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A Solution Method for Semidefinite Variational Inequality with Coupled Constraints *

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Abstract The semidefinite variational inequalities with coupled constraints are introduced. The properties of the symmetric matrices value functions that form coupled constraints are discussed. A method involving the augmented Lagrange function is proposed for solving the semidefinite variational inequalities with coupled constraints. The convergence of the method is proved.

1. Introduction and Preliminaries

In this paper, let $\mathfrak{R}^n$ be the n-dimensional Euclidean space with the normal inner product $\langle x, y\rangle = x^\top y$ and norm $\|x\| = \sqrt{x^\top y}$ for any $x, y \in \mathfrak{R}^n$. Let S^p denote the space of $p \times p$ symmetric matrices with the inner product $\langle A, B\rangle := \mathrm{tr} A^\top B$ and the norm $\| A \| := \sqrt{\langle A, A\rangle}$ for all $A, B \in S^p$. Let S^p_+ be the cone of positive semidefinite matrices in the space S^p. For a linear mapping $M : \mathfrak{R}^n \to S^p$, we denote the adjoint of M by $M^* : S^p \to \mathfrak{R}^n$, that is, $\langle A, Mh\rangle = \langle M^* A, h\rangle$ for $h \in \mathfrak{R}^n$ and $A \in S^p$.

We consider the semidefinite variational inequality with coupled constraints: Find $\bar{x} \in K$ such that

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } y \in K. \tag{1.1}$$

where $K = \{\text{for all } y \in \Omega \mid G(\bar{x}, y) \preceq 0\}$, $F : \mathfrak{R}^n \to \mathfrak{R}^n$ is a mapping, $G(x, y) : \mathfrak{R}^n \times \mathfrak{R}^n \to S^p$ is a matrix value mapping, and $\Omega \subseteq \mathfrak{R}^n$ is a convex closed set.

If $G(x, y) = \mathrm{diag}(g(x, y))$, where $g(x, y) : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^p$ is a mapping, then Problem 1.1 is equivalent to find a vector $\bar{x} \in K$ such that

$$\langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \text{for all } \ y \in K. \tag{1.2}$$

where $K = \{\text{for all } y \in \Omega \mid g(\bar{x}, y) \leq 0\}$.

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Problem (1.2) is called variational inequality with coupled constraints, which were introduced and studied by Antipin [1].

Antipin [1] introduced the existence of the problems with coupled constraints which arise in economic equilibrium models, n-person game, equilibrium programming, hierarchical programming problems, mathematical physics and other fields (see [2], [3], [4], [5], [6] and the references therein). This short list of problems shows that the coupled constraints are characteristic of a wide class of problems. For this reason, the development of methods for the problems with coupled constraints is a very important task. Antipin [1] considered variational inequality with coupled constraints (1.2), and introduced a class of symmetric vector functions that formed coupled constraints. The explicit and implicit prediction-type gradient and proximal methods were proposed for solving variational inequalities with coupled constraints. And the convergence of these methods were proved.

Semidefinite programming is an extension of linear programming. In recent years, the theory and algorithm for semidefinite programming have developed greatly, and its most important applications are found in combinatorial optimization, system engineering and electrical engineering. Semidefinite programming is a new and important research field in mathematical programming. In the study of semidefinite programming, semidefinite variational inequality has been given more concern.

Inspired by the papers cited above, in this paper, we firstly introduce a new class of semidefinite variational inequality with coupled constraints (1.1). The constraint function $G(x,y) : \Re^n \times \Re^n \to S^p$ is a matrix value function which is new and different from that in [1]. In Section 2, the properties of the symmetric matrix value function are discussed. It easy to see that Problem (1.1) can be viewed as the minimization problem for the linear function $f(y) = \langle F(\bar{x}), y - \bar{x}\rangle$ and $f(y) \geq 0$ on the set K. By using Lagrange function, we study the semidefinite variational inequality with coupled constraints (1.1) and give the other equivalent transformations in Section 3. In Section 4, based on the transformations of Problem (1.1), we propose the method involving augmented Lagrange function for solving the semidefinite variational inequalities with coupled constraints (1.1). Furthermore, we prove that a accumulation point of the sequence generated by the method is a solution to the variational inequality with coupled constraints.

Let us firstly recall the following definitions and theorems.

