Communications in Mathematics and Applications
Vol. 12, No. 3, pp. 737–748, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i3.1597

Research Article

Strong Upper Geodetic Number of Graphs

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Received: May 15, 2021 **Accepted:** August 3, 2021

Abstract. Let $G=(V,E)$ be a graph. A set $S\in G$ is a strong geodetic set, if each vertex $v\in G/S$ lies on a fixed geodetic between the pair of vertices of S. A set S is called a minimal strong geodetic set, if no proper subset of S is a strong geodetic set. The maximum cardinality of a minimal strong geodetic set is the strong upper geodetic number. It is denoted by $sg^+(G)$. In this paper, we have proved the NP-completeness of strong upper geodetic number. Some results on strong upper geodetic number and strong upper edge geodetic number of graphs were also found.

Keywords. Geodetic set; Strong geodetic set; Upper geodetic number; Strong Upper geodetic number

Mathematics Subject Classification (2020). 05C30; 6R07; 94C15; 05C10

1. Introduction

Let $G(V,E)$ be a connected graph. The distance $d(u,v)$ between any two vertices u,v of the graph G is the length of the shortest $u-v$ path in G. The concept of geodetic number of a graph was introduced in [4]. For any two vertices $u,v\in V(G)$ the set $I[u,v]$ consist of u,v and all vertices lying on some $u-v$ geodesic in G. For a non-empty subset $S\in G$, $I[S]=\bigcup_{u,v\in S} I[u,v]$. A set $S\subset V(G)$ is a geodetic set of G if $I[S]=V(G)$. The minimum cardinality of a geodetic set of G is the geodetic number $g(G)$ of G. A geodetic set S in G is called a minimal geodetic set if no proper subset S is a geodetic set of G. The maximum cardinality of the minimal geodetic set of G is the upper geodetic number of G and is denoted by $g^+(G)$. For each pair of vertices

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$x, y \subseteq S$, $x \neq y$, let $\widetilde{g}(x,y)$ be a selected fixed shortest path between x and y and

$$\widetilde{I}(S) = \{\widetilde{g}(x,y) : x, y \in S\}$$

and

$$V(\widetilde{I}(S)) = \bigcup_{\widetilde{p} \in \widetilde{I}(S)} V(\widetilde{P}).$$

If $V(\widetilde{I}(S)) = V$ for some $\widetilde{I}(S)$, then the set S is called a strong geodetic set. The minimum cardinality of a strong geodetic set is the strong geodetic number and is denoted by $sg(G)$. The strong geodetic problem is a NP-complete [8]. The strong edge geodetic number is defined as follows. If G is a graph, then $S \subseteq V(G)$ is called a strong edge geodetic set if to any pair $x, y \in S$ one can assign a shortest x,y-path P_{xy} such that

$$\bigcup_{\{x,y\} \in \binom{S}{2}} E(P_{x,y}) = E(G).$$

The cardinality of the smallest strong edge geodetic set S will be called the strong edge geodetic number of G and denoted by $Sg_e(G)$. The Upper edge geodetic number of graphs was introduced in [10]. A vertex v is said to be a simplicial vertex (extreme vertex) if the subgraph induced by its adjacent vertices is a clique.

Definition 1. A strong geodetic set S in a graph G is a minimal strong geodetic set if no proper subset of S is a strong geodetic set of G. The maximum cardinality of a minimal strong geodetic set of G is the strong upper geodetic number of G and is denoted by $sg^+(G)$.

Definition 2. A strong edge geodetic set S in a graph G is a minimal strong edge geodetic set if no proper subset of S is a strong edge geodetic set of G. The maximum cardinality of a minimal strong edge geodetic set of G is the strong upper edge geodetic number of G and is denoted by $sg_e^+(G)$.

We have derived certain bounds on strong upper geodetic number and strong upper edge geodetic number. Also, the strong upper geodetic number for generalized Petersen graph, Circulant network, Bipartite graph and certain graphs were found. The Realization results for the strong upper geodetic and strong upper edge geodetic number of graphs have also been presented in this paper.

