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Research Article

Fixed Point Theorem in *G*-Metric Space for Auxiliary Functions

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Abstract. In this paper, fixed point results satisfying generalized contractive condition with new auxiliary functions are proved in G-metric spaces.

Keywords. Fixed point; Auxiliary functions; G-metric space

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1. Introduction

In 2006, Mustafa and Sims [9] introduced a new notion of generalized metric space (L,G) called G-metric space.

Definition 1.1 ([9])**.** Let L be a nonempty set, and $G : L^3 \to \mathbb{R}^+$ be a function satisfying the following properties:

(1) $G(l,m,n) = 0$, if $l = m = n$,

(2) $0 < G(l,l,m)$, for all $l,m \in M$, with $l \neq m$,

(3) $G(l,l,m) \leq G(l,m,n)$, for all $l,m,n \in M$, with $n \neq m$,

(4) $G(l,m,n) = G(l,n,m) = G(m,n,l) = \dots$ (symmetry in all three variables),

(5) $G(l,m,n) \leq G(l,a,a) + G(a,m,n)$, for all $l,m,n,a \in M$, (rectangular inequality).

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Then the function G is called a generalized metric, or, more specifically, a G-metric on L. The pair (L,G) is called a *G-metric space*.

Definition 1.2 ([7])**.** For a G-metric space (L,G), a mapping $T:L\to L$ is called a contraction mapping on L if for any real number λ with $0\le\lambda<1$, the following inequality holds:

$$G(Tl,Tm,Tn)\le\lambda G(l,m,n),\quad \text{for all } l,m,n\in L.$$

Remark 1.3. It can be easily seen that the geographical distance between the images of any three points of a given set is contracting by a uniform factor $\lambda<1$.

Example 1.4. Let $L=\mathbb{R}^3$ be a set equipped with standard G-metric G (i.e. $G(l,m,n)=|l_1-l_2|+|l_2-l_3|+|m_1-m_2|+|m_2-m_3|+|n_1-n_2|+|n_2-n_3|$ for all $l,m,n\in L$) and $T:\mathbb{R}^3\to\mathbb{R}^3$ be the mapping defined as $Tl=\frac{5}{8}l$ for all $l\in\mathbb{R}^3$. Then T is a contraction on L as $G(l,m,n)=\frac{5}{8}\{|l_1-m_1-n_1|+|l_2-m_2-n_2|+|l_3-m_3-n_3|\}=\frac{5}{8}G(l,m,n)$.

Theorem 1.5 ([9])**.** *Let (L,G) be a complete G-metric space and T be the contraction mapping defined on L. Then T possesses a unique fixed point l in L, i.e., $Tl=l$.*

Theorem 1.6 ([8])**.** *Let (L,G) be a complete G-metric space and T be the self mapping defined on L which satisfy the condition*

$$G(Tl,Tm,Tn)\le\alpha G(l,Tl,Tl)+\beta G(m,Tm,Tm)+\gamma G(n,Tn,Tn)+\delta G(l,m,n)$$

for all $l,m,n\in L$ and α, β, γ, δ non-negative with $\alpha+\beta+\gamma+\delta<1$. Then T admits a unique fixed point in L.

Definition 1.7 ([1])**.** Let Ψ be the family of all functions $\psi:[0,+\infty)\to[0,+\infty)$ satisfying the following properties:

(1) $\sum_{p=1}^{+\infty}\psi^p(t)<+\infty$ for every $t>0$, where ψ^p is the pth iterate of ψ;

(2) ψ is nondecreasing.

Definition 1.8 ([2])**.** A mapping $B:[0,\infty)^2\to\Re$ is called C-class function if it is continuous and satisfies the following conditions:

(1) $B(x,y)\le x$ for all $x,y\in[0,\infty)$;

(2) $B(x,y)=x$ implies that either $x=0$ or $y=0$;

Let us consider:

$\Phi_1=\{\phi_1:[0,\infty)\to[0,\infty)$ is a continuous and non-decreasing function such that $\phi_1(m)=0$ if and only if $m=0\}$,