We say that F is monotone if $\langle F(x) - F(y), x - y\rangle \geq 0$ for any $x, y \in \Re^n$.

Let C be a closed convex set, for every $x \in \Re^n$, there is a unique $\hat{x}$ in C such that

$$\|x - \hat{x}\| = \min\{\|x - y\| \mid y \in C\}.$$

The point $\hat{x}$ is the projection of x onto C, denoted by $\Pi_C(x)$. Therefore, the projection mapping $\Pi_C : \Re^n \to C$ is well defined for every $x \in \Re^n$, which is a nonexpensive mapping. Similarly, for every $A \in S^p$, we can get the projection mapping $\Pi_{S^p_+} : S^p \to S^p_+$. For the projection mappings, we recall the following well-known results.

Lemma 1.1 ([7]). *Let H be a real Hilbert space and C be a closed convex set. For a given $z \in H$, $u \in C$ satisfies the inequality*

$$\langle u - z, x - u \rangle \geq 0, \quad \text{for all } x \in C,$$

if and only if

$$u = \Pi_C(z),$$

where Π_C is the projection of H to C.

Han [8] proved the following lemma.

Lemma 1.2 ([8]). *$A \in S^p_+$, $B \in S^p_+$, $\langle A, B \rangle = 0$ if and only if $E(A) = 0$, where $E(A) = A - \Pi_{S^p_+}(A - B)$ and $\Pi_{S^p_+}(A - B)$ is the operator projection $A - B$ onto the space of $p \times p$ positive semidefinite symmetric matrices.*

2. The Properties of Symmetric Matrix Value Functions

In this section, firstly the definition of symmetric matrix value function is introduced. Then the properties of this symmetric matrix value function are discussed which play an important role in demonstrating the convergence theorem.

Definition 2.1. *A matrix value function $G(x, y) : \Re^n \times \Re^n \to S^p$ is said to be symmetric if it satisfies the following*

$$G(x, y) = G(y, x), \quad \text{for all } x,\ y \in \Re^n. \tag{2.1}$$

Remark 2.1. *In* (2.1)*, if $G(x, y) = \mathrm{diag}(g(x, y))$ and $g(x, y) : \Re^n \times \Re^n \to \Re^p$, then we get the definition of the symmetric function which was introduced in* [1]*. Hence the symmetric function definition in* [1] *is a special case of Definition* 2.1.

Next we analyze the properties of symmetric matrix value function (2.1).

Property 2.1. *Let C be a closed convex subset of $\Re^n$ and $G : C \times C \to S^p$ be a symmetric matrix value mapping. Suppose that $D_y G(x, y)$ and $D_x G(y, x)$ denote the partial differential operators of $G(x, y)$ and $G(y, x)$ with respect to y and x, respectively. Then these two partial differential operators of their restrictions to the diagonal of the square $C \times C$ are equal. That is, for any $h \in \Re^n$,*

$$D_y G(x, x)h = D_x G(x, x)h, \quad \text{for all } x \in C. \tag{2.2}$$

Proof. By the definition of partial differential operator of matrix value function, for any $h \in \Re^n$, we differentiate $G(x, y)$ and $G(y, x)$ with respect to y and x, respectively, to obtain

$$D_y G(x, y)h = h_1 \frac{\partial G(x, y)}{\partial y_1} + h_2 \frac{\partial G(x, y)}{\partial y_2} + \cdots + h_n \frac{\partial G(x, y)}{\partial y_n}, \quad \text{for all } x,\ y \in C \tag{2.3}$$

and

$$D_x G(y,x)h = h_1 \frac{\partial G(y,x)}{\partial x_1} + h_2 \frac{\partial G(y,x)}{\partial x_2} + \cdots + h_n \frac{\partial G(y,x)}{\partial x_n}, \quad \text{for all } x,\ y \in C, \tag{2.4}$$

respectively.