Observation 1.1. *Let G be a graph with order n.Then every simplicial vertex belongs to minimal strong geodetic and minimal strong edge geodetic set of G.*

Example 1.1. Consider the graph G in Figure 1. Let $S_1 = \{v_1, v_6, v_8, v_9, v_7\}$ and $S_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ are the strong geodetic sets of G and $S_3 = \{v_3, v_7, v_{10}, v_{11}\}$ is the upper geodetic number g^+ of G. Since no proper subset of S_1 and S_2 are strong geodetic sets of G, it is clear that S_1 and S_2 are the minimal strong geodetic sets of G. The maximum cardinality of the minimal strong geodetic set is the strong upper geodetic number. Therefore, S_2 is the strong upper geodetic number $sg^+(G)$.

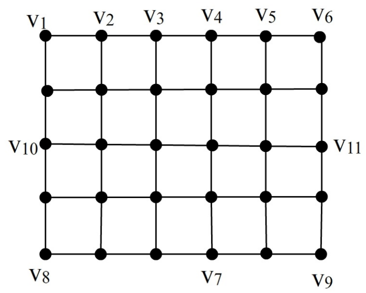


Figure 1

2. Computational Complexity

For a graph $G(V,E)$, a subset S of V is a dominating set if every vertex $v \in V - S$ is adjacent to al least one member of S. A dominating set $\gamma(G)$ is minimal if no proper subset is dominating. The maximum cardinality over the minimal dominating set is the Upper domination number of G and it is NP-complete [3]. We prove that the strong upper geodetic problem is NP-complete. The proof is the polynomial reduction from the Upper domination number.

Theorem 2.1. *The strong upper geodetic problem is NP-complete for general graphs.*

Proof. We construct a graph $\bar{G}(\bar{V},\bar{E})$ as follows. Let $\bar{V} = V \cup V' \cup V''$ be the vertex set and $\bar{E} = E \cup E' \cup E'' \cup E'''$ be the edge set of the graph $\bar{G}(\bar{V},\bar{E})$. For $\bar{V}$ let V be the vertex set of the graph G, $V' = \{v', v \in V\}$, $V'' = \{v'', v \in V\}$. For $\bar{E}$, E be the edge set of the graph G, $E' = \{vv', v \in V\}$, $E'' = \{v'_i v'_j, v \in V/1 \le i \le n, 1 \le j \le n; i \ne j\}$ and $E''' = \{v'v'', v \in V\}$. Thus the graph $\bar{G} = (\bar{V},\bar{E})$ consist of three layers (refer Figure 2). First layer is the graph G. The second layer is the complete graph which is induced by V'. The Independent set of vertices V'' forms the third layer.

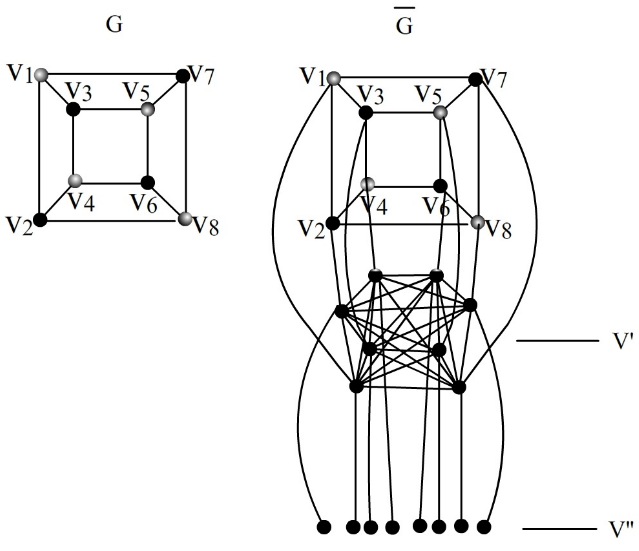


Figure 2. Computational complexity of strong upper geodetic number

Let S be a upper dominating set of G. Then it is easy to see that $S \cup V''$ is a strong upper geodetic set of $\bar{G}$.

