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Invariant Submanifolds of Sasakian Manifolds Admitting Semi-symmetric Metric Connection

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Abstract The object of this paper is to study invariant submanifolds M of Sasakian manifolds $\widetilde{M}$ admitting a semi-symmetric metric connection and to show that M admits semi-symmetric metric connection. Further it is proved that the second fundamental forms σ and $\overline{\sigma}$ with respect to Levi-Civita connection and semi-symmetric metric connection coincide. It is shown that if the second fundamental form σ is recurrent, 2-recurrent, generalized 2-recurrent and M has parallel third fundamental form with respect to semi-symmetric metric connection, then M is totally geodesic with respect to Levi-Civita connection.

1. Semi-symmetric Metric Connection

The geometry of invariant submanifolds M of Sasakian manifolds $\widetilde{M}$ is carried out from 1970's by M. Kon [12], D. Chinea [8], K. Yano and M. Kon [17]. It is proved that invariant submanifold of Sasakian structure also carries Sasakian structure. Also the authors B.S. Anitha and C.S. Bagewadi [1] have studied and the same authors [2] have studied on Invariant submanifolds of Sasakian manifolds admitting semi-symmetric non-metric connection. In this paper we extend the results to invariant submanifolds M of Sasakian manifolds admitting semi-symmetric metric connection.

We know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M if $\nabla g = 0$ otherwise it is non-metric. Further it is said to be semi-symmetric if its torsion tensor $T(X,Y) = 0$, i.e., $T(X,Y) = w(Y)X - w(X)Y$, where w is a 1-form. In 1924, A. Friedmann and J.A. Schouten [10] introduced the idea of semi-symmetric linear connection on differentiable manifold. In 1932, H.A. Hayden [11] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by K. Yano [16] in

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1970. After that the properties of semi-symmetric metric connection have studied by many authors like K.S. Amur and S.S. Pujar [3], C.S. Bagewadi, D.G. Prakasha and Venkatesha [4, 5], A. Sharfuddin and S.I. Hussain [14], U.C. De and G. Pathak [9] etc. If $\overline{\nabla}$ denotes semi-symmetric metric connection on a contact metric manifold, then it is given by [4]

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{1.1}$$

where $\eta(Y) = g(Y,\xi)$.

The covariant differential of the pth order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇, is denoted by $\nabla^p T$. The tensor T is said to be *recurrent* and 2*-recurrent* [13], if the following conditions hold on M, respectively,

$$(\nabla T)(X_1,\ldots,X_k;X)T(Y_1,\ldots,Y_k) = (\nabla T)(Y_1,\ldots,Y_k;X)T(X_1,\ldots,X_k), \tag{1.2}$$
$$(\nabla^2 T)(X_1,\ldots,X_k;X,Y)T(Y_1,\ldots,Y_k) = (\nabla^2 T)(Y_1,\ldots,Y_k;X,Y)T(X_1,\ldots,X_k),$$

where $X,Y,X_1,Y_1,\ldots,X_k,Y_k \in TM$. From (1.2) it follows that at a point $x \in M$, if the tensor T is non-zero, then there exists a unique 1-form ϕ and a $(0,2)$-tensor ψ, defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|) \tag{1.3}$$

and

$$\nabla^2 T = T \otimes \psi, \tag{1.4}$$

hold on U, where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T,T)$. The tensor T is said to be *generalized* 2*-recurrent* if

$$((\nabla^2 T)(X_1,\ldots,X_k;X,Y) - (\nabla T \otimes \phi)(X_1,\ldots,X_k;X,Y))T(Y_1,\ldots,Y_k)$$
$$= ((\nabla^2 T)(Y_1,\ldots,Y_k;X,Y) - (\nabla T \otimes \phi)(Y_1,\ldots,Y_k;X,Y))T(X_1,\ldots,X_k),$$

hold on M, where ϕ is a 1-form on M. From this it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi, \tag{1.5}$$

holds on U.