$\Phi_2=\{\phi_2:[0,\infty)\to[0,\infty)$ is a continuous function such that $\phi_2(0)=0$ and $\phi_2(m)>0$ for $m>0\}$,

$$\Phi_3 = \Big\{\phi_3 : [0,\infty) \to [0,\infty) \text{ is a Lebesgue-integrable function, summable on each compact subset of } R^+, \text{ non-negative, and such that for each } \epsilon > 0, \int_0^{\epsilon} \phi(t)dt > 0\Big\}.$$

2. Main Results

Theorem 2.1. *Let (L,G) be a G-metric space and h be a self map on L be a mapping satisfying*

$$\phi_1\left(\int_0^{G(hx,hy,hz)} \phi(t)\right)dt \le B\left(\phi_1\left(\int_0^{N(x,y,z)} \phi(t)dt\right), \phi_2\left(\int_0^{N(x,y,z)} \phi(t)dt\right)\right), \tag{2.1}$$

where B is a C-class function $\phi_1 \in \Phi_1$, $\phi_2 \in \Phi_2$, $\phi \in \Phi_3$ and

$$N(x,y,z) = \max\{G(x,y,z), G(x,hx,hx), G(y,hy,hy), G(z,hz,hz)\}. \tag{2.2}$$

Then h has a unique fixed point.

Proof. Suppose that $x_0 \in L$. Choose a point $x_1 \in L$ such that $x_1 = hx_0$.
In general, choose x_{n+1} such that $x_{n+1} = hx_n$ for $n = 0,1,2,\dots$.
Suppose that $x_n \ne x_{n+1}$ for each integer $n > 1$, then from (2.1)

$$\phi_1\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right) \le B\left(\phi_1\left(\int_0^{N(x_{n-1},x_n,x_n)} \phi(t)dt\right), \phi_2\left(\int_0^{N(x_{n-1},x_n,x_n)} \phi(t)dt\right)\right), \tag{2.3}$$

where from (2.2),

$$\begin{aligned} N(x_{n-1},x_n,x_n) &= \max\{G(x_{n-1},x_n,x_n), G(x_{n-1},hx_{n-1},hx_{n-1}), G(x_n,hx_n,hx_n), G(x_n,hx_n,hx_n)\} \\ &= \max\{G(x_{n-1},x_n,x_n), G(x_n,x_{n+1},x_{n+1})\}. \end{aligned} \tag{2.4}$$

If $\max\{G(x_{n-1},x_n,x_n), G(x_n,x_{n+1},x_{n+1})\} = G(x_n,x_{n+1},x_{n+1})$.
From (2.3) and (2.4), we have

$$\phi_1\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right) \le B\left\{\phi_1\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right), \phi_2\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right)\right\}. \tag{2.5}$$

Thus by definition of $B \in C$, we get
either

$$\phi_1\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right) = 0$$

or

$$\phi_2\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right) = 0.$$

From definition of ϕ_1 and ϕ_2 it is possible only if

$$\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt = 0.$$

This is a contraction to our hypothesis.
Thus $N(x_{n-1},x_n,x_n) = G(x_{n-1},x_n,x_n)$, this implies

$$\phi_1\left(\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\right) \le B\phi_1\left(\int_0^{G(x_{n-1},x_n,x_n)} \phi(t)dt\right), \phi_2\left(\int_0^{G(x_{n-1},x_n,x_n)} \phi(t)dt\right)$$

$$\leq \phi_1\left(\int_0^{G(x_{n-1},x_n,x_n)} \phi(t)dt\right).$$

Since ϕ_1 is continuous and non-decreasing, therefore

$$\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt \leq \int_0^{G(x_{n-1},x_n,x_n)} \phi(t)dt,$$

thus $\{\int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt\}$ is monotone decreasing and lower bounded sequence.
Therefore, there exist $\widehat{r} \geq 0$ such that

$$\lim_{n\to\infty} \int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt = \widehat{r}. \tag{2.6}$$

