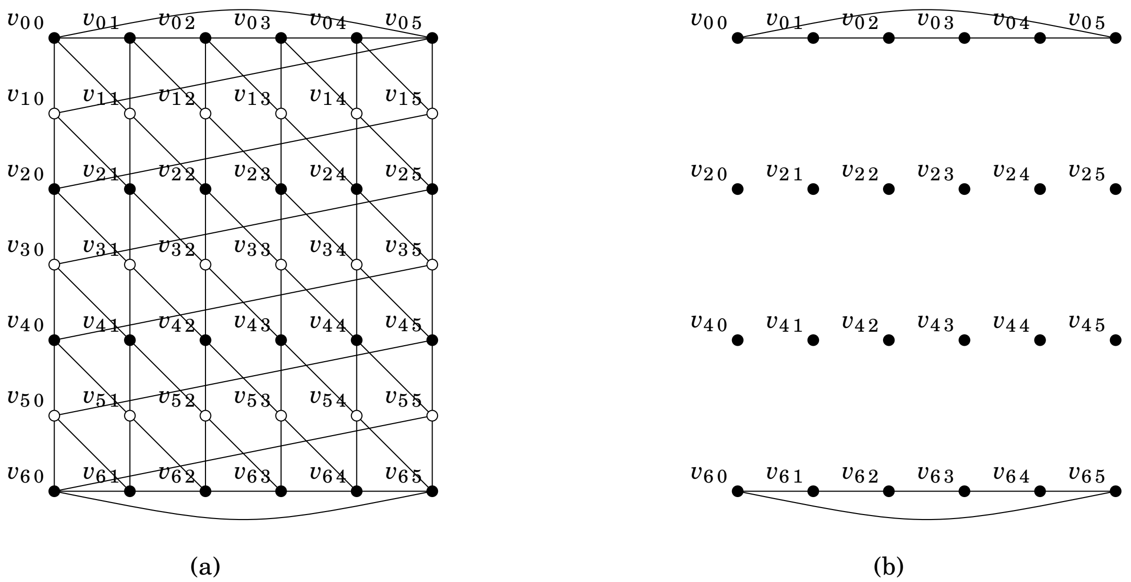


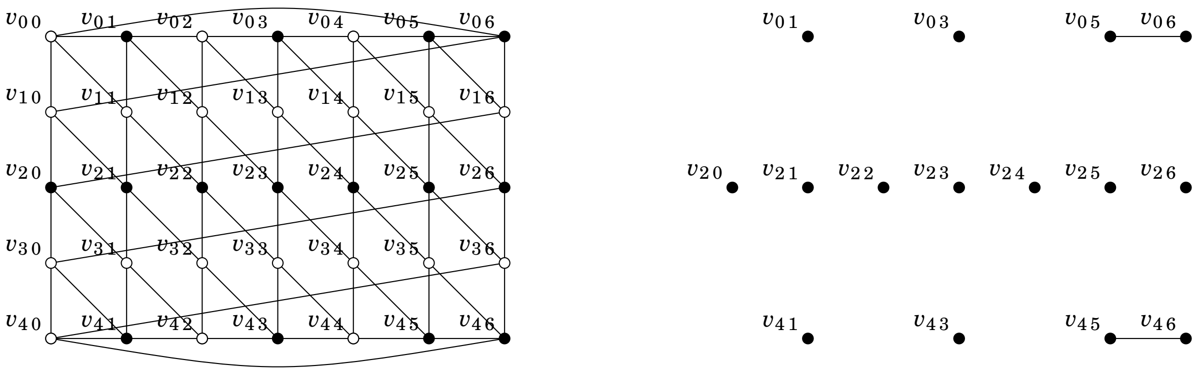
Figure 7. (a) Cutset S of the Bloom graph $B_{7,6}$; (b) Components of $B_{7,6} \setminus S$

Case (ii): When $n > m$

Subcase (i): n odd

Without loss of generality, let

$$S^0 = S \cup \{v_{ij} \mid i = 0, m-1, j = 0, 2, 4, \dots, n-3\}. \tag{4.7}$$



(a) (b)

Figure 8. (a) Cutset S^0 of the Bloom graph; $B_{5,7}$, (b) Components of $B_{5,7} \setminus S^0$

Then, $n > m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{n\lfloor \frac{m}{2} \rfloor + n - 1}{mn - n\lfloor \frac{m}{2} \rfloor - n + 1} < \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2}.$$

Since, the components of $B_{m,n} \setminus S^0$ are isomorphic to K_1,

$$\tau(B_{m,n}) = \frac{n\lfloor \frac{m}{2} \rfloor + n - 1}{mn - n\lfloor \frac{m}{2} \rfloor - n + 1}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{n(m+1) - 2}{n(m-1) - 2}.$$

Subcase (ii): n even

Without loss of generality, let

$$S^0 = \{v_{ij} \mid i = 1, 3, 5, \dots, m-2, 0 \le j \le n-1\} \cup \{v_{ij} \mid i = 0, m-1, j = 0, 2, 4, \dots, n-2\} \tag{4.8}$$

$n > m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{n\lfloor \frac{m}{2} \rfloor + n}{mn - n\lfloor \frac{m}{2} \rfloor - n} < \frac{n\lfloor \frac{m}{2} \rfloor}{mn - n\lfloor \frac{m}{2} \rfloor - 2n + 2}.$$

Since, the components of $B_{m,n} \setminus S^0$ are isomorphic to K_1,

$$\tau(B_{m,n}) = \frac{n\lfloor \frac{m}{2} \rfloor + n}{mn - n\lfloor \frac{m}{2} \rfloor - n}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{m+1}{m-1}.$$ □