2. **Isometric Immersion**

Let $f : (M,g) \to (\widetilde{M},\widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M,g) into $(n+d)$-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote by ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n

and $\widetilde{M}^{n+d}$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y),N) = g(A_N X,Y),$$

for tangent vector fields X,Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z), \tag{2.3}$$

$$\begin{aligned}(\widetilde{\nabla}^2\sigma)(Z,W,X,Y) &= (\widetilde{\nabla}_X\widetilde{\nabla}_Y\sigma)(Z,W), \\ &= \nabla^{\perp}_X((\widetilde{\nabla}_Y\sigma)(Z,W)) - (\widetilde{\nabla}_Y\sigma)(\nabla_X Z,W) \\ &\quad - (\widetilde{\nabla}_X\sigma)(Z,\nabla_Y W) - (\widetilde{\nabla}_{\nabla_X Y}\sigma)(Z,W)\end{aligned} \tag{2.4}$$

respectively, where $\widetilde{\nabla}$ is called the *vander Waerden-Bortolotti connection* of M [7]. If $\widetilde{\nabla}\sigma = 0$, then M is said to have *parallel second fundamental form* [7].

3. **Sasakian Manifolds**

An n-dimensional differential manifold M is said to have an almost contact structure (ϕ,ξ,η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η on M respectively such that

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \eta\circ\phi = 0, \quad \phi\xi = 0. \tag{3.1}$$

Thus a manifold M equipped with this structure is called an almost contact manifold and is denoted by (M,ϕ,ξ,η). If g is a Riemannian metric on an almost contact manifold M such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{3.2}$$

where X,Y are vector fields defined on M, then M is said to have an almost contact metric structure (ϕ,ξ,η,g) and M with this structure is called an almost contact metric manifold and is denoted by (M,ϕ,ξ,η,g).

If on (M,ϕ,ξ,η,g) the exterior derivative of 1-form η satisfies,

$$\Phi(X,Y) = d\eta(X,Y) = g(X,\phi Y), \tag{3.3}$$

then (ϕ,ξ,η,g) is said to be a contact metric structure and together with manifold M is called contact metric manifold and Φ is a 2-form. The contact metric structure

(M,ϕ,ξ,η,g) is said to be normal if

$$[\phi,\phi](X,Y)+2d\eta\otimes\xi=0. \tag{3.4}$$

If the contact metric structure is normal, then it is called a Sasakian structure and M is called a Sasakian manifold. Note that an Almost contact metric manifold defines Sasakian structure if and only if

$$(\nabla_X\phi)Y=g(X,Y)\xi-\eta(Y)X, \tag{3.5}$$

$$\nabla_X\xi=-\phi X. \tag{3.6}$$

Example of Sasakian manifold. Consider the 3-dimensional manifold $M=\{(x,y,z)\in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let $\{E_1,E_2,E_3\}$ be linearly independent global frame field on M given by

$$E_1=\frac{\partial}{\partial x},\quad E_2=\frac{\partial}{\partial y}+2xe^z\frac{\partial}{\partial z},\quad E_3=e^z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2)=g(E_1,E_3)=g(E_2,E_3)=0,$$
$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1.$$

The (ϕ,ξ,η) is given by

$$\eta=-2xdy+e^{-z}dz,\quad \xi=E_3=\frac{\partial}{\partial z},$$
$$\phi E_1=E_2,\qquad \phi E_2=-E_1,\qquad \phi E_3=0.$$

The linearity property of ϕ and g yields

$$\eta(E_3)=1,\qquad \phi^2U=-U+\eta(U)E_3,$$
$$g(\phi U,\phi W)=g(U,W)-\eta(U)\eta(W),\quad g(U,\xi)=\eta(U),$$

for any vector fields U,W on M. By definition of Lie bracket, we have

$$[E_1,E_2]=2E_3.$$

The Levi-Civita connection with respect to above metric g and be given by Koszula formula

$$2g(\nabla_XY,Z)=X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))$$
$$-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).$$