By [8, Property 4.2], if X is a strong geodetic set of $\bar{G}$, then there exist a strong geodetic set Y with $|Y| \le |X|$ such that $Y = S \cup V''$ and $S \subseteq V$.

Conversely, let T is a strong upper geodetic set of $\bar{G}$. Since T is a minimal strong geodetic set by above property, $T \cap V' = \phi$. Since the vertices of V'' are simplicial vertices, $V'' \subset T$. It is straight forward that T/V'' is a upper domination set of G. □

3. Main Rsults

Preposition 3.1. *For a graph G with strong upper geodetic number $sg^+(G)$ and strong geodetic number $sg(G)$, $sg(G) \le sg^+(G)$.*

Proof. Since every minimal strong geodetic set of G is a strong geodetic set, $sg(G) \le sg^+(G)$. □

Remark 3.1. The above bound is sharp for a cycle C_n in which $sg(C_n) = sg^+(C_n) = 3$.

Preposition 3.2. *For any graph G of order n, $2 \le sg(G) \le sg^+(G) \le n$.*

Result 3.1. *For a graph G the strong upper geodetic set need not be a upper geodetic set.*

For example, consider the graph in Figure 3. The set $\{b, f, k, d, i\}$ forms the strong upper geodetic number and the set $\{b, f, k\}$ forms the upper geodetic number.

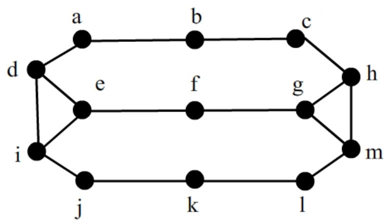


Figure 3

Preposition 3.3. *For a connected graph G of order n, $2 \le sg_e(G) \le sg_e^+(G) \le n$.*

Remark 3.2. The bounds given above are sharp. Consider the graphs in Figure 4 and 5. For a path graph, $sg^+(P_n) = sg_e(P_n) = 2$ and for a complete graph $sg^+(K_n) = sg_e^+(K_n) = n$.

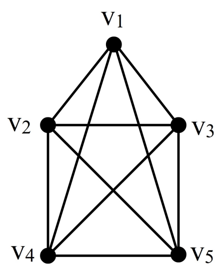


Figure 4. $sg^+(K_5) = sg_e^+(K_n) = 5$

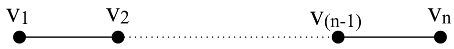


Figure 5. $sg^+(P_n) = sg_e(P_n) = 2$

Result 3.2. *For a uniform theta graph $\theta(l,n)$ with l levels n number of vertices in each level, the strong upper geodetic number is*

$$sg^+(\theta(l,1)) = l,$$
$$sg^+(\theta(l,n)) = l+1, \quad \text{for } n \geq 2.$$

Corollary 3.1. *Let G be a hexagonal silicate network. The set of simplicial vertices of G forms the $sg_e^+(G)$ set.*

Proof. Let S be the set of simplicial vertices of the hexagonal silicate network G. Clearly the set S gives the strong upper geodetic number of G. The result follows form Observation 1.1. □

Theorem 3.1. *If G has the unique minimal strong geodetic set, then $sg(G) = sg^+(G)$.*

Proof. Let S be a minimum strong geodetic set and $|S| = sg(G)$. Since S is a minimum strong geodetic set, this implies S is a minimal strong geodetic set. Since G has a unique minimal strong geodetic set, S is the only minimal geodetic set and therefore S is the strong upper geodetic set and $sg^+(G) = |S|$. □

Preposition 3.4. *Let G be a graph then $sg^+(G) = 2$ if and only if $G = P_n$.*

Theorem 3.2. *For a graph G of order n, $sg(G) = n$ if and only if $sg^+(G) = n$.*

Proof. Let $sg(G) = n$. Then by Preposition 3.2, $sg^+(G) = n$. Conversely, let $sg^+(G) = n$. This implies $V(G)$ is the minimal strong geodetic set. Suppose $sg(G) = n-1$. Let $S \subseteq V(G)$ be a minimum strong geodetic set such that $|S| = n-1$. This implies $V(G)$ is not a minimal strong geodetic set. Therefore $sg(G) = n$. □