In view of (2.1), we have

$$\frac{\partial G(x,y)}{\partial y_i} = \frac{\partial G(y,x)}{\partial x_i}, \quad \text{for } i = 1,2,\cdots,n.$$

Contrasting (2.3) with (2.4), we conclude that for any $h \in \Re^n$, $D_y G(x,x)h = D_x G(x,x)h$, for all $x \in C$. This completes the proof. □

Property 2.2. *Suppose that C and G satisfy the conditions of Property 2.1. The partial differential operator $2D_y G(x,y)|_{x=y}$ is equal to the differential operator $DG(x,x)$ of the restriction of the symmetric function $G(x,y)$ to the $C \times C$ square's diagonal. That is, for any $h \in \Re^n$,*

$$2D_y G(x,x)h = DG(x,x)h, \quad \textit{for all } \ x \in C. \tag{2.5}$$

Proof. By the definition of a differentiable matrix value function $G(x,y)$ which is defined on C, for any $h,k \in \Re^n$, we have

$$G(x+h, y+k) = G(x,y) + D_x G(x,y)h + D_y G(x,y)k + \gamma(h,k), \quad \text{for all } x,y \in C, \tag{2.6}$$

where $\gamma : \Re^n \times \Re^n \to \Re^n$ and $\gamma(h,k)/(\|h\|^2 + \|k\|^2)^{1/2} \to 0$ as $\|h\|^2 + \|k\|^2 \to 0$.

Letting $y = x$ and $h = k$ in (2.6) and using (2.2) and (2.6), we obtain that

$$G(x+h, x+h) = G(x,x) + 2D_y G(x,x)h + \gamma(h,h), \tag{2.7}$$

where $\gamma(h,h)/\|h\| \to 0$ as $\|h\| \to 0$.

From (2.7), it is easy to see that the restriction of the partial differential operator to the diagonal of $C \times C$ is $DG(x,x)$, that is, $2D_y G(x,x)h = DG(x,x)h$, for any $h \in \Re^n$. This completes the proof. □

Next we introduce the other definition of antisymmetric matrix value function.

Definition 2.2. *A matrix value function $G : \Re^n \times \Re^n \to S^p$ is said to be antisymmetric if it satisfies the following condition*

$$G(x,y) = -G(y,x), \quad \textit{for all } x,\ y \in \Re^n. \tag{2.8}$$

Now we show that the antisymmetric coupled constraints have no effect on the solution to Problem (1.1), and hence we can drop it.

In fact, let us consider a pair of problems

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \ \in \Omega \tag{2.9}$$

and

$$\langle F(\bar{x}), y-\bar{x}\rangle \geq 0, \quad \text{for all } y \in \Omega,\ G(\bar{x}, y) \preceq 0, \tag{2.10}$$

where $G(x,y)$ is an antisymmetric matrix value function.

Since G is an antisymmetric matrix value function, we get that $G(x,x)=\mathbf{0}$ if we set $x=y$ in $G(x,y)=-G(y,x)$, where $\mathbf{0}$ is a $p\times p$ null matrix. Thus $G(x,x)\preceq 0$ is always to be satisfied for any $x\in\Omega$. Observe that if $y=\bar{x}$ is the solution to (2.9), then it satisfies the constraints of (2.10). Thus the antisymmetric coupled constraints in problem (2.10) can be dropped.

Generally, when $G(x,y)$ is neither symmetric nor antisymmetric, the coupled constraints in problem (1.1) can be symmetrized as follows.

$$G(x,y)=S(x,y)+K(x,y),$$

where

$$S(x,y)=\frac{1}{2}(G(x,y)+G(y,x)), \quad K(x,y)=\frac{1}{2}(G(x,y)-G(y,x)).$$

It is clear that $S(x,y)$ and $K(x,y)$ are a symmetric and an antisymmetric function, respectively.

From above analysis, we can present the coupled constraints of the problem (1.1) as $K=\{\text{for all } y\in\Omega \mid G(\bar{x},y)=S(\bar{x},y)+K(\bar{x},y)\preceq 0\}$. Since the antisymmetric part of the constraints can be dropped, the problem (1.1) can become to find $\bar{x}\in K$ such that

$$\langle F(\bar{x}), y-\bar{x}\rangle \geq 0, \quad \text{for all } y\in K, \tag{2.11}$$

where $K=\{\text{for all } y\in\Omega \mid S(\bar{x},y)\preceq 0\}$. Hence, in order to find a solution to the problem (1.1), we can solve the symmetrized problem (2.11).

3. Saddle Point Problems from Semidefinite Programming

In this section, we give some transformations of the semidefinite variational inequality with coupled constraints problem (1.1), which will be used in the subsequent analysis.