Then, we have

$$\nabla_{E_1}E_1=0,\quad \nabla_{E_1}E_2=E_3,\quad \nabla_{E_1}E_3=-E_2,$$
$$\nabla_{E_2}E_1=-E_3,\quad \nabla_{E_2}E_2=0,\quad \nabla_{E_2}E_3=E_1,$$
$$\nabla_{E_3}E_1=-E_2,\quad \nabla_{E_3}E_2=E_1,\quad \nabla_{E_3}E_3=0.$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations (3.1), (3.2), (3.5) and (3.6). Thus M is a Sasakian manifold. Further the following relations hold:

$$R(X,Y)Z = \{g(Y,Z)X - g(X,Z)Y\}, \tag{3.7}$$

$$R(X,Y)\xi = \{\eta(Y)X - \eta(X)Y\}, \tag{3.8}$$

$$R(\xi,X)Y = \{g(X,Y)\xi - \eta(Y)X\}, \tag{3.9}$$

$$R(\xi,X)\xi = \{\eta(X)\xi - X\}, \tag{3.10}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{3.11}$$

$$Q\xi = (n-1)\xi, \tag{3.12}$$

for all vector fields, X, Y, Z and where ∇ denotes the operator of covariant differentiation with respect to g, ϕ is a $(1,1)$ tensor field, S is the Ricci tensor of type $(0,2)$ and R is the Riemannian curvature tensor of the manifold.

4. Invariant Submanifolds of Sasakian Manifolds admitting Semi-symmetric Metric Connection

If $\widetilde{M}$ is a Sasakian manifold with structure tensors $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$, then we know that its invariant submanifold M has the induced Sasakian structure (ϕ, ξ, η, g).

A submanifold M of a Sasakian manifold $\widetilde{M}$ with a semi-symmetric metric connection is called an invariant submanifold of $\widetilde{M}$ with a semi-symmetric metric connection, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. For an invariant submanifold of a Sasakian manifold with a semi-symmetric metric connection, we have

$$\sigma(X,\xi) = 0, \tag{4.1}$$

for any vector X tangent to M.

Let $\widetilde{M}$ be a Sasakian manifold admitting a semi-symmetric metric connection $\widetilde{\nabla}$.

Lemma 1. *Let M be an invariant submanifold of contact metric manifold $\widetilde{M}$ which admits semi-symmetric metric connection $\overline{\widetilde{\nabla}}$ and let σ and $\overline{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and semi-symmetric metric connection, then* (a) *M admits semi-symmetric metric connection,* (b) *the second fundamental forms with respect to $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ are equal.*

Proof. We know that the contact metric structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ on $\widetilde{M}$ induces (ϕ, ξ, η, g) on invariant submanifold. By virtue of (1.1), we get

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi. \tag{4.2}$$

By using (2.1) in (4.2), we get

$$\overline{\widetilde{\nabla}}_X Y = \nabla_X Y + \sigma(X,Y) + \eta(Y)X - g(X,Y)\xi. \tag{4.3}$$

Now Gauss formula (2.1) with respect to semi-symmetric metric connection is given by

$$\overline{\widetilde{\nabla}}_X Y = \overline{\nabla}_X Y + \overline{\sigma}(X,Y). \tag{4.4}$$

Equating (4.3) and (4.4), we get (1.1) and

$$\overline{\sigma}(X,Y) = \sigma(X,Y). \tag{4.5}$$

□

5. Recurrent Invariant Submanifolds of Sasakian Manifolds Admitting Semi-symmetric Metric Connection

We consider invariant submanifolds of a Sasakian manifold when σ is recurrent, 2-recurrent, generalized 2-recurrent and M has parallel third fundamental form with respect to semi-symmetric metric connection. We write the equations (2.3) and (2.4) with respect to semi-symmetric metric connection in the form

$$(\overline{\widetilde{\nabla}}_X \sigma)(Y,Z) = \overline{\nabla}^{\perp}_X(\sigma(Y,Z)) - \sigma(\overline{\nabla}_X Y, Z) - \sigma(Y, \overline{\nabla}_X Z), \tag{5.1}$$