Suppose that $\widehat{r} > 0$. Taking $\lim\limits_{n\to\infty}$ on both sides of equation (2.5) and using (2.6), we get

$$\phi_1(\widehat{r}) \leq B((\phi_1(\widehat{r}), \phi_2(\widehat{r})),$$

implies from definition of $B \in C$ that
either

$$\phi_1(\widehat{r}) = 0$$

or

$$\phi_2(\widehat{r}) = 0.$$

From definition of ϕ_1 and ϕ_2, we get $\widehat{r} = 0$.
Hence from equation (2.6), we obtain

$$\lim_{n\to\infty} \int_0^{G(x_n,x_{n+1},x_{n+1})} \phi(t)dt = 0, \tag{2.7}$$

implies

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.8}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence.
Let, if possible, it is not.
Therefore, for an $\epsilon > 0$, there exists two subsequences $\{x_{m(p)}\}$ and $\{x_{n(p)}\}$ of $\{x_n\}$ with $m(p) < n(p) < m(p+1)$ such that

$$G(x_m(p), x_n p, x_n p) \geq \epsilon, \quad G(x_m(p), x_{np-1}, x_{np-1}) < \epsilon. \tag{2.9}$$

Consider

$$\phi_1\left(\int_0^{\epsilon} \phi(t)dt\right) \leq \phi_1\left(\int_0^{G(x_m(p),x_n p,x_n p)} \phi(t)dt\right)$$
$$\leq B\left\{\phi_1\left(\int_0^{N(x_{m(p)-1},x_{n(p)-1},x_{n(p)-1})} \phi(t)dt\right), \phi_2\left(\int_0^{N(x_m(p),x_n p,x_n p)} \phi(t)dt\right)\right\}. \tag{2.10}$$

Using (2.2)

$$N(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1})$$
$$= \max\{G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1}), G(x_{m(p)-1}, hx_{m(p)-1}, hx_{m(p)-1}),$$
$$G(x_{n(p)-1}, hx_{n(p)-1}, hx_{n(p)-1}), G(x_{n(p)-1}, hx_{n(p)-1}, hx_{n(p)-1})\}$$

$$= \max\{G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1}), G(x_{m(p)-1}, x_{m(p)}, x_{m(p)}), G(x_{n(p)-1}, x_{n(p)}, x_{n(p)})\}. \quad (2.11)$$

Thus

$$\int_0^{N(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1})} \phi(t)dt$$
$$= \int_0^{\max\{G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1}), G(x_{m(p)-1}, x_{m(p)}, x_{m(p)}), G(x_{n(p)-1}, x_{n(p)}, x_{n(p)})\}} \phi(t)dt$$
$$= \max\left\{\int_0^{G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1})} \phi(t)dt, \int_0^{G(x_{m(p)-1}, x_{m(p)}, x_{m(p)})} \phi(t)dt, \int_0^{G(x_{n(p)-1}, x_{n(p)}, x_{n(p)})} \phi(t)dt\right\}. \quad (2.12)$$

Using (2.9) and triangle inequality, we get

$$G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1}) \le G(x_{m(p)-1}, x_{m(p)}, x_{m(p)}) + G(x_{m(p)}, x_{n(p)-1}, x_{n(p)-1})$$
$$< G(x_{m(p)-1}, x_{m(p)}, x_{m(p)}) + \epsilon.$$

Therefore,

$$\lim_{p\to\infty} \int_0^{G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1})} \phi(t)dt \le \int_0^{\epsilon} \phi(t)dt. \quad (2.13)$$

Taking $\lim_{p\to\infty}$ on both sides of (2.10) and using (2.11), (2.12), (2.13), we get

$$\phi_1\left(\int_0^{\epsilon} \phi(t)dt\right) \le B\left(\phi_1\left(\int_0^{\epsilon} \phi(t)dt\right), \phi_2\left(\int_0^{\epsilon} \phi(t)dt\right)\right).$$

Again from definition of $B \in C$, we get

either

$$\phi_1\left(\int_0^{\epsilon} \phi(t)dt\right) = 0$$

or

$$\phi_2\left(\int_0^{\epsilon} \phi(t)dt\right) = 0.$$

It is possible only if $\int_0^{\epsilon} \phi(t)dt = 0$.

This is a contraction to our hypothesis, therefore $\{x_n\}$ is a Cauchy sequence, ξ be the limit such that

$$\lim_{n\to\infty} hx_{n-1} = \xi. \quad (2.14)$$

Next, we prove that ξ is the fixed point of map h.

That is $h\xi = \xi$, suppose it is not.