Preposition 3.5. *For a connected graph G with $n \geq 3$, if $sg(G) = n-1$ then $sg^+(G) = n-1$.*

Proof. Let $sg(G) = n-1$. By Preposition 3.2, $sg^+(G) = n$ or $n-1$. If $sg^+(G) = n$, then $sg(G) = n$ (refer Theorem 3.3) which is a contradiction. Therefore $sg^+(G) = n-1$. □

Remark 3.3. The converse of the above theorem is not true. Consider the graph G in Figure 6, where $sg^+(G) = n-1$ and $sg(G) \neq n-1$.

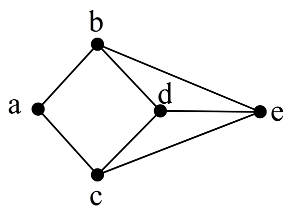


Figure 6

Theorem 3.3. *For a connected graph G of order n, $sg_e(G) = n$ if and only if $sg_e^+(G) = n$.*

Proof. The proof is similar to Theorem 3.2. So we omit the proof. □

Preposition 3.6. *If G is a graph with $sg_e^+ = 2$ if and only if $G = P_n$.*

Theorem 3.4. *Let G be a connected graph and S be the minimal strong geodetic set of G. If there exist a cut vertex $v \in G$, then every component of $G - v$ contains an element of S.*

Proof. G is a connected graph and let A be a component of $G - v$ such that A does not have an element of S. By Observation 1.1 it is clear that A does not contain any vertex $u \in G$ such that $\deg(u) = 1$. Since S is a minimal strong geodetic set, every vertex of $G - S$ lies on a unique geodesic between the vertices of S. For every $v_1 \in A$ there exist $x, y \in S$ such that v_1 lies on a fixed $x - y$ geodesic. It is clear that $x - y$ is a path. Since v is a cut vertex of G, both $x - v_1$ and $v_1 - y$ should contain v. Which implies $x - y$ is not a path. This is a contradiction. Hence every component of $G - v$ contains an element of S. □

Result 3.3. *The set of simplicial vertices of a Block graph forms the minimal strong geodetic set.*

Theorem 3.5. *For a Petersen graph, $sg^+(P(5,2)) = 6$.*

Proof. Let G be the Petersen graph. We have the following observations. Consider the graph in Figure 7. Three vertices on the outer cycle covers all other vertices of the outer cycle and three vertices from inner cycle covers all vertices of the inner cycle. Therefore $sg^+(G) \leq 6$. Also, it is easy to see that G has a strong minimal geodetic set with 6 vertices. Therefore $sg^+(G) = 6$. □

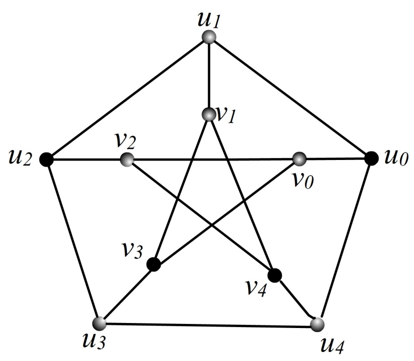


Figure 7

Theorem 3.6. *For a Petersen graph, $sg_e^+(P(5,2)) = 6$.*

Proof. Let G be a Petersen graph. It is easy to observe that three vertices from outer cycle will cover all the edges of the edges of the outer cycle and three vertices from inner cycle will cover all the edges of the inner cycle. Let it be A and B. The paths between the vertices of A and B will cover the edges connecting both the cycles. Therefore $sg_e^+(G) \leq 6$. Clearly, G has a strong minimal edge geodetic set with 6 vertices. Therefore $sg_e^+(G) = 6$. □

Theorem 3.7. *The strong upper geodetic number for a circulant network* $C_n\{1,2\}$ *is*

$$sg^+(C_n\{1,2\}) = n, \quad \text{for } 1 \le n \le 5,$$
$$sg^+(C_n\{1,2\}) = 4, \quad \text{for } 6 \le n \le 7,$$
$$sg^+(C_n\{1,2\}) = 5, \quad \text{for } n \ge 8.$$

Proof. Let G be a circulant graph $C_n\{1,2\})$ (refer Figure 8 for odd and even cases) and S be the minimal strong geodetic set of G.
The proof follows as cases:

Case (i): $1 \le n \le 5$.
For $1 \le n \le 5$, G is a complete graph and hence $|S| = n$.