$$\begin{aligned}(\overline{\widetilde{\nabla}}^2 \sigma)(Z,W,X,Y) &= (\overline{\widetilde{\nabla}}_X \overline{\widetilde{\nabla}}_Y \sigma)(Z,W), \\ &= \overline{\nabla}^{\perp}_X((\overline{\widetilde{\nabla}}_Y \sigma)(Z,W)) - (\overline{\widetilde{\nabla}}_Y \sigma)(\overline{\nabla}_X Z, W) \\ &\quad - (\overline{\widetilde{\nabla}}_X \sigma)(Z, \overline{\nabla}_Y W) - (\overline{\widetilde{\nabla}}_{\overline{\nabla}_X Y} \sigma)(Z,W).\end{aligned} \tag{5.2}$$

We prove the following theorems

Theorem 1. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting semi-symmetric metric connection. Then σ is recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be recurrent with respect to semi-symmetric metric connection. Then from (1.3) we get

$$(\overline{\widetilde{\nabla}}_X \sigma)(Y,Z) = \phi(X)\sigma(Y,Z),$$

where ϕ is a 1-form on M. By using (5.1) and $Z = \xi$ in the above equation, we have

$$\overline{\nabla}^{\perp}_X \sigma(Y,\xi) - \sigma(\overline{\nabla}_X Y, \xi) - \sigma(Y, \overline{\nabla}_X \xi) = \phi(X)\sigma(Y,\xi), \tag{5.3}$$

which by virtue of (4.1) reduces to

$$-\sigma(\overline{\nabla}_X Y, \xi) - \sigma(Y, \overline{\nabla}_X \xi) = 0. \tag{5.4}$$

Using (1.1), (3.1), (3.6) and (4.1) in (5.4), we get

$$\sigma(Y,\phi X)-\sigma(Y,X)=0. \tag{5.5}$$

Replace X by ϕX and by virtue of (3.1) and (4.1) in (5.5), we get

$$-\sigma(Y,X)-\sigma(Y,\phi X)=0. \tag{5.6}$$

Adding equation (5.5) and (5.6), we obtain $\sigma(X,Y)=0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Theorem 2. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting semi-symmetric metric connection. Then M has parallel third fundamental form with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M has parallel third fundamental form with respect to semi-symmetric metric connection. Then we have

$$(\overline{\overline{\nabla}}_X\overline{\overline{\nabla}}_Y\sigma)(Z,W)=0.$$

Taking $W=\xi$ and using (5.2) in the above equation, we have

$$\overline{\nabla}^{\perp}_X((\overline{\overline{\nabla}}_Y\sigma)(Z,\xi))-(\overline{\overline{\nabla}}_Y\sigma)(\overline{\nabla}_XZ,\xi)-(\overline{\overline{\nabla}}_X\sigma)(Z,\overline{\nabla}_Y\xi)-(\overline{\overline{\nabla}}_{\overline{\nabla}_XY}\sigma)(Z,\xi)=0. \tag{5.7}$$

By using (4.1) and (5.1) in (5.7), we get

$$\begin{aligned}0&=-\overline{\nabla}^{\perp}_X\{\sigma(\overline{\nabla}_YZ,\xi)+\sigma(Z,\overline{\nabla}_Y\xi)\}-\overline{\nabla}^{\perp}_Y\sigma(\overline{\nabla}_XZ,\xi)+\sigma(\overline{\nabla}_Y\overline{\nabla}_XZ,\xi)\\&\quad+2\sigma(\overline{\nabla}_XZ,\overline{\nabla}_Y\xi)-\overline{\nabla}^{\perp}_X\sigma(Z,\overline{\nabla}_Y\xi)+\sigma(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi)+\sigma(\overline{\nabla}_{\overline{\nabla}_XY}Z,\xi)\\&\quad+\sigma(Z,\overline{\nabla}_{\overline{\nabla}_XY}\xi).\end{aligned} \tag{5.8}$$