Let $f(y)=\langle F(\bar{x}), y-\bar{x}\rangle$ and $f(y)\geq 0$, then the problem (1.1) can be transformed into the following semidefinite programming problem

$$(P) \qquad \min f(y) \qquad \text{s.t. } y\in\Omega,\ G(\bar{x},y)\preceq 0. \tag{3.1}$$

The Lagrange function of problem (3.1) is of the form

$$\mathscr{L}(\bar{x},y,Q)=\langle F(\bar{x}), y-\bar{x}\rangle+\langle Q, G(\bar{x},y)\rangle, \quad \text{for all } y\in\Omega,\ Q\in S_{+}^{p}.$$

The Lagrange dual function of problem (3.1) is

$$(D) \qquad \max_{Q\succeq 0}\Big\{\inf_{x\in\Omega}\mathscr{L}(\bar{x},y,Q)\Big\}. \tag{3.2}$$

In order to get the other transformations, we use the famous duality theorem below.

Theorem 3.1 ([9]). *Let (P) and (D) be the primal and dual problem* (3.1) *and* (3.2)*, respectively. Then $Val(D) \leq Val(P)$. Moreover, $Val(P) = Val(D)$ and $\bar{x}$ and $\bar{Q}$ are optimal solutions of (P) and (D), respectively, if and only if $(\bar{x}, \bar{Q})$ is a saddle point of the Lagrange function $\mathscr{L}(\bar{x}, y, Q)$ if and only if the following conditions*

$$\bar{x} \in \operatorname*{argmin}_{y \in \Omega} \mathscr{L}(\bar{x}, y, Q), \quad \langle \bar{Q}, G(\bar{x}, \bar{x}) \rangle = 0, \quad G(\bar{x}, \bar{x}) \preceq 0, \quad \bar{Q} \succeq 0. \tag{3.3}$$

Since $\bar{x}$ is the minimum solution of $f(y)$, the pair $(\bar{x}, \bar{Q})$ is a saddle point of the Lagrange function $\mathscr{L}(\bar{x}, y, Q)$ by Theorem 3.1. Hence we get the following inequality from the definition of saddle point.

$$\begin{aligned} \langle F(\bar{x}), \bar{x} - \bar{x} \rangle + \langle Q, G(\bar{x}, \bar{x}) \rangle &\leq \langle F(\bar{x}), \bar{x} - \bar{x} \rangle + \langle \bar{Q}, G(\bar{x}, \bar{x}) \rangle \\ &\leq \langle F(\bar{x}), y - \bar{x} \rangle + \langle \bar{Q}, G(\bar{x}, y) \rangle, \end{aligned} \tag{3.4}$$

for all $y \in \Omega$ and for all $Q \in S_+^p$. For the arbitrariness of Q and y, in view of the first inequality and the second inequality of (3.4), we can represent (3.4) in an equivalent manner in the following

$$\begin{aligned} &\bar{x} \in \arg\min \{ \langle F(\bar{x}), y - \bar{x} \rangle + \langle \bar{Q}, G(\bar{x}, y) \rangle \mid y \in \Omega \}, \\ &\bar{Q} \in \arg\max \{ \langle Q, G(\bar{x}, \bar{x}) \rangle \mid Q \in S_+^p \}. \end{aligned} \tag{3.5}$$

If $G(x, y)$ is differentiable with respect to y for any x, by computing, we can transform the system (3.5) in the form of the system of semidefinite variational inequalities as follows.

$$\begin{aligned} &\langle F(\bar{x}) + D_y G(\bar{x}, \bar{x})^* \bar{Q}, y - \bar{x} \rangle \geq 0, && \text{for all } y \in \Omega, \\ &\langle -G(\bar{x}, \bar{x}), Q - \bar{Q} \rangle \geq 0, && \text{for all } Q \in S_+^p, \end{aligned} \tag{3.6}$$

where $D_y G(\bar{x}, \bar{x})^*$ is the adjoint operator of $D_y G(\bar{x}, \bar{x})$, that is, $\langle D_y G(\bar{x}, \bar{x})^* \bar{Q}, y - \bar{x} \rangle = \langle \bar{Q}, D_y G(\bar{x}, \bar{x})(y - \bar{x}) \rangle$.