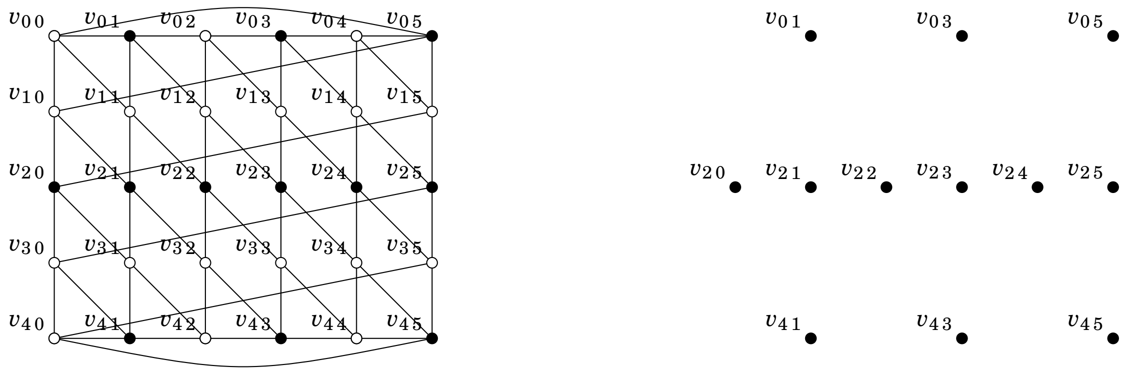


Figure 9. (a) Cutset S^0 of the Bloom graph $B_{5,6}$; (b) Components of $B_{5,6} \setminus S^0$

4.1 τ-Tough Sets of $B(m,n)$, $m \geq 4$, $n > 4$, m odd

Case (i). When $n \leq m$

Since, S defined in eq. (4.3) is the only cutset satisfying eqs. (4.5) and (4.6), the τ-tough set of $B(m,n)$, $m \geq 4$, $n \leq m$, m odd is given by

$$S_\tau = \{v_{i\,j} \mid i = 1,3,5,\ldots,m-2,\ 0 \leq j \leq n-1\}. \tag{4.9}$$

Case (ii). When $n > m$

Subcase (i): n *odd*

Since, S^0 defined in eq. (4.7) is a cutset attaining $\tau(B_{m,n})$ derived in eq. (4.5),

$$S_\tau^1 = S \cup \{v_{i\,j} \mid i = 0, m-1,\ j = 0,2,4,\ldots,n-3\},$$

where

$$S = \{v_{i\,j} \mid i = 1,3,5,\ldots,m-2,\ 0 \leq j \leq n-1\}$$

is a τ-tough set of $B_{m,n}$, $m \geq 4$, $n > m$, m odd. Let

$$V_0 = \{v_{0\,j} \mid j = 0,2,4,\ldots,n-3\},$$

$$V_{m-1} = \{v_{m-1\,j} \mid j = 0,2,4,\ldots,n-3\}.$$

Then,

$$S_\tau^1 = S \cup V_0 \cup V_{m-1}.$$

Since, S is unique, the remaining τ-tough sets can be obtained with respect to V_0 and V_{m-1}. By the definition of V_0 and V_{m-1}, they are cutsets of the components of $B_{m,n}$ isomorphic to cycle graph C_n. In general, cutsets of C_n analogous to V_0 and V_{m-1} are as follows:

$$V_0' = \{v_{0\,j+2}\, v_{0\,j+4}\, \ldots\, v_{0\,j-3}\, v_{0\,j-1},\ 0 \leq j \leq n-1\}$$

such that $(v_{0\,j+2}\ v_{0\,j+4}\ \ldots\ v_{0\,j-3}\ v_{0\,j-1})(v_{0\,j}\ v_{0\,j+1})$, $0 \leq j \leq n-1$ generates the components of C_n, namely the trivial components $v_{0\,j+2}$, $v_{0\,j+4}$, $\ldots$, $v_{0\,j-3}$, $v_{0\,j-1}$ and the component $(v_{0\,j}, v_{0\,j+1})$ isomorphic to K_2. Similarly,

$$V_{m-1}' = \{v_{m-1\,j+2}\, v_{m-1\,j+4}\, \ldots\, v_{m-1\,j-3}\, v_{m-1\,j-1},\ 0 \leq j \leq n-1\}$$

such that $(v_{m-1\,j+2}\ v_{m-1\,j+4}\ \dots\ v_{m-1\,j-3}\ v_{m-1\,j-1})(v_{m-1\,j}\ v_{m-1\,j+1}), 0 \le j \le n-1$ generates the cut vertices of C_n, namely, $v_{m-1\,j+2}, v_{m-1\,j+4}, \dots, v_{m-1\,j-3}, v_{m-1\,j-1}$ and the component $(v_{m-1\,j}, v_{m-1\,j+1})$ isomorphic to K_2.