In view of (1.1), (3.1), (3.6) and (4.1) the above result (5.8) gives

$$\begin{aligned}0&=2\overline{\nabla}^{\perp}_X\sigma(Z,\phi Y)-2\overline{\nabla}^{\perp}_X\sigma(Z,Y)-\sigma(Z,\nabla_X\phi Y)-\sigma(Z,\nabla_X\eta(Y)\xi)\\&\quad-\sigma(Z,\phi\nabla_XY)-\eta(Y)\sigma(Z,\phi X)+2\sigma(Z,\nabla_XY)+\eta(Y)\sigma(Z,X)\\&\quad-2\sigma(\nabla_XZ,\phi Y)+2\sigma(\nabla_XZ,Y)-2\eta(Z)\sigma(X,\phi Y)+2\eta(Z)\sigma(X,Y).\end{aligned} \tag{5.9}$$

Put $Y=\xi$ and using (3.1), (3.6), (4.1) in (5.9), we get

$$0=-2\sigma(Z,\phi X). \tag{5.10}$$

Replacing X by ϕX and using (3.1) and (4.1) in (5.10) to obtain $\sigma(X,Z)=0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Corollary 1. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting semi-symmetric metric connection. Then σ is 2-recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be 2-recurrent with respect to semi-symmetric metric connection. From (1.4), we have

$$(\bar{\tilde{\nabla}}_X \bar{\tilde{\nabla}}_Y \sigma)(Z,W) = \sigma(Z,W)\phi(X,Y).$$

Taking $W = \xi$ and using (5.2) in the above equation, we have

$$\begin{aligned}\overline{\nabla}^{\perp}_X((\bar{\tilde{\nabla}}_Y\sigma)(Z,\xi)) - (\bar{\tilde{\nabla}}_Y\sigma)(\overline{\nabla}_X Z,\xi) - (\bar{\tilde{\nabla}}_X\sigma)(Z,\overline{\nabla}_Y\xi) - (\bar{\tilde{\nabla}}_{\overline{\nabla}_X Y}\sigma)(Z,\xi) \\ = \sigma(Z,\xi)\phi(X,Y). \qquad (5.11)\end{aligned}$$

In view of (4.1) and (5.1) we write (5.11) in the form

$$\begin{aligned}0 = &-\overline{\nabla}^{\perp}_X\{\sigma(\overline{\nabla}_Y Z,\xi) + \sigma(Z,\overline{\nabla}_Y\xi)\} - \overline{\nabla}^{\perp}_Y\sigma(\overline{\nabla}_X Z,\xi) + \sigma(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi) \\ &+ 2\sigma(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi) - \overline{\nabla}^{\perp}_X\sigma(Z,\overline{\nabla}_Y\xi) + \sigma(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi) \\ &+ \sigma(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi) + \sigma(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi). \qquad (5.12)\end{aligned}$$

Using (1.1), (3.1), (3.6) and (4.1) in (5.12), we get

$$\begin{aligned}0 = &\,2\overline{\nabla}^{\perp}_X\sigma(Z,\phi Y) - 2\overline{\nabla}^{\perp}_X\sigma(Z,Y) - \sigma(Z,\nabla_X\phi Y) \\ &- \sigma(Z,\nabla_X\eta(Y)\xi) - \sigma(Z,\phi\nabla_X Y) - \eta(Y)\sigma(Z,\phi X) + 2\sigma(Z,\nabla_X Y) \\ &+ \eta(Y)\sigma(Z,X) - 2\sigma(\nabla_X Z,\phi Y) + 2\sigma(\nabla_X Z,Y) \\ &- 2\eta(Z)\sigma(X,\phi Y) + 2\eta(Z)\sigma(X,Y). \qquad (5.13)\end{aligned}$$