With the help of projection operators, it follows from Theorem 3.1, Lemma 1.1 and Lemma 1.2 that

$$\begin{aligned} &\bar{x} = \Pi_\Omega(\bar{x} - \alpha(F(\bar{x}) + D_y G(\bar{x}, \bar{x})^* \bar{Q})), \\ &\bar{Q} = \Pi_{S_+^p}(\bar{Q} + \alpha G(\bar{x}, \bar{x})), \end{aligned} \tag{3.7}$$

where $\alpha > 0$ is a parameter, and Π_Ω and $\Pi_{S_+^p}$ are the operators that project a vector and a matrix onto the set Ω and the space S_+^p, respectively.

By the definition of the adjoint of $D_y G(\bar{x}, \bar{x})$, we can transform the first inequality of the system (3.6) as

$$\langle F(\bar{x}), y - \bar{x} \rangle + \langle \bar{Q}, D_y G(\bar{x}, \bar{x})(y - \bar{x}) \rangle \geq 0. \tag{3.8}$$

When the function $G(x,y)\mid_{x=y}$ on the diagonal of the square $\Omega\times\Omega$ is convex, using Property 2.2, we transform (3.8) as

$$\begin{aligned}&\langle F(\bar{x}), y-\bar{x}\rangle + \langle \bar{Q}, D_y G(\bar{x},\bar{x})(y-\bar{x})\rangle\\ &\quad = \langle F(\bar{x}), y-\bar{x}\rangle + \frac{1}{2}\langle \bar{Q}, DG(\bar{x},\bar{x})(y-\bar{x})\rangle\\ &\quad \le \langle F(\bar{x}), y-\bar{x}\rangle + \frac{1}{2}\langle \bar{Q}, G(y,y)-G(\bar{x},\bar{x})\rangle.\end{aligned}$$

From the above inequality, (3.6) can be represented as

$$\begin{aligned}&\langle F(\bar{x}), y-\bar{x}\rangle + \tfrac{1}{2}\langle \bar{Q}, G(y,y)-G(\bar{x},\bar{x})\rangle \ge 0, && \text{for all } y\in\Omega,\\ &\langle -G(\bar{x},\bar{x}), Q-\bar{Q}\rangle \ge 0, && \text{for all } Q\in S_+^p.\end{aligned} \tag{3.9}$$

Thus the semidefinite variational inequality with coupled constraints reduces to the saddle point problem (3.9).

Remark 3.1. *The above discussion yields that, under some conditions, $\bar{x}$ is the solution of* (1.1) *if and only if $\bar{x}$ satisfies the relations* (3.4)-(3.7) *and* (3.9). *Furthermore, if $G(x,y)$ is differentiable with respect to y for any x and is convex on the the diagonal of the square $\Omega\times\Omega$,* (3.4)-(3.7) *and* (3.9) *are equivalent from each other. Hence, by solving* (3.9)*, we can obtain the solution of* (1.1)*. The methods for solving* (3.9) *will be shown in the rest of sections.*

4. The Method Involving the Augmented Lagrange Function

In this section, we state the augmented Lagrange method for solving the system of variational inequalities (3.9) which is equivalent to the semidefinite variational inequality with coupled constraints problem (1.1) as follows.

Let $x^1\in\Omega$, $Q^1\in S_+^p$ be initial estimated solution and Lagrange multiplier. At n-th iteration ($x^n\in\Omega, Q^n\in S_+^p$ are known), determine (x^{n+1}, Q^{n+1}) by

$$\begin{aligned}&x^{n+1}\in\arg\min\left\{\frac{1}{2}\|y-x^n\|^2+\alpha\mathscr{M}(x^{n+1},y,Q^n)\,\middle|\, y\in\Omega\right\},\\ &Q^{n+1}=\Pi_{S_+^p}(Q^n+\alpha G(x^{n+1},x^{n+1})),\quad \alpha>0,\end{aligned} \tag{4.1}$$

where

$$\mathscr{M}(x,y,Q)=\langle F(x), y-x\rangle+\frac{1}{2\alpha}\|\Pi_{S_+^p}(Q+\alpha G(x,y))\|^2-\frac{1}{2\alpha}\|Q\|^2$$

is the augmented Lagrangian function for problem (1.1).