Hence, following are the τ - tough sets of $B_{m,n}$, $m \ge 4$, $n > m$, n odd, m odd:

$$S_\tau = S \cup V_0' \cup V_{m-1}'. \tag{4.10}$$

Subcase (ii): n even

Since, S^0 defined in eq. (4.8) is a cutset attaining $\tau(B_{m,n})$ derived in eq. (4.6),

$$S_\tau^1 = S \cup \{v_{i\,j} \mid i = 0, m-1, j = 0, 2, 4, \dots, n-2\},$$

where

$$S = \{v_{i\,j} \mid i = 1, 3, 5, \dots, m-2, 0 \le j \le n-1\}$$

is a τ-tough set of $B_{m,n}$, $m \ge 4$, $n > m$, m odd. Let

$$V_0 = \{v_{0\,j} \mid j = 0, 2, 4, \dots, n-2\},$$

$$V_{m-1} = \{v_{m-1\,j} \mid j = 0, 2, 4, \dots, n-2\}.$$

Then,

$$S_\tau^1 = S \cup V_0 \cup V_{m-1}.$$

Since, S is unique, the remaining τ-tough sets can be obtained with respect to V_0 and V_{m-1}. By the definition of V_0 and V_{m-1}, they are cutsets of the components of $B_{m,n}$ isomorphic to cycle graph C_n. In general, cutsets of C_n analogous to V_0 and V_{m-1} are as follows:

$$V_0' = \{v_{0\,j}, v_{0\,j+2}, v_{0\,j+4}, \dots, v_{0\,j-2} \mid 0 \le j \le n-1\},$$

$$V_{m-1}' = \{v_{m-1\,j}, v_{m-1\,j+2}, v_{m-1\,j+4}, \dots, v_{m-1\,j-2} \mid 0 \le j \le n-1\}.$$

Hence, following are the τ-tough sets of $B_{m,n}$, $m \ge 4$, $n > m$, n even, m odd:

$$S_\tau = S \cup V_0' \cup V_{m-1}'. \tag{4.11}$$

5. Toughness and τ-Tough Sets of $B_{m,n}$, $m \ge 4$, $n > 4$, m even

Theorem 7. *Let $B_{m,n}$, $m \ge 3$, $n \ge 3$ be the bloom graph on mn vertices. Then, the minimum toughness of the bloom graph $B_{m,n}$, $m \ge 4$, $n > 4$, m even is given by*

$$\tau(B_{m,n}) = \begin{cases} \frac{mn}{n(m-2)+2} & n \le m, \\ \frac{n(m+1)-1}{n(m-1)-1} & n > m,\ n \text{ odd}, \\ \frac{m+1}{m-1} & n > m,\ n \text{ even}. \end{cases}$$

Proof. Consider the bloom graph $B_{m,n}$, $m < n-1$, $n > 4$. The bloom graph is planar [15] and it is easy to verify that $\kappa(B_{m,3}) = 4$. Therefore, Theorem 1 and Theorem 3 imply that

$$1 < \tau(B_{m,n}) \le 2.$$

Therefore, there exists a τ tough set S of $B_{m,n}$, such that

$$|S| \ge \frac{mn}{2}$$

which implies

$$\tau(B_{m,n}) > \frac{\frac{mn}{2}}{mn - \frac{mn}{2}}.$$

Consider the following cutset:

$$S = \{v_{ij} \mid i = 1,3,5,\dots,m-1, 0 \le j \le n-1\}. \tag{5.1}$$

Clearly, $|S| = \frac{mn}{2}$ and $B_{m,n} \setminus S$ yields $\frac{mn}{2} - n + 1$ components. Therefore,

$$\frac{|S|}{\omega(B_{m,4} \setminus S)} = \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}.$$

Also, cutsets of $B_{m,n}$ analogous to S satisfy the following equation:

$$1 \le \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1} < \tau(B_{m,n}) \le 2.$$

From, $B_{m,n} \setminus S$ it is clear that for $S \cup \{v_{ij}\}$ where, $v_{ij} \in \{v_{ij} \mid i = 2,4,6,\dots,m-2, 0 \le j \le n-1\}$, we have

$$\frac{|S \cup \{v_{ij}\}|}{\omega(B_{m,n} \setminus S \cup \{v_{ij}\})} = \frac{\frac{mn}{2} + 1}{\frac{mn}{2} - n} > \frac{|S|}{\omega(B_{m,n} \setminus S)}$$

Then, depending on the vertices that can be included in S, we have the following cases:

Case (i): When $n \le m$

Subcase (i): When n odd

Without loss of generality, let

$$S^0 = S \cup \{v_{ij} \mid i = 0, j = 0,2,4,\dots,n-3\}.$$

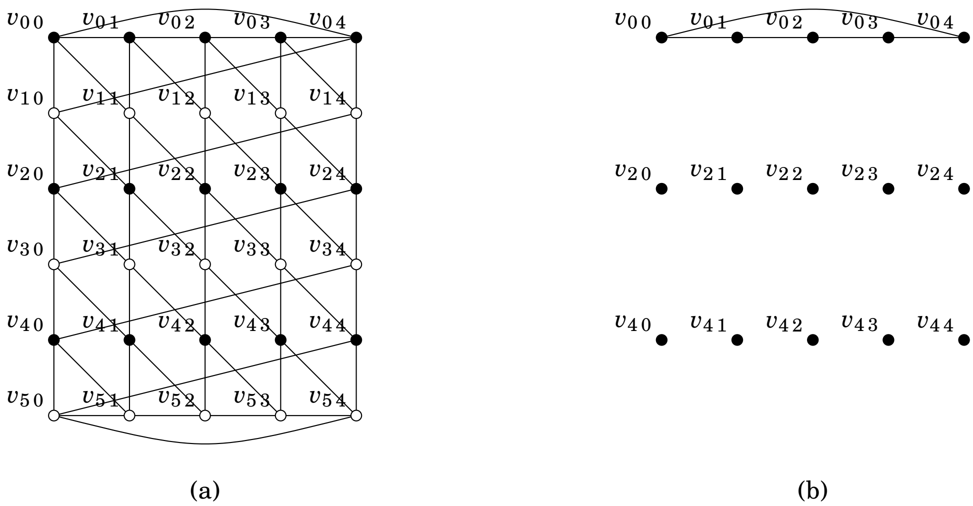


Figure 10. (a) Cutset S of the Bloom graph $B_{6,5}$; (b) Components of $B_{6,5} \setminus S$