Taking $Y = \xi$ and using (3.1), (3.6), (4.1) in (5.13), we get

$$0 = -2\sigma(Z,\phi X). \qquad (5.14)$$

Replacing X by ϕX and using (3.1) and (4.1) in (5.14) to obtain $\sigma(X,Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Theorem 3. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting semi-symmetric metric connection. Then σ is generalized 2-recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be generalized 2-recurrent with respect to semi-symmetric metric connection. From (1.5), we have

$$(\bar{\tilde{\nabla}}_X \bar{\tilde{\nabla}}_Y \sigma)(Z,W) = \psi(X,Y)\sigma(Z,W) + \phi(X)(\bar{\tilde{\nabla}}_Y\sigma)(Z,W), \qquad (5.15)$$

where ψ and ϕ are 2-recurrent and 1-form respectively. Taking $W = \xi$ in (5.15) and using (4.1), we get

$$(\bar{\tilde{\nabla}}_X \bar{\tilde{\nabla}}_Y \sigma)(Z,\xi) = \phi(X)(\bar{\tilde{\nabla}}_Y\sigma)(Z,\xi).$$

Using (5.2) and (4.1) in above equation, we get

$$\overline{\nabla}_X^{\perp}((\overline{\widetilde{\nabla}}_Y\sigma)(Z,\xi))-(\overline{\widetilde{\nabla}}_Y\sigma)(\overline{\nabla}_X Z,\xi)-(\overline{\widetilde{\nabla}}_X\sigma)(Z,\overline{\nabla}_Y\xi)-(\overline{\widetilde{\nabla}}_{\overline{\nabla}_X Y}\sigma)(Z,\xi)$$
$$=-\phi(X)\{\sigma(\overline{\nabla}_Y Z,\xi)+\sigma(Z,\overline{\nabla}_Y\xi)\}. \tag{5.16}$$

In view of (4.1) and by virtue of (5.1), the above result gives (5.16), we get

$$-\overline{\nabla}_X^{\perp}\{\sigma(\overline{\nabla}_Y Z,\xi)+\sigma(Z,\overline{\nabla}_Y\xi)\}-\overline{\nabla}_Y^{\perp}\sigma(\overline{\nabla}_X Z,\xi)+\sigma(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi)+2\sigma(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi)$$
$$-\overline{\nabla}_X^{\perp}\sigma(Z,\overline{\nabla}_Y\xi)+\sigma(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi)+\sigma(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi)+\sigma(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi)$$
$$=-\phi(X)\{\sigma(\overline{\nabla}_Y Z,\xi)+\sigma(Z,\overline{\nabla}_Y\xi)\}. \tag{5.17}$$

Using (1.1), (3.1), (3.6) and (4.1) in (5.17), we get

$$2\overline{\nabla}_X^{\perp}\sigma(Z,\phi Y)-2\overline{\nabla}_X^{\perp}\sigma(Z,Y)-\sigma(Z,\nabla_X\phi Y)-\sigma(Z,\nabla_X\eta(Y)\xi)$$
$$-\sigma(Z,\phi\nabla_X Y)-\eta(Y)\sigma(Z,\phi X)+2\sigma(Z,\nabla_X Y)+\eta(Y)\sigma(Z,X)$$
$$-2\sigma(\nabla_X Z,\phi Y)+2\sigma(\nabla_X Z,Y)-2\eta(Z)\sigma(X,\phi Y)+2\eta(Z)\sigma(X,Y)$$
$$=-\phi(X)\{-\sigma(Z,\phi Y)+\sigma(Z,Y)\}. \tag{5.18}$$

Choosing $Y=\xi$ and using (3.1), (3.6), (4.1) in (5.18), we get

$$0=-2\sigma(Z,\phi X). \tag{5.19}$$

Replacing X by ϕX and using (3.1) and (4.1) in (5.19) to obtain $\sigma(X,Z)=0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Using Theorems 5.1 to 5.3 and Corollary 5.1, we have the following result

Corollary 2. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting semi-symmetric metric connection. Then the following statements are equivalent*

(i) *σ is recurrent.*
(ii) *σ is 2-recurrent.*
(iii) *σ is generalized 2-recurrent.*
(iv) *M has parallel third fundamental form.*

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