The system (4.1) can be represented equivalently as the following variational inequalities.

$$\begin{aligned}&\langle x^{n+1}-x^n+\alpha(F(x^{n+1})+D_yG(x^{n+1},x^{n+1})^*\Pi_{S_+^p}(Q^n+\alpha G(x^{n+1},x^{n+1}))), y-x^{n+1}\rangle\\ &\qquad \ge 0,\end{aligned} \tag{4.2}$$

and

$$\langle Q^{n+1}-Q^n-\alpha G(x^{n+1},x^{n+1}), Q-Q^{n+1}\rangle \ge 0, \tag{4.3}$$

for all $y \in \Omega$ and for all $Q \in S_+^p$.

In what follows, we demonstrate the convergence theorem of the augmented Lagrange method for solving problem (1.1).

Theorem 4.1. *Suppose that the solution set Ω_0 of problem* (1.1) *is non-empty, $F(x)$ is a monotone mapping and $G(x,y)$ is a symmetric matrix value function. Assume that $\Omega \subseteq R^n$ is a convex closed set, $G(x,y)$ is differentiable with respect to y for any x and is convex on the diagonal of the square $\Omega \times \Omega$, and $\alpha > 0$. Then, the accumulation of the sequence x^n constructed by method* (4.1) *is a solution to the variational inequality with coupled constraints problem* (1.1).

Proof. Setting $y = \bar{x} \in \Omega_0$ in (4.2) and taking into account the second equation (4.3), we obtain that

$$\langle x^{n+1} - x^n + \alpha(F(x^{n+1}) + D_y G(x^{n+1}, x^{n+1})^* Q^{n+1}), \bar{x} - x^{n+1}\rangle \geq 0.$$

The above inequality can be transformed as follows:

$$\langle x^{n+1} - x^n, \bar{x} - x^{n+1}\rangle + \alpha\langle F(x^{n+1}), \bar{x} - x^{n+1}\rangle + \alpha\langle D_y G(x^{n+1}, x^{n+1})^* Q^{n+1}, \bar{x} - x^{n+1}\rangle \geq 0. \tag{4.4}$$

By the convexity of $G(x,x)$ and Property 2.2, the last term in (4.4) can be transformed as

$$\begin{aligned}\langle D_y G(x^{n+1}, x^{n+1})^* Q^{n+1}, \bar{x} - x^{n+1}\rangle &= \langle Q^{n+1}, D_y G(x^{n+1}, x^{n+1})(\bar{x} - x^{n+1})\rangle \\ &= \frac{1}{2}\langle Q^{n+1}, DG(x^{n+1}, x^{n+1})(\bar{x} - x^{n+1})\rangle \\ &\leq \frac{1}{2}\langle Q^{n+1}, G(\bar{x}, \bar{x}) - G(x^{n+1}, x^{n+1})\rangle. \end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4), we have

$$\langle x^{n+1} - x^n, \bar{x} - x^{n+1}\rangle + \alpha\langle F(x^{n+1}), \bar{x} - x^{n+1}\rangle + \frac{\alpha}{2}\langle Q^{n+1}, G(\bar{x}, \bar{x}) - G(x^{n+1}, x^{n+1})\rangle \geq 0. \tag{4.6}$$

Setting $y = x^{n+1}$ in the first inequality in (3.9), we get that

$$\langle F(\bar{x}), x^{n+1} - \bar{x}\rangle + \frac{1}{2}\langle \bar{Q}, G(x^{n+1}, x^{n+1}) - G(\bar{x}, \bar{x})\rangle \geq 0. \tag{4.7}$$

Summing (4.6) and (4.7), we have

$$\begin{aligned}&\langle x^{n+1} - x^n, \bar{x} - x^{n+1}\rangle + \alpha\langle F(x^{n+1}) - F(\bar{x}), \bar{x} - x^{n+1}\rangle \\ &+ \frac{\alpha}{2}\langle Q^{n+1} - \bar{Q}, G(\bar{x}, \bar{x}) - G(x^{n+1}, x^{n+1})\rangle \geq 0.\end{aligned} \tag{4.8}$$

Letting $Q = \bar{Q}$ in (4.3) and using $\langle Q^{n+1}, G(\bar{x}, \bar{x})\rangle \leq 0$ and $\langle \bar{Q}, G(\bar{x}, \bar{x})\rangle = 0$, we obtain that

$$\frac{1}{2}\langle Q^{n+1} - Q^n, \bar{Q} - Q^{n+1}\rangle - \frac{\alpha}{2}\langle G(x^{n+1}, x^{n+1}) - G(\bar{x}, \bar{x}), \bar{Q} - Q^{n+1}\rangle \geq 0. \tag{4.9}$$