Case (ii): $6 \le n \le 7$.

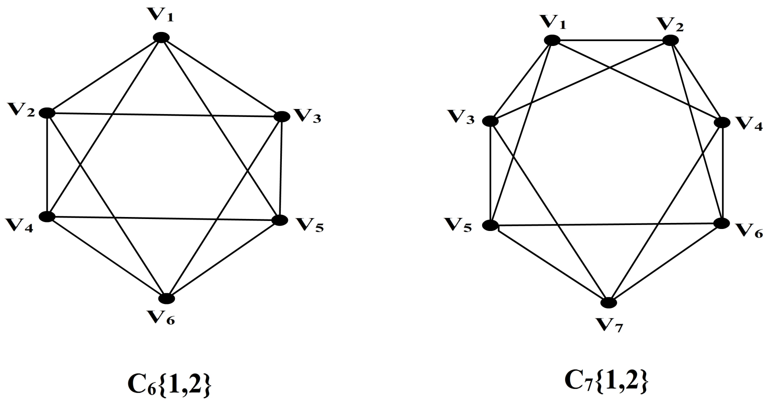


Figure 8. Circulent graph

For $n = 6$, consider the vertex v_2 which is adjacent to v_1, v_3, v_4 and v_6. Here the fixed geodesic between $v_2 - v_5$ will cover almost one vertex say v_4. Also, the vertex v_1 is covered by the fixed geodesic between $v_6 - v_3$. Hence $S = \{v_2, v_3, v_5, v_6\}$ is the minimal strong geodetic set. The proof is same for the case $n = 7$, where the vertex v_7 is covered by the fixed geodesic path $v_6 - v_2$. Therefore $|S| = 4$.

Case (iii): $n \ge 8$.
For $n \ge 8$ we consider even and odd cases.

Case (iii)(a): n is even
In circulant graph $C_n\{1,2\}$ where n is even, we have two cycles C_{n1} and C_{n2} of equal lengths. Let $S = S_1 \cup S_2$ where S_1 and S_2 are the minimal strong geodetic sets of C_{n1} and C_{n2}. Since $sg(C_n) = 3$ and we have two cycles in this case, let us assume that $C_n\{1,2\} = 6$. Now, let us consider any circulant graph with $n = 2n$, $n \ge 8$. A set of three vertices $\{a, b, c\} \in S_1$ will cover all other vertices of C_{n1} in a fixed geodesic path. For the cycle C_{n2} let $\{x, y\} \in S_2$. Then the $x - y$ geodesic will cover half the vertices of the cycle C_{n2}. Since we have considered $sg^+(C_n\{1,2\}) = 6$, $\{x, y, z\} \in S_2$ be the minimal strong geodetic set of C_{n2}. In a circulant graph, each vertex $v \in C_n$, is adjacent to $\{u, w\} \in C_{n2}$ and vice versa. Hence the paths $y - b$ and $b - x$ will cover the other

half vertices of the cycle C_{n2}. Hence the subset $S' = \{a,b,c,x,y\}$ will form the strong geodetic set. Which implies $S = \{a,b,c,x,y,z\}$ is not a minimal strong geodetic set. This contradicts our assumption. Therefore $sg^+(C_n\{1,2\}) = 5$ for $n = 2n$.

