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Research Article

A Numerical Approach for Fredholm Delay Integro Differential Equation

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Abstract. This paper deal with the initial-value problem for a linear first order Fredholm delay integro differential equation. To solve this problem numerically, a finite difference scheme is presented, which based on the method of integral identities with the use of exponential form basis function. As a result of the error analysis, it is proved that the method is first-order convergent in the discrete maximum norm. Finally, an example is provided that supports the theoretical results.

Keywords. Fredholm delay integro differential equation; Finite difference method; Error estimate

Mathematics Subject Classification (2020). 34K06; 45J05; 65L05; 65L12; 65L20

1. Introduction

Integro-differential equations have many applications of science and engineering. They are an important subject of mathematical physics and biology, which are modelled in many areas such as the elasticity, fluid dynamics, electromagnetic, heat and mass transfer, biomechanics, propagation of nervous impulse, population dynamics, polymeric liquids [11, 13, 14, 16, 18–22].

Motivated by these works, we deal with the following *Fredholm integro-differential equation* (FIDE) with delay in the interval $\bar{\Omega} = [0, l]$:

$$Lu := u'(x) + a(x)u(x) + b(x)u(x-r) = f(x) + \lambda \int_0^l K(x,t)u(t-r)dt, \quad x \in \Omega, \tag{1.1}$$

$$u(x) = \varphi(x), \quad x \in \Omega_0 \tag{1.2}$$

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where $\Omega = (0,l] = \bigcup_{p=1}^{m} \Omega_p$, $\Omega_p = \{x : r_{p-1} < x \le r_p,\ 1 \le p \le m\}$, $\bar{\Omega} = [0,l]$, $r_k = kr$, $0 \le k \le m$ and $\Omega_0 = [-r,0]$ (for simplicity we suppose that l/r is integer; i.e., $l = mr$). Also, λ is a real parameter and r is a constant delay term. $a(x) \ge \alpha > 0$, $b(x)$, $f(x)$, $\varphi(x)$ and $K(x,t)$ are given sufficiently smooth functions. Under the conditions in Lemma 2.1, the existence and uniqueness of the solution to problem (1.1)-(1.2) are guaranteed.

There are many methods for the numerical approximation of FIDEs, such as the collocation method [6, 23], the variational iteration method [10], the Wavelet method [9], the direct method [15], the Tau method with polynomial bases [17], the finite difference method [2, 12], the reproducing kernel Hilbert space method [5], and references there in. Although, there are many studies on FIDEs without delay in the literature, there are few works on FIDEs including delay terms. Especially, in the last two decades, interest in these equations has been remarkable. For example, to solve FIDEs with delay, various types of computational methods have been proposed, such as the collocation technique with the Haar wavelet [1], the Jacobi matrix method with collocation points [8], the Tau method with the some special polynomials [24, 25], the polynomial approach with spline functions [7].

Even if the equation (1.1) is linear, it may not always be possible to find its analytical solution. Because, it is not possible to benefit from Taylor's expansion because of the delay term is large. Therefore, it is necessary to develop suitable numerical solutions for such equations.

In this paper, we have developed a new approach to solve (1.1)-(1.2), numerically. This approach is based on the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form for differential part and the composite integration method for integral part. As a result of this, the local truncation errors occur that include up to first order derivative of the exact solution and so facilitates examination of the convergence.

The remainder of this paper is as follows. In Section 2, a priori estimate of the exact solution is given. The finite difference scheme is constructed in Section 3. In Section 4, the error analysis for the approximate solution is presented and convergence is proved in the discrete maximum norm. In Section 5, the iterative algorithm for solving the discrete problem is formulated and numerical results are presented, which computationally verify the theoretical analysis. The paper ends with a summary of the main conclusions.

Throughout the paper, C denotes a generic positive constant. Some specific, fixed constants of this kind are indicated by subscripting C.