Then, $n \le m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{\frac{mn}{2} + \lfloor\frac{n}{2}\rfloor}{\frac{mn}{2} - n + \lfloor\frac{n}{2}\rfloor} > \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}.$$

Also, including every pair of independent vertices from $\{v_{ij} \mid i=0, j=0,2,4,\dots,n-3\}$ increases the ratio of number of vertices in S to the number of components in $B_{m,n}\setminus S$ in the ratio $2:1$. Hence,

$$\tau(B_{m,n}) = \frac{\frac{mn}{2}}{\frac{mn}{2}-n+1}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{mn}{n(m-2)+2}.$$

Subcase (ii): When n even

Without loss of generality, let

$$S^0 = S \cup \{v_{ij} \mid i=0, j=0,2,4,\dots,n-2\}.$$

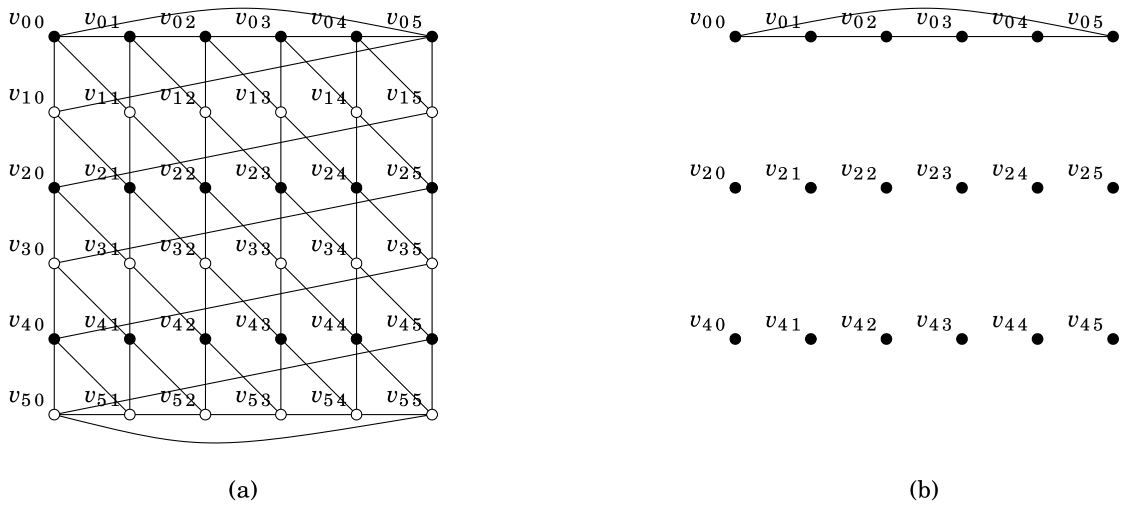


Figure 11. (a) Cutset S of the Bloom graph $B_{6,6}$; (b) Components of $B_{6,6}\setminus S$

Then, $n \le m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n}\setminus S^0)} = \frac{\frac{mn}{2}+\frac{n}{2}}{\frac{mn}{2}-\frac{n}{2}} > \frac{\frac{mn}{2}}{\frac{mn}{2}-n+1}.$$

Also, including every pair of independent vertices from $\{v_{ij} \mid i=0, j=0,2,4,\dots,n-2\}$ increases the ratio of number of vertices in S to the number of components in $B_{m,n}\setminus S$ in the ratio $2:1$. Hence,

$$\tau(B_{m,n}) = \frac{\frac{mn}{2}}{\frac{mn}{2}-n+1}.$$

On simplifying,

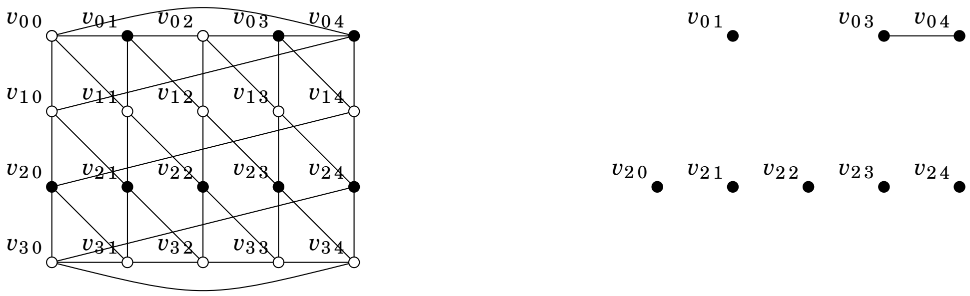
$$\tau(B_{m,n}) = \frac{mn}{n(m-2)+2}.$$

Case (ii): When $n > m$

Subcase (i): n odd

Without loss of generality, let

$$S^0 = S \cup \{v_{ij} \mid i=0, j=0,2,4,\dots,n-3\}. \tag{5.2}$$



(a) (b)

Figure 12. (a) Cutset S^0 of the Bloom graph $B_{4,5}$; (b) Components of $B_{4,5} \setminus S^0$

Then, $n > m$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{\frac{mn}{2} + \lfloor \frac{n}{2} \rfloor}{\frac{mn}{2} - n + \lfloor \frac{n}{2} \rfloor} < \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}.$$

Since, the components of $B_{m,n} \setminus S^0$ are isomorphic to K_1,

$$\tau(B_{m,n}) = \frac{\frac{mn}{2} + \lfloor \frac{n}{2} \rfloor}{\frac{mn}{2} - n + \lfloor \frac{n}{2} \rfloor}.$$