Case (iii)(b): n is odd
For $C_n\{1,2\}$ where n is odd, we have two cycles C_{n1} and C_{n2}. The cycle C_{n1} is from v_1 to v_n which covers the odd vertices. The cycle C_{n2} starts from the vertex v_n which passes through the even vertices and ends at the vertex v_1. Hence the two cycles are connected. The proof is similar to the previous case and hence omitted. Therefore $sg^+(C_n\{1,2\}) = 5$ for $n = 2n-1$. □

Theorem 3.8. *For a grid graph $P_m \square P_n$ with $2 \le n \le m$ and $n \ge \lceil \frac{m}{2} \rceil$, the strong upper geodetic number is $sg^+(P_m \square P_n) \ge m+1$.*

Proof. Let $G = P_m \square P_n$. Let $V(P_m \square P_n) = \{(u_i,v_j)/1 \le i \le m, 1 \le j \le n\}$. Consider $S = \{(u_1,v_j)/1 \le j \le n\} \cup \{(u_i,v_k)/i = n, k = \lceil \frac{m}{2} \rceil\}$. Let $P = \{P_t/1 \le t \le m+1\}$ be the unique geodesics between the vertices of S. The path P_1 starting from the vertex (u_1,v_1) covers the first uncovered row (u_j,v_1) and reaches the vertex (u_i,v_k) covering the column corresponding to it. Likely the path P_2 from (u_1,v_2) covers the row (u_i,v_2) until (u_{i-1},v_2) and reaches (u_i,v_k) covering the column (u_{i-1},v_2) (refer Figure 9).

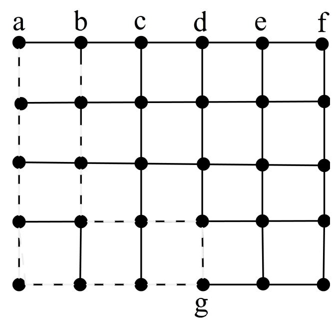


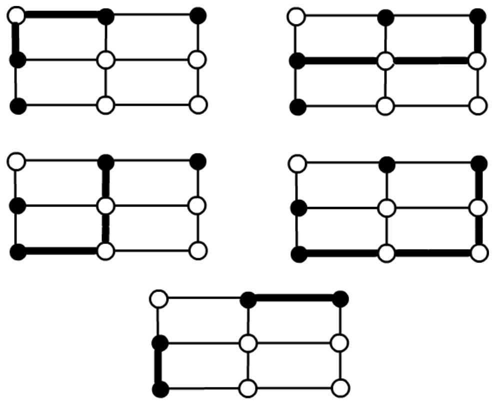
Figure 9

The procedure is repeated for each pair of vertices of S and each vertex of $G-S$ will lie on a unique geodesic between the vertices of S. Hence it is clear that the set S needs at least $m+1$ vertices to cover the graph in a unique geodesic. It is necessary to prove that S is the minimal strong geodetic set of G. Let S_1 be the subset of S such that $S_1 = S - a_1$ where $a_1 = (u_1,v_1)$. Let P_1 be the path between a_1 and (u_i,v_k). It is clear that the row $\{(u_1,v_1),(u_2,v_1),\ldots,(u_i,v_1)\}$ will not be covered by any of the paths $P_t, 2 \le t \le m+1$. Which implies there is no $S_1 \in S$ and S is the minimal strong geodetic set. Therefore $sg^+(P_m \square P_n) \ge m+1$. □

Theorem 3.9. *For a grid graph $P_m \square P_n$ with $n,m \ge 3$, the strong upper edge geodetic number $sg_e^+(P_m \square P_n) \ge (m+n)-2$.*

Proof. Let G be a grid graph $P_m \square P_n$ and S be the minimal strong edge geodetic set of G. A grid graph G is the cartesian product of two paths P_n and P_m. Let $V(P_m \square P_n) = \{(u_i v_j)/1 \le i \le m, 1 \le j \le j\}$. If $(u_1 v_1)$ $(u_2 v_2)$ are two vertices then $u_, u_2 \in V(P_m)$ and $v_1, v_2 \in V(P_n)$. An edge

$(u_1v_1)(u_2v_2) \in E(P_m \square P_n)$ is horizontal if $u_1 = u_2$ and vertical if $v_1 = v_2$. Hence the ith row is the vertex set $\{(u_iv_1), \ldots, (u_mv_j)\}$ with horizontal edges and the jth column is the vertex set $\{(u_1v_j), \ldots, (u_iv_n)\}$ with vertical edges (refer Figure 10).