2. The Statement of the Problem

In this section, we present some estimates for the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution. For any continuous function $g(t)$, we use $\|g\|_\infty$ for the continuous maximum norm on the corresponding interval. In particular, we will use $\|g\|_{\infty,p} = \|g\|_{\infty,\bar{\Omega}_p} := \max_{x\in\bar{\Omega}_p} |g(x)|$, $0 \le p \le m$ and $\bar{\Omega}_p^* = \{x : 0 < x \le r_p\}$, $1 \le p \le m$.

Lemma 2.1. *Let* $a,b,f \in C(\bar{\Omega})$, $K \in C(\bar{\Omega}\times\bar{\Omega})$, $\varphi \in C(\Omega_0)$ *and* $|\lambda| < \alpha/(\bar{K}l)$. *Then for* u *which the solution of the problem* (1.1)-(1.2) *the following estimates hold:*

$$\|u\|_{\infty,\bar{\Omega}_p^*} \le C_p, \tag{2.1}$$

$$\|u'\|_{\infty,\bar{\Omega}_p^*} \le \|a\|_\infty C_p + (\|b\|_\infty + |\lambda|\bar{K}l)C_{p-1} + \|f\|_\infty, \tag{2.2}$$

where

$$\begin{aligned} C_p &= \|\varphi\|_{\infty,0}[(\alpha-|\lambda|\bar{K}l)^{-1}\|b\|_\infty]^p \\ &\quad + (1-\alpha^{-1}|\lambda|\bar{K}l)^{-1}\left[|\varphi(0)| + \alpha^{-1}\|f\|_\infty + \alpha^{-1}|\lambda|\bar{K}\int_{-r}^{0}|\varphi(s)|ds\right] \\ &\quad \times \sum_{k=1}^{p}[(\alpha-|\lambda|\bar{K}l)^{-1}\|b\|_\infty]^{p-k}, \quad p=1,2,\dots,m \end{aligned}$$

and

$$\bar{K} = \max_{(x,t)\in\bar{\Omega}\times\bar{\Omega}} |K(x,t)|.$$

Proof. From (1.1)-(1.2) we can write

$$u'(x) + a(x)u(x) = F(x), \quad x \in \bar{\Omega}_p^*, \tag{2.3}$$

where

$$F(x) = f(x) - b(x)u(x-r) + \lambda\int_0^l K(x,t)u(t-r)dt.$$

Then

$$|F(x)| \le \|f\|_{\infty,\bar{\Omega}_p^*} + \|b\|_{\infty,\bar{\Omega}_p^*}\|u\|_{\infty,\bar{\Omega}_{p-1}^*} + |\lambda|\bar{K}\int_0^l |u(t-r)|dt$$

and based on assumptions of this lemma and from (2.3) it follows that

$$|u(x)| \le |\varphi(0)| + \alpha^{-1}\left[\|f\|_{\infty,\bar{\Omega}_p^*} + \|b\|_{\infty,\bar{\Omega}_p^*}\|u\|_{\infty,\bar{\Omega}_{p-1}^*}\right] + \alpha^{-1}|\lambda|\bar{K}\int_0^l |u(t-r)|dt.$$

After replacing $s = t-r$ we arrive at

$$\begin{aligned} |u(x)| &\le \gamma_p + \alpha^{-1}|\lambda|\bar{K}\int_{-r}^{0}|\varphi(s)|ds + \alpha^{-1}|\lambda|\bar{K}\int_0^{l-r}|u(s)|ds \\ &\le \gamma_p + \alpha^{-1}|\lambda|\bar{K}\int_{-r}^{0}|\varphi(s)|ds + \alpha^{-1}|\lambda|\bar{K}\int_0^{l}|u(s)|ds, \end{aligned}$$

where

$$\gamma_p = |\varphi(0)| + \alpha^{-1}\left[\|f\|_{\infty,\bar{\Omega}_p^*} + \|b\|_{\infty,\bar{\Omega}_p^*}\|u\|_{\infty,\bar{\Omega}_{p-1}^*}\right].$$