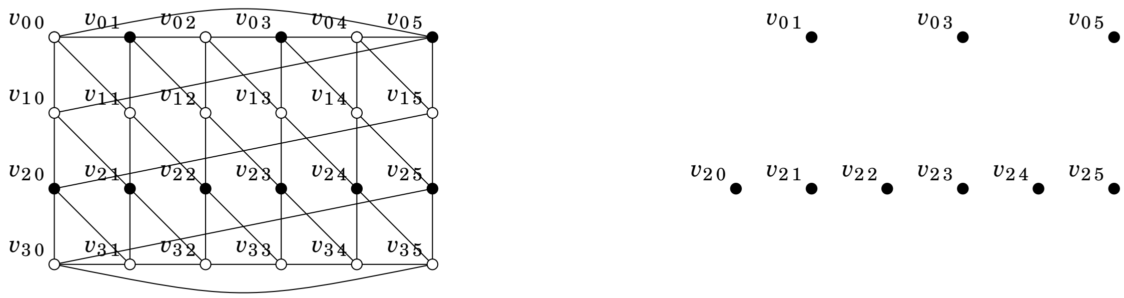
On simplifying,

$$\tau(B_{m,n}) = \frac{n(m+1)-1}{n(m-1)-1}. \tag{5.3}$$

Subcase (ii): n even

Without loss of generality, let

$$S^0 = S \cup \{v_{ij} \mid i = 0, j = 0, 2, 4, \dots, n-2\}. \tag{5.4}$$



(a) (b)

Figure 13. (a) Cutset S^0 of the Bloom graph $B_{4,6}$; (b) Components of $B_{4,6} \setminus S^0$

Then, $m < n$ implies that

$$\frac{|S^0|}{\omega(B_{m,n} \setminus S^0)} = \frac{\frac{mn}{2} + \frac{n}{2}}{\frac{mn}{2} - n + \frac{n}{2}} < \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}.$$

Since, the components of $B_{m,n} \setminus S^0$ are isomorphic to K_1,

$$\tau(B_{m,n}) = \frac{\frac{mn}{2} + \frac{n}{2}}{\frac{mn}{2} - n + \frac{n}{2}}.$$

On simplifying,

$$\tau(B_{m,n}) = \frac{m+1}{m-1} \tag{5.5}$$

□

5.1 τ-Tough Sets of $B(m,n)$, $m \geq 4$, $n > 4$, m even

Case (i): When $n \leq m$

Since, S defined in eq. (5.1) is a cutset attaining $\tau(B_{m,n})$,

$$S_\tau^1 = \{v_{ij} \mid i = 1,3,5,\dots,m-1, 0 \leq j \leq n-1\}.$$

is a τ-tough set of $B_{m,n}$, $m \geq 4$, $n \leq m$, m even. Since m is even, it is possible to find a cutset analogous to S such that it attains $\tau(B_{m,n})$, say S_τ^2. Then,

$$S_\tau^2 = \{v_{ij} \mid i = 0,2,4,\dots,m-2, 0 \leq j \leq n-1\}. \tag{5.6}$$

Case (ii). When $n > m$

Subcase (i): n *odd*

Since, S^0 defined in eq. (5.2) is a cutset attaining $\tau(B_{m,n})$ derived in eq. (5.3),

$$S_\tau^1 = S \cup \{v_{ij} \mid i = 0, j = 0,2,4,\dots,n-3\},$$

where

$$S = \{v_{ij} \mid i = 1,3,5,\dots,m-1, 0 \leq j \leq n-1\}$$

is a cutset of $B_{m,n}$, $m \geq 4$, $n > m$, m even.

Let

$$V_0 = \{v_{0j} \mid j = 0,2,4,\dots,n-3\}.$$

Then,

$$S_\tau^1 = S \cup V_0.$$

Since, S is unique, the remaining τ-tough sets can be obtained with respect to V_0. By the definition of V_0, it is the cutset of the component of $B_{m,n}$ isomorphic to cycle graph C_n. In general, cutsets of C_n analogous to V_0 are as follows:

$$V_0' = \{v_{0\,j+2}\, v_{0\,j+4} \,\dots\, v_{0\,j-3}\, v_{0\,j-1}, 0 \leq j \leq n-1\}$$

such that $(v_{0\,j+2}\, v_{0\,j+4} \,\dots\, v_{0\,j-3}\, v_{0\,j-1})(v_{0\,j}\, v_{0\,j+1})$, $0 \leq j \leq n-1$ generates the cut vertices of C_n, namely, $v_{0\,j+2}, v_{0\,j+4}, \dots, v_{0\,j-3}, v_{0\,j-1}$ and the component $(v_{0\,j}, v_{0\,j+1})$ isomorphic to K_2. Hence, following are the generalization of S_τ^1:

$$S_\tau^1 = S \cup V_0'. \tag{5.7}$$

Since, m is even, it is possible to find a cutset analogous to S, say S'.

$$S' = \{v_{ij} \mid i = 0,2,4,\dots,m-2, 0 \leq j \leq n-1\}.$$

Then, the cutset analogous to S^0 defined in eq. (5.2) attaining $\tau(B_{m,n})$ can be obtained by including the cutset V_{m-1} of the component of $B_{m,n}$ isomorphic to cycle graph C_n.

$$S_\tau^2 = S' \cup V_{m-1},$$

where

$$V_{m-1} = \{v_{0\,j} \mid j = 0,2,4,\dots,n-3\}.$$

Also, the generalization of S_τ^2 is as follows:

$$S_\tau^2 = S' \cup V'_{m-1}, \tag{5.8}$$

where

$$V'_{m-1} = (v_{m-1\,j+2}\, v_{m-1\,j+4}\, \dots\, v_{m-1\,j-3}\, v_{m-1\,j-1})(v_{m-1\,j}\, v_{m-1\,j+1}),\ 0 \le j \le n-1.$$

Hence, S_τ^1 and S_τ^2 obtained in eq. (5.7) and eq. (5.8) are the τ-tough sets of $B_{m,n}$, $m \ge 4$, $n > m$, n odd, m even.