$P_3 \square P_3$

Figure 10

Let $S = \{(u_2v_1), (u_3v_1), \ldots, (u_mv_1), (u_1v_2)(u_1v_3), \ldots, (u_1v_n)\}$. The edges adjacent to the vertex (u_1v_1) is obviously covered by the unique path P_1 between (u_1v_2) and (u_2v_1). The path between (u_1v_2) and (u_3v_2) will traverse through the horizontal edges form (u_1v_2) and reaches (u_3v_1) by covering the vertical edge adjacent to it. Let it be P_2. Similarly, the paths $\{P_t/1 \leq t \leq |S|\}$ will cover every edge of the graph G in a unique geodesic. Hence S is a strong edge geodetic set of G. Now, it is necessary to prove that S is the minimal strong edge geodetic set. Let $S_1 \subseteq S$ be a subset of S such that $S_1 = S/(u_2v_1)$. The either the vertical edge $(u_2v_{n-1})(u_2v_n)$ or the edges adjacent to (u_mv_n) will be left uncovered. Which implies S_1 is not a strong geodetic set of G. Then S is the minimal strong geodetic set and $|S| = (m-1)+(n-1)$. Therefore $sg_e^+(P_m \square P_n) \geq (m+n)-2$. □

Result 3.4. *The strong upper geodetic number of a Fan graph $F_{1,n}$ is $sg^+(F_{1,n}) = \lceil \frac{n}{2} \rceil$ for $n \geq 4$.*

Result 3.5. *For a windmill graph, $sg^+(Wd(k,n)) = n-1$.*

Result 3.6. *The strong upper geodetic number for a Wheel graph is $sg^+(W_{1,n}) = \lceil \frac{n}{2} \rceil$.*

Result 3.7. *The strong upper geodetic number of a Fan graph F_n is $sg^+(F_n) = n-1$.*

Theorem 3.10. *Let $a, b \geq 2$ be any two integers with $2 \leq a \leq b$ then there exist a graph G such that $sg^+(G) = a$ and $|V(G)| = b$.*

Proof. If $a = b = 2$ then $G = K_2$. For $G = K_3$, $a = b = 3$. Let $b = a$. Consider the complete graph K_n. We know that $sg^+(K_n) = b = a$. Let $a < b$. Consider the graph $k(n,1)/n = a$. Attach a path $P_n = \{v_1, v_2, \ldots v_{b-a}\}$ to the $u_1 \in k(n,1)$ (Figure 11). It is clear to see that $|V(G)| = b$. The pendent vertices of the graph will form the minimal strong geodetic set of G. Hence $sg^+(G) = a$. Hence the proof. □

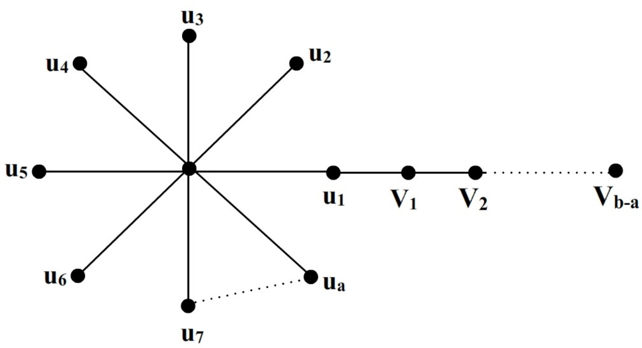


Figure 11. Realization result of strong Upper geodetic number

Theorem 3.11. *Let $a,b \geq 2$ be any two integers with $2 \leq a \leq b$ then there exist a graph G such that $sg_e^+(G) = a$ and $|V(G)| = b$.*

Proof. The proof follows from Theorem 3.10. □

Theorem 3.12. *Let $a,b \geq 2$ be any two positive integers. For $2 \leq a \leq b$, there exist a graph G such that $g^+(G) = a$ and $sg^+(G) = b$.*

Proof. Given that $a,b \geq 2$ are positive integers and $2 \leq a \leq b$ then there exist a graph G, such that $g^+(G) = a$ and $sg^+(G) = b$. Depending upon the integers a and b, we consider two cases $a = b$ and $a < b$. Let $|S|$ be the minimal geodetic set and $|S'|$ be the minimal strong geodetic set of G.