Since $F(x)$ is monotone, summing (4.9) and (4.8), we get that

$$\langle x^{n+1}-x^n, \bar{x}-x^{n+1}\rangle + \frac{1}{2}\langle Q^{n+1}-Q^n, \bar{Q}-Q^{n+1}\rangle \geq 0. \tag{4.10}$$

By using the identity for arbitrary x_1, x_2 and x_3 as follows

$$\|x_1-x_3\|^2 = \|x_1-x_2\|^2 + 2\langle x_1-x_2, x_2-x_3\rangle + \|x_2-x_3\|^2,$$

which yields that

$$\langle x_1-x_2, x_2-x_3\rangle = \frac{1}{2}\|x_1-x_3\|^2 - \frac{1}{2}\left[\|x_1-x_2\|^2 + \|x_2-x_3\|^2\right] \tag{4.11}$$

In view of (4.11), we get from (4.10) that

$$\begin{aligned}&\|x^{n+1}-x^n\|^2 + \|\bar{x}-x^{n+1}\|^2 + \frac{1}{2}\|Q^{n+1}-Q^n\|^2 + \frac{1}{2}\|\bar{Q}-Q^{n+1}\|^2\\ &\quad \leq \|\bar{x}-x^n\|^2 + \frac{1}{2}\|\bar{Q}-Q^n\|^2.\end{aligned} \tag{4.12}$$

Summing (4.12) from $n=0$ to $n=N$, we have

$$\begin{aligned}&\sum_{k=0}^{k=N}\|x^{k+1}-x^k\|^2 + \frac{1}{2}\sum_{k=0}^{k=N}\|Q^{k+1}-Q^k\|^2 + \|x^{N+1}-\bar{x}\|^2 + \frac{1}{2}\|Q^{N+1}-\bar{Q}\|^2\\ &\quad \leq \|x^0-\bar{x}\|^2 + \frac{1}{2}\|Q^0-\bar{Q}\|^2.\end{aligned} \tag{4.13}$$

The inequality (4.13) implies the boundedness of the trajectory $\{(x^i, Q^i) : i = 1, 2, \cdots\}$, that is,

$$\|x^{N+1}-\bar{x}\|^2 + \frac{1}{2}\|Q^{N+1}-\bar{Q}\|^2 \leq \|x^0-\bar{x}\|^2 + \frac{1}{2}\|Q^0-\bar{Q}\|^2, \tag{4.14}$$

and also the convergence of the series

$$\sum_{k=0}^{\infty}\|x^{k+1}-x^k\|^2 < \infty, \qquad \sum_{k=0}^{\infty}\|Q^{k+1}-Q^k\|^2 < \infty,$$

therefore, $\|x^{n+1}-x^n\|^2 \to 0$, $\|Q^{n+1}-Q^n\|^2 \to 0$ as $n \to \infty$. Since the sequence (x^n, Q^n) is bounded, there exists an element (x', Q') such that $x^{n_i} \to x'$ and $Q^{n_i} \to Q'$ as $n_i \to \infty$. Moreover

$$\|x^{n_i+1}-x^{n_i}\|^2 \to 0, \quad \|Q^{n_i+1}-Q^{n_i}\|^2 \to 0.$$

Considering (4.2) and (4.3) with $n = n_i$ and passing to the limit as $n_i \to \infty$ produces

$$\begin{aligned}&\langle F(x') + D_yG(x', x')^*Q', y-x'\rangle \geq 0, && \text{for all } y \in \Omega,\\ &Q' = \Pi_{S_+^P}(Q' + \alpha G(x', x')), \langle -G(x', x'), Q-Q'\rangle \geq 0, && \text{for all } Q \succeq 0.\end{aligned}$$

Since these relations coincide with (3.6), therefore the limit of the subsequence $\{x^n\}$ is a solution to problem (1.1). That is, the accumulation of the $\{x^n\}$ is a solution to the variational inequality with coupled constraints. This completes the proof. □

5. Conclusions

In this paper, we introduce a class of semidefinite variational inequality with coupled constraints which generalized the variational inequality with coupled constraints in [1]. Furthermore, we address the properties of the symmetric matrix value function that formed the coupled constraints. A method involving the augmented Lagrange function is discussed and the convergence theorem is demonstrated.

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