Then $G(h\xi, \xi, \xi) > 0$.

Let $\sigma = G(h\xi, \xi, \xi)$.

Consider,

$$\phi_1\left(\int_0^{\sigma} \phi(t)dt\right) = \phi_1\left(\int_0^{G(h\xi,\xi,\xi)} \phi(t)dt\right)$$
$$\le B\left\{\phi_1\left(\int_0^{N(\xi, x_n, x_n)} \phi(t)dt\right), \phi_2\left(\int_0^{N(\xi, x_n, x_n)} \phi(t)dt\right)\right\}, \quad (2.15)$$

where

$$N(\xi,x_n,x_n) = \max\{G(\xi,x_n,x_n),G(\xi,h\xi,h\xi),G(x_n,hx_n,hx_n),G(x_n,hx_n,hx_n)\}. \tag{2.16}$$

Since,

$$\lim_{n\to\infty} G(\xi,x_n,x_n) = \lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0. \tag{2.17}$$

Taking $\lim_{n\to\infty}$ in (2.15) and by using (2.14), (2.16), (2.17), we get

$$\begin{aligned}\phi_1\left(\int_0^\sigma \phi(t)dt\right) &\le B\left\{\phi_1\left(\int_0^{max\{G(\xi,h\xi,h\xi)\}}\right)\phi(t)dt, \phi_2\left(\int_0^{max\{G(\xi,h\xi,h\xi)\}}\right)\phi(t)dt\right\} \\ &\le B\left\{\phi_1\left(\int_0^\sigma \phi(t)dt\right), \phi_2\left(\int_0^\sigma \phi(t)dt\right)\right\}. \end{aligned} \tag{2.18}$$

Thus, we obtain

either

$$\phi_1\left(\int_0^\sigma \phi(t)dt\right) = 0$$

or

$$\phi_2\left(\int_0^\sigma \phi(t)dt\right) = 0$$

that is

$$\int_0^\sigma \phi(t)dt = 0.$$

Hence $\sigma = 0$ which implies that $P(h\xi,\xi,\xi) = 0$.

Therefore ξ is the fixed point of map h. □

3. Applications

For the application purpose some important corollaries have been derived from our main result. If we put $\phi(t) = t$ in Theorem 2.1, we get a new result.

Corollary 3.1. *Let (L,G) be a complete G-metric space and h be a self map on L, such that for each $x,y,z \in L$,*

$$\phi_1\left(\int_0^{G(hx,hy,hz)} \phi(t)dt\right) \le B\left(\left(\int_0^{N(x,y,z)} \phi(t)dt\right), \phi_2\left(\int_0^{N(x,y,z)} \phi(t)dt\right)\right),$$

where $N(x,y,z)$ is given in (2.2), B is a C-class function, $\phi_2 \in \Phi_2$, $\phi \in \Phi_3$.

Corollary 3.2. *Let (L,G) be a complete G-metric space and h be a self map on L, such that for each $x,y,z \in L$,*

$$\phi_1\left(\int_0^{G(hx,hy,hz)} \phi(t)dt\right) \le \lambda\phi_1\left(\int_0^{N(x,y,z)} \phi(t)dt\right), \tag{3.1}$$

where $N(x,y,z)$ is given in (2.2), $\lambda \in (0,1)$, $\phi_1 \in \Phi_1$, $\phi \in \Phi_3$.

Then h has a unique fixed point.

Corollary 3.3. *Let (L,G) be a complete G-metric space and h be a self map on L, such that for each $x,y,z \in L$,*

$$\phi_1\left(\int_0^{G(hx,hy,hz)} \phi(t)dt\right) \le \phi_1\left(\int_0^{N(x,y,z)} \phi(t)dt\right) - \phi_2\left(\int_0^{N(x,y,z)} \phi(t)dt\right), \tag{3.2}$$

where $N(x,y,z)$ is given in (2.2), $\phi_1 \in \Phi_1$, $\phi_2 \in \Phi_2$, $\phi \in \Phi_3$.
Then h has a unique fixed point.

4. Conclusion

With the aid of new auxiliary functions, some fixed point results are proved for generalized contractive conditions in the setting of G-metric spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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