Case (i): $a = b$
Consider a path P_n. Now attach $a-2$ vertices $\{u_1,u_2,\ldots\ldots,u_{a-1}\}$ to P_n by joining $\{u_i/1 \leq i \leq a-2\}$ to v_1 and v_3 respectively (refer Figure 12). Since v_n is a simplicial vertex, $v_4 \in S$. Also $S = \{v_2,u_1,u_2,\ldots,u_{a-2}\} \cup v_n$ forms a minimal geodetic set of the graph. Therefore $S = \{v_2,u_1,u_2,\ldots,u_{a-2}\} = a$. Since each vertex of the graph lies on a fixed geodesic between the pair of vertices of S, $S = S' = a = b$.

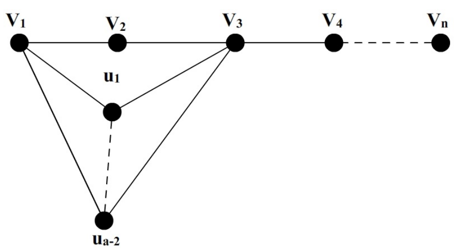


Figure 12. Realization result of strong upper geodetic number

Case (ii): $b \geq a+1$
Consider a path P_5. Now add $a-2$ vertices $\{u_1,u_2,\ldots,u_{a-2}\}$ to v_2 and v_4, respectively. Let $T = \{x_1,x_2,\ldots,x_{a-3}\}$ be the vertex set such that the vertex $x_1 \in T$ is adjacent to v_3 and u_1. Every other vertex of T are added in between $\{u_i/1 \leq i \leq a-3\}$ such that each $\{x_i/2 \leq i \leq a-3\}$ is

adjacent to u_i and u_{i-1}. A set of vertices $T' = \{y_1, y_2, \dots, y_{b-a+1}\}$ are added which are adjacent to u_{a-3} and u_{a-2} (refer the graph in Figure 13). Here the vertex set $\{v_1, v_3, v_5, u_1, \dots u_{a-2}\}$ forms the minimal geodetic set. Hence $|S| = a+1$. It is necessary to cover all the vertices of a graph in a unique geodetic path for a strong geodetic set. The path between u_{a-3} and u_{a-2} can cover only vertex say y_1 in a unique geodesic. Hence every other vertices of the set T' belongs to the minimal strong geodetic set. Which implies $S' = \{v_1, v_3, v_5, u_1, \dots, u_{a-2}, y_2, y_3, \dots, y_{b-a+1}\}$. Therefore $b \geq a+1$. □

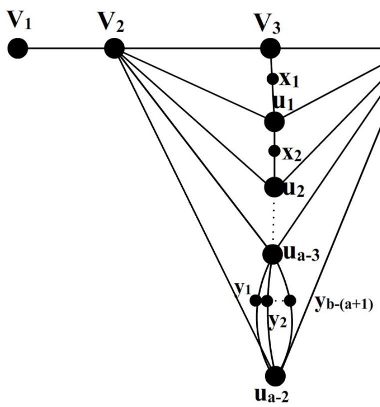


Figure 13. Realization result of strong upper geodetic number

Theorem 3.13. *Let a and b be any two positive integers where $a, b \geq 2$. Then for $2 \leq a \leq b$, there exist a graph G such that $g_e^+(G) = a$ and $sg_e^+(G) = b$.*

Proof. The construction of the graph G and the proof is similar to Theorem 3.12. □

4. Conclusion

We have proved that the strong upper geodetic number of graphs is NP-complete. Some results for strong upper geodetic number and strong upper edge geodetic number of graphs were found. The realization result for strong upper geodetic number and strong upper edge geodetic number were also derived.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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