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Finiteness of the Cyclic Group Related to the Group Inverse of A Matrix and Finite Markov Chains

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Abstract The Group inverse is one of the generalized inverses possessing the properties the closest to the normal inverse. The positive and negative powers of a given matrix A (the latter being interpreted as powers of $A^{\#}$, the group inverse of A), together with the projector $AA^{\#}$ as the unit element, constitute an Abelian group. In this paper, we are going to give equivalent conditions so that this group is finite, and applying the result in finite Markov chains.

1. Introduction

The most of the properties of the generalized inversion were handled in the work of A. Ben Isra l and T.N.E. Greville [1], also in the work of Z. Nashed [3]. Some algebraic structures on the set of generalized inverses of matrices are studied in [4]. Some algebraic properties are widely studied in [5]. The Group inverse is one of the generalized inverses possessing the properties the closest to the normal inverse. The positive and negative powers of a given matrix A (the latter being interpreted as powers of $A^{\#}$ the Group inverse of A), together with the projector $AA^{\#}$ as the unit element, constitute an Abelian group. In this paper, we are going to give equivalent conditions so that this group is finite. In all the content, $\mathbb{K}$ design the real or the complex field. $r(A)$, $R(A)$, $N(A)$ denote respectively, the rank, the range and the null space of a matrix A.

2. The Group Inverse of a Matrix

The definition of the Group inverse was firstly given by I. Erd lyi in [2].

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Definition 1. Let A be a square matrix of order n. The Group inverse of A denoted by $A^{\#}$, is the square matrix satisfying the following equations

$$\begin{cases} AA^{\#}A = A, \\ A^{\#}AA^{\#} = A^{\#}, \\ AA^{\#} = A^{\#}A. \end{cases} \tag{1}$$

Remark 2. From the third equation, we get

$$R(A) = R(A^{\#}) \quad \text{and} \quad N(A) = N(A^{\#}).$$

2.1. *Existence and uniqueness*

Lemma 3. *Let A be a square matrix. The following conditions are equivalent*

(i) $r(A) = r(A^2)$.
(ii) $R(A) = R(A^2)$.
(iii) $N(A) = N(A^2)$.
(iv) $R(A) \oplus N(A) = \mathbb{K}^n$ *($R(A)$ and $N(A)$ are complementary).*
(v) $A^{\#}$ *exists and unique.*

Proof. Equivalences between (i), (ii) and (iii) are obvious, as we have $r(X) + \dim N(X) = n$, for any matrix X of order n.

We will prove equivalence between (i) and (iv). In fact, it is sufficient to prove equivalence between (i) and $N(A) \cap R(A) = \{0\}$. Let $x \in N(A) \cap R(A)$, then there exists y such that $0 = Ax = A^2y$, which implies that $y \in N(A^2) = N(A)$. Consequently, $x = Ay = 0$.

Conversely, let $N(A) \cap R(A) = \{0\}$, then if $x \in R(A)$ such that $Ax = 0$, then $x = 0$. Which means that, the restriction of A to $R(A)$ (as an endomorphism) is an automorphism. Therefore $R(A^2) = A(R(A)) = R(A)$.

Now we prove the equivalence between (ii) and (v). Suppose that $R(A) = R(A^2)$, then, there exists a matrix X such that

$$A = A^2X. \tag{2}$$

Let $A = BC$ be the rank factorization of A, and let B_l and C_r be left and right inverses of B and C respectively. Then equation (2) yields

$$A = BC = BCBCX. \tag{3}$$

Thus, multiplying (3) on the left by B_l and on the right by C_r gives

$$(B_lB)(CC_r) = I = (CB)CXC_r \tag{4}$$

which means that (CB) is invertible. Let $X = B(CB)^{-2}C$. By substituting in system (1), we obtain

$$AXA = BCB(CB)^{-2}CBC = BC = A,$$

$$XAX = B((CB)^{-2}CBCB)(CB)^{-2}C = B(CB)^{-2}C = X,$$

and

$$AX = BCB(CB)^{-2}C = B(CB)^{-1}C = XA.$$

It follows that $X = A^{\#}$. Inversely, if $A^{\#}$ exists, then equation $A^2X = A$ is equivalent to $R(A) = R(A^2)$.

Uniqueness. Suppose that there are two Group inverses G_1 and G_2. We apply successively system (1), we obtain $G_1 = G_1^2A = G_1^2(A^2G_2) = G_1(G_1A^2)G_2 = G_1AG_2 = G_1(A^2G_2)G_2 = (G_1A^2)G_2^2 = AG_2^2 = G_2$. □

2.2. *Characterization of the Group Inverse by the Index of the Matrix*

Definition 4. Let A be a square matrix. The smallest positive integer k for which

$$R(A^k) = R(A^{k+1}), \tag{5}$$

called the index of A and denoted by $\operatorname{Ind} A$.

Lemma 5 ([1]). *Let A be a square matrix. Then the following statements are equivalent:*

(i) $\operatorname{Ind} A = k$.

(ii) *The smallest positive exponent for which* (5) *holds is k.*

(iii) *If A is singular and $m(\lambda)$ is its minimal polynomial, k is the multiplicity of $\lambda = 0$ as a zero of $m(\lambda)$.*

From Definition 4, and Lemma 5, it follows:

Corollary 6. *A square matrix A has the group inverse if and only if* $\operatorname{Ind} A = 1$.

2.3. *The Cyclic Group*

The name "Group inverse" was first given by I. Erdelyi [2], because the positive and the negative powers of a given matrix A (the negative powers of A being interpreted as positive powers of $A^{\#}$), together with the projector $AA^{\#}$ as the unit element, constitute an abelian group under matrix multiplication.