Subcase (ii): n *even*

Since, S^0 defined in eq. (5.4) is a cutset attaining $\tau(B_{m,n})$ derived in eq. (5.5),

$$S_\tau^1 = S \cup \{v_{i\,j} \mid i = 0,\ j = 0,2,4,\dots,n-2\},$$

where

$$S = \{v_{i\,j} \mid i = 1,3,5,\dots,m-1,\ 0 \le j \le n-1\}$$

is a cutset of $B_{m,n}$, $m \ge 4$, $n > m$, m even.

Let

$$V_0 = \{v_{0\,j} \mid j = 0,2,4,\dots,n-2\}.$$

Then,

$$S_\tau^1 = S \cup V_0.$$

Since, S is unique, the remaining τ-tough sets can be obtained with respect to V_0. By the definition of V_0, it is the cutset of the component of $B_{m,n}$ isomorphic to cycle graph C_n. In general, cutsets of C_n analogous to V_0 are as follows:

$$V'_0 = \{v_{0\,j} \mid j = 0,2,4,\dots,n-2\}.$$

Hence, following are the generalization of S_τ^1:

$$S_\tau^1 = S \cup V'_0. \tag{5.9}$$

Since, m is even, it is possible to find a cutset analogous to S, say S'.

$$S' = \{v_{i\,j} \mid i = 0,2,4,\dots,m-2,\ 0 \le j \le n-1\}.$$

Then, the cutset analogous to S^0 defined in eq. (5.4) attaining $\tau(B_{m,n})$ can be obtained by including the cutset V_{m-1} of the component of $B_{m,n}$ isomorphic to cycle graph C_n.

$$S_\tau^2 = S' \cup V_{m-1},$$

where

$$V_{m-1} = \{v_{0\,j} \mid j = 0,2,4,\dots,n-2\}.$$

Also, the generalization of S_τ^2 is as follows:

$$S_\tau^2 = S' \cup V'_{m-1}, \tag{5.10}$$

where

$$V'_{m-1} = \{v_{m-1\,j} \mid j = 0,2,4,\dots,n-2\}.$$

Hence, S_τ^1 ans S_τ^2 obtained in eq. (5.9) and eq. (5.10) are the τ-tough sets of $B_{m,n}$, $m \geq 4$, $n > m$, n even, m even.

6. Maximum Extension of Certain 2-Tough Sets of $B_{m,n}$, $m \geq 3$, $n \geq 3$

6.1 Maximum Extension of Certain 2-Tough Sets of $B_{m,n}$, $m > 3$, $n \geq 3$, m odd

In this section, we have investigated the conditions for maximum extension of certain 2-tough sets of the bloom graph $B_{m,n}$, $m \geq 3$, $n \geq 3$ and later extend the same to any t-tough set such that $t > \tau$.

Theorem 8. *Let $B_{m,n}$, $m \geq 3$, $n \geq 3$ be the bloom graph on mn vertices. Then, every 2 - tough set of $B_{m,n}$ $m > 3$, $n \geq 3$, m odd given by*

$$S_2 = \{v_{i\,j}, v_{i\,j+1}, v_{i+2\,j+1}v_{i+2\,j+2} \mid i = 1,3,5,\dots,m-2, 0 \leq j \leq n-1\}$$

has a maximum extension to a τ-tough set of $B_{m,n}$.

Proof. Consider the bloom graph $B_{m,n}$ $m > 3$, $n \geq 3$, m odd. Without loss of generality, let

$$S^1 = \{v_{1\,j}, v_{1\,j+1}, v_{3\,j+1}v_{3\,j+2} \mid 0 \leq j \leq n-1\}, \quad t_{S^1} = 2.$$

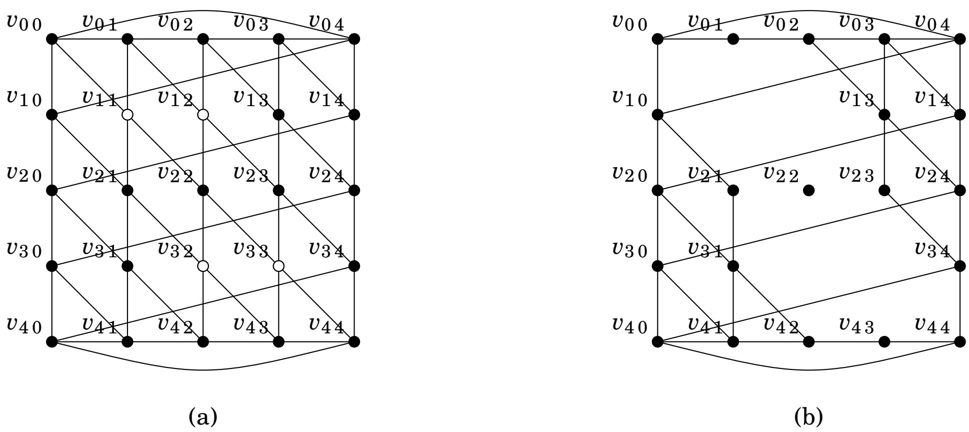


Figure 14. (a) 2-tough set $S_2 \subset S_\tau$ of the Bloom graph $B_{5,5}$; (b) Components of $B_{5,5} \setminus S_2$

The components of $B_{m,n} \setminus S^1$ are $\{v_{2\,j+1}\} \simeq K_1$ and $B_{m,n}[V \setminus \{S^1 \cup v_{2\,j+1}\}]$. Then, an extension of S^1 can be obtained by including vertices adjacent to vertices of minimum degree in $B_{m,n}[V \setminus \{S^1 \cup v_{2\,j+1}\}]$, recursively.

Consider the following tough sets and their corresponding values of toughness:

$$S^2 = \{v_{i\,j} \mid i = 1,3, 0 \leq j \leq n-1\}, \quad t_{S^2} = \frac{2n}{n+2}$$

$$S^3 = \{v_{i\,j} \mid i = 1,3,5, 0 \leq j \leq n-1\}, \quad t_{S^3} = \frac{3n}{2n+2}$$

$$S^4 = \{v_{i\,j} \mid i = 1,3,5,7, 0 \leq j \leq n-1\}, \quad t_{S^4} = \frac{4n}{3n+2}$$

$$\vdots$$

$S^{\lfloor\frac{m}{2}\rfloor}=\{v_{ij} \mid i=1,3,5,\dots,m-2,\ 0\le j\le n-1\}$, $t_{S^{\lfloor\frac{m}{2}\rfloor}}=\dfrac{n\lfloor\frac{m}{2}\rfloor}{mn-n\lfloor\frac{m}{2}\rfloor-2n+2}$

Clearly, $S^1\subset S^2\subset S^3\subset\dots\subset S^{\lfloor\frac{m}{2}\rfloor}$ and $t_{S^2}>t_{S^3}t_{S^4}>\dots>t_{S^{\lfloor\frac{m}{2}\rfloor}}$. Hence, $S^{\lfloor\frac{m}{2}\rfloor}$ is an extension of S^1.

Case (i): When $m=3$, $n\ge4$

Since, $B_{3,n}$ is 2-tough, S^1 is the maximum extension of itself.

Case (ii): When $m\ge4$, $n\ge4$, m odd

Subcase (i): When $n\le m$, the τ-tough set is given by eq. (4.9).

$$S_\tau=\{v_{ij} \mid i=1,3,5,\dots,m-2,\ 0\le j\le n-1\}=S^{\lfloor\frac{m}{2}\rfloor}.$$

Hence, S_τ is the maximum extension of S^1.

Subcase (ii): When $n>m$, n odd, without loss of generality consider the following τ-tough set obtained from eq. (4.10).

$$S_\tau=\{v_{ij} \mid i=1,3,5,\dots,m-2,\ 0\le j\le n-1\}\cup\{v_{ij} \mid i=0,m-1,\ j=0,2,4,\dots,n-3\},$$

where

$$\tau=\frac{n(m+1)-2}{n(m-1)-2}.$$

Clearly, $S^{\lfloor\frac{m}{2}\rfloor}\subset S_\tau$ and $\tau<t_{S^{\lfloor\frac{m}{2}\rfloor}}$. Hence, S_τ is the maximum extension of S^1.

Subcase (iii): When $n>m$, n even, without loss of generality consider the following τ-tough set obtained from eq. (4.11).

$$S_\tau=\{v_{ij} \mid i=1,3,5,\dots,m-2,\ 0\le j\le n-1\}\cup\{v_{ij} \mid i=0,m-1,\ j=0,2,4,\dots,n-2\},$$

where

$$\tau=\frac{m+1}{m-1}$$

Clearly, $S^{\lfloor\frac{m}{2}\rfloor}\subset S_\tau$ and $\tau<t_{S^{\lfloor\frac{m}{2}\rfloor}}$. Hence, S_τ is the maximum extension of S^1. □

Corollary 8.1. *Let $B_{m,n}$, $m\ge3$, $n\ge3$, m odd be a bloom graph on mn vertices. Then, every t-tough set S_t of $B_{m,n}$ has a maximum extension to S_τ if and only if $S_t\subseteq S_\tau$.*

As a consequence of the corollary, suppose $S_t\not\subseteq S_\tau$ for some $t\ge\tau$. Then,

$$S_t\subseteq\{v_{ij} \mid i=0,2,4,\dots,m-1,\ 0\le j\le n-1\}.$$

Let

$$S'=\{v_{ij} \mid i=0,2,4,\dots,m-1,\ 0\le j\le n-1\}.$$

Then, every $S_t\not\subseteq S_\tau$ has a maximum extension to S' since it is maximal with respect to the components of $B_{m,n}\setminus S'$, (i.e.), the components of $B_{m,n}\setminus S'$ are isomorphic to K_1.

6.2 Maximum Extension of Certain 2-Tough Sets of $B_{m,n}$, $m > 3$, $n \geq 3$, m even

Theorem 9. *Let $B_{m,n}$, $m \geq 3$, $n \geq 3$ be the bloom graph on mn vertices. Then, every 2-tough set of $B_{m,n}$ $m > 3$, $n \geq 3$, m even given by*

$$S_2^1 = \{v_{i\,j}, v_{i\,j+1}, v_{i+2\,j+1}v_{i+2\,j+2} \mid i = 0,2,4,\dots,m-2, 0 \leq j \leq n-1\},$$
$$S_2^2 = \{v_{i\,j}, v_{i\,j+1}, v_{i+2\,j+1}v_{i+2\,j+2} \mid i = 1,3,5,\dots,m-1, 0 \leq j \leq n-1\}$$

has a maximum extension to a τ-tough set of $B_{m,n}$.