In fact, let A be a square matrix and $A^{\#}$ its Group inverse. Set $G = \{A^k, k \in \mathbb{Z}\}$. As $AA^{\#} = A^{\#}A$, it is sufficient to check multiplication only in the left side. Remark that for any $k \in \mathbb{Z}$, $(AA^{\#})A^k = AA^{\#}AA^{k-1} = AA^{k-1} = A^k$, which means that $AA^{\#}$ is the unit of G. As $AA^{\#} = A^{\#}A$ then A and $A^{\#}$ are the inverses of each other (within the meaning of the inversion in the group). Therefore the negative powers of A are the positive ones of $A^{\#}$. Notice also that G is cyclic, generated by$A$ or $A^{\#}$, so we designate this group by $C(A)$.

Problem 7. *Does it exist a positive integer k for which $A^k = AA^{\#}$?*

The following theorem will give equivalent conditions for the existence.

Theorem 8. *Let A be a square matrix of order n such that $A^{\#}$ exists. The following conditions are equivalent*

(i) *There exists a positive integer k for which $A^k = AA^{\#}$.*

(ii) *$C(A)$ is a finite group of order k.*

(iii) *A is similar to diagonal block matrix $\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$ where S is a k-root of unity.*

Proof. The first two conditions are just a simple consequence of cyclic groups.

Now if $A^k = AA^{\#}$ then A^k is a projector on $R(A)$. Therefore $A^k = I$ on $R(A)$ and $A^k = 0$ on $N(A)$. By Jordan form, this means that the non zero eigenvalues of A are the roots of unity of order k on $R(A)$ and, $A = N$ a nilpotent block matrix of index k on $N(A)$. Then, there exists an invertible matrix P such that

$$A = P^{-1}\begin{pmatrix} S & 0 \\ 0 & N \end{pmatrix} P\,,$$

where S is a root of unity of order k. If we take

$$X = P^{-1}\begin{pmatrix} S^{-1} & 0 \\ 0 & N^t \end{pmatrix} P\,,$$

then X is a $\{1,2\}$-inverse of A (i.e. X satisfies the first two equations of system (1)). As AX is idempotent we have

$$AX = (AX)^k = P^{-1}\begin{pmatrix} I_r & 0 \\ 0 & (NN^t)^k \end{pmatrix} P\,. \tag{6}$$

To have the Group inverse, X should satisfy $AX = XA$,which implies that $NN^t = N^tN$, and this yields $(NN^t)^k = N^k(N^t)^k = 0$. From the equality in (5), we obtain $NN^t = (NN^t)^k = 0$, and so $N = 0$ because NN^t is positive semidefinite. Finally,

$$A = P^{-1}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} P\,.$$

Conversely, if

$$A = P^{-1}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} P\,,$$

then, the characteristic polynomial of A is $t^{(n-k)}(t^k - 1)$. Consequently,

$$\mathbb{K}^n = \ker(A^k - I) \oplus \ker A^{n-k}\,.$$

As

$$\ker(A^k - I) = R(A),$$
$$\ker A^{n-k} = N(A)\,,$$

we have

$$\begin{cases} A^k = I \text{ on } R(A)\,, \\ A^k = 0 \text{ on } N(A) \end{cases}$$

which means that $A^k = AA^{\#}$. □

3. **Application in Markov Chains**

Definition 9. A square matrix $P = (p_{ij})$ of order n satisfying

$$p_{ij} \geq 0,\ i,j \in \{1,\ldots,n\} \quad \text{and} \quad \sum_{j=1}^{n} p_{ij} = 1$$

is called stochastic.

Lemma 10 ([1]). *Every stochastic matrix is similar to*

$$\begin{pmatrix} I_k & 0 \\ 0 & K \end{pmatrix} \tag{7}$$

where $1 \notin s(K)$ *(*$s(K)$ *is the spectrum of* K*).*

Lemma 11. *Let P be a stochastic matrix. Then $I - P$ has the Group inverse.*

Proof. It follows by lemma (10) that, there exists a nonsingular matrix Q such that

$$I - P = Q^{-1}\begin{pmatrix} 0 & 0 \\ 0 & I-K \end{pmatrix} Q\,.$$

As $1 \notin s(K)$, it follows that $I - K$ is nonsingular. It is then fairly easy to show that

$$(I-P)^{\#} = Q^{-1}\begin{pmatrix} 0 & 0 \\ 0 & (I-K)^{-1} \end{pmatrix} Q \square$$

Definition 12. A square matrix is said to be reducible if by a rearrangement of rows and columns can be brought to the block matrix of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Theorem 13. *Let P be a stochastic matrix such that the cyclic group $C(I-P)$ is finite, then*

(i) *P has the Group inverse.*
(ii) *P is reducible.*

Proof. (i) By Theorem 8, there exists a smallest positive integer s (the order of $C(I-P)$) such that

$$(I-K)^s = I\,.$$

It follows that the minimal polynomial of K is

$$m(\lambda) = \lambda(1 + (1-\lambda) + \ldots + (1-\lambda)^{s-1}),$$

and 1 is the multiplicity of $\lambda = 0$ as a zero of $m(\lambda)$, from which it follows by Lemma 5 that $\operatorname{Ind} K = 1$. By Corollary 6, $K^{\#}$ exists, and therefore $P^{\#}$ exists.

(ii) As $\lambda = 0 \in s(K)$, then, the form (7) shows that P has at least one zero row, which by a rearrangement of rows and columns can be brought to the bottom left corner of P, say

$$P = Q^{-1} \begin{pmatrix} I_k & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{pmatrix} Q,$$

where Q is a permutation matrix, and $1, 0 \notin s(R)$. Therefore, P is reducible. □

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