Proof. Consider the bloom graph $B_{m,n}$ $m > 3$, $n \geq 3$, m odd. Without loss of generality, let

$$S^1 = \{v_{0\,j}, v_{0\,j+1}, v_{2\,j+1}v_{2\,j+2} \mid 0 \leq j \leq n-1\}, \quad t_{S^1} = 2.$$

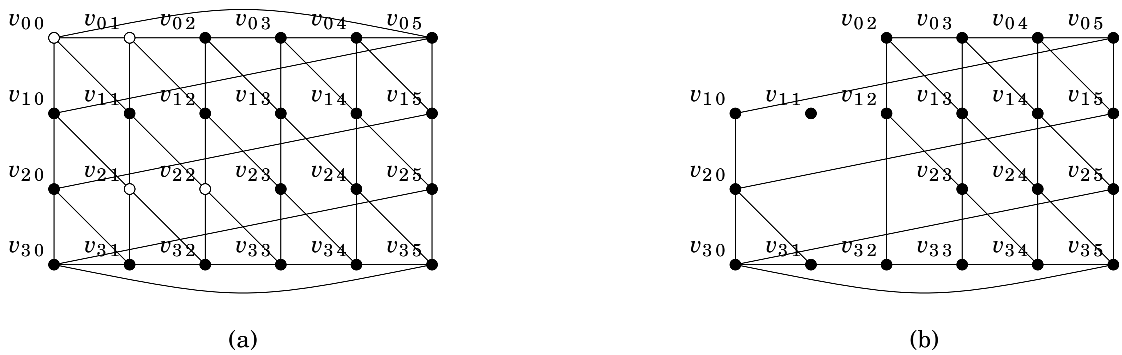


Figure 15. (a) 2-tough set S^1 of the Bloom graph $B_{5,6}$ (b); Components of $B_{5,6} \setminus S^1$

The components of $B_{m,n} \setminus S^1$ are $\{v_{1\,j+1}\} \simeq K_1$ and $B_{m,n}[V \setminus \{S^1 \cup v_{2\,j+1}\}]$. Then, an extension of S^1 can be obtained by including vertices adjacent to vertices of minimum degree in $B_{m,n}[V \setminus \{S^1 \cup v_{1\,j+1}\}]$, recursively.

Consider the following tough sets and their corresponding values of toughness:

$$S^2 = \{v_{i\,j} \mid i = 0,2, 0 \leq j \leq n-1\}, \quad t_{S^2} = \frac{2n}{n+1}$$
$$S^3 = \{v_{i\,j} \mid i = 0,2,4, 0 \leq j \leq n-1\}, \quad t_{S^3} = \frac{3n}{2n+1}$$
$$S^4 = \{v_{i\,j} \mid i = 0,2,4,6, 0 \leq j \leq n-1\}, \quad t_{S^4} = \frac{4n}{3n+1}$$
$$\vdots$$
$$S^{\frac{m}{2}} = \{v_{i\,j} \mid i = 0,2,4,\dots,m-2, 0 \leq j \leq n-1\}, \quad t_{S^{\frac{m}{2}}} = \frac{\frac{mn}{2}}{\frac{mn}{2} - n + 1}$$

Clearly, $S^1 \subset S^2 \subset S^3 \subset \dots \subset S^{\frac{m}{2}}$ and $t_{S^2} > t_{S^3} t_{S^4} > \dots > t_{S^{\frac{m}{2}}}$. Hence, $S^{\frac{m}{2}}$ is an extension of S^1.

Case (i): When $n \leq m$, the τ-tough set is given by eq. (5.6).

$$S_\tau = \{v_{i\,j} \mid i = 0,2,4,\dots,m-2, 0 \leq j \leq n-1\} = S^{\frac{m}{2}}$$

Hence, S_τ is the maximum extension of S^1.

Case (ii): When $n > m$, n odd, without loss of generality consider the following τ - tough set obtained from eq. (5.8).

$$S_\tau = \{v_{ij} \mid i = 0,2,4,\dots,m-2,\ 0 \le j \le n-1\} \cup \{v_{ij} \mid i = m-1,\ j = 0,2,4,\dots,n-3\},$$

where

$$\tau = \frac{n(m+1)-1}{n(m-1)-1}.$$

Clearly, $S^{\frac{m}{2}} \subset S_\tau$ and $\tau < t_{S^{\frac{m}{2}}}$. Hence, S_τ is the maximum extension of S^1.

Case (iii): When $n > m$, n even, without loss of generality consider the following τ - tough set obtained from eq. (5.10).

$$S_\tau = \{v_{ij} \mid i = 0,2,4,\dots,m-2, 0 \le j \le n-1\} \cup \{v_{ij} \mid i = m-1, j = 0,2,4,\dots,n-2\},$$

where

$$\tau = \frac{m+1}{m-1}.$$

Clearly, $S^{\frac{m}{2}} \subset S_\tau$ and $\tau < t_{S^{\frac{m}{2}}}$. Hence, S_τ is the maximum extension of S^1.

Similarly, it can be proved that 2-tough set S_2^2 has a maximum extension to a τ-tough set of $B_{m,n}$ □

7. Conclusion

We have proposed toughness to be a measure for measuring the efficiency of the data broadcasting under fault tolerant conditions and maximum extension of a t-tough set to be a fault propagation warning. We have investigated and settled the problem of toughness and maximum extension of all t-tough sets, $t \ge \tau$, for the bloom graph $B_{m,n}$, $m \ge 3$, $n \ge 3$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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