So, we have

$$\begin{aligned} \|u\|_{\infty,\bar{\Omega}_p^*} &\le (1-\delta l)^{-1}\left[|\varphi(0)| + \alpha^{-1}\|f\|_{\infty,\bar{\Omega}_p^*} + \delta\int_{-r}^{0}|\varphi(s)|ds\right] \\ &\qquad + \alpha^{-1}(1-\delta l)^{-1}\|b\|_{\infty,\bar{\Omega}_p^*}\|u\|_{\infty,\bar{\Omega}_{p-1}^*}, \end{aligned}$$

where

$$\delta = \alpha^{-1}|\lambda|\bar{K}.$$

From here, the following first order difference inequality follows

$$v_p \le q v_{p-1} + \rho$$

with

$$v_p = \|u\|_{\infty,\bar{\Omega}_p^*}, \quad \delta = \alpha^{-1}|\lambda|\bar{K},$$

$$q = \alpha^{-1}(1-\delta l)^{-1}\|b\|_{\infty,\bar{\Omega}_p^*},$$

$$\rho = (1-\delta l)^{-1}\left[|\varphi(0)| + \alpha^{-1}\|f\|_{\infty,\bar{\Omega}_p^*} + \delta\int_{-r}^{0}|\varphi(s)|ds\right],$$

which yields the estimate

$$v_p \le v_0 q^p + \rho\sum_{k=1}^{p} q^{p-k}$$

so we arrive at (2.1).

Now, from (1.1), we easily get

$$|u'(x)| \le |a(x)||u(x)| + |b(x)||u(x-r)| + |f(x)| + |\lambda|\int_0^l |K(x,t)||u(t-r)|dt$$

and from hence we obtain

$$\|u'\|_{\infty,\bar{\Omega}_p^*} \le \|a\|_{\infty,\bar{\Omega}_p^*}\|u\|_{\infty,\bar{\Omega}_p^*} + \|b\|_{\infty,\bar{\Omega}_p^*}\|u\|_{\infty,\bar{\Omega}_{p-1}^*} + \|f\|_{\infty,\bar{\Omega}_p^*} + |\lambda|\|u\|_{\infty,\bar{\Omega}_{p-1}^*}\bar{K}l,$$

which completes the proof. □

3. The Difference Scheme and Mesh

Let $\bar{\omega}_{N_0}$ be a uniform mesh on $\bar{\Omega}$:

$$\omega_{N_0} = \{x_i = ih,\ i = 1,2,\dots,N,\ h = l/N_0 = r/N\}, \quad \bar{\omega}_{N_0} = \omega_{N_0}\cup\{0\},$$

which contains by N mesh point at each subinterval Ω_p $(1 \le p \le m)$:

$$\omega_{N_p} = \{x_i : (p-1)N + 1 \le i \le pN,\ 1 \le p \le m\}$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m}\omega_{N_p}$$

and also

$$\omega_{N_p}^* = \{x_i : 1 \le i \le pN\}, \quad (1 \le p \le m).$$

For any mesh function $g(x)$, we use $g_i = g(x_i)$ and moreover y_i denotes an approximation of $u(x)$ at x_i and

$$g_{\bar{x},i} = (g_i - g_{i-1})/h, \quad \|g\|_{\infty,p} = \|g\|_{\infty,\omega_{N_p}} := \max_{(p-1)N \le i \le pN}|g_i|,$$

$$\|g\|_{\infty,\omega_{N_p}^*} := \max_{1 \le i \le pN}|g_i|, \quad (1 \le p \le m).$$

For the difference approximation the problem (1.1)-(1.2), we are using the following identity

$$h^{-1}\int_{x_{i-1}}^{x_i} Lu(x)\phi_i(x)dx = h^{-1}\int_{x_{i-1}}^{x_i}[f(x) + \lambda\int_0^l K(x,s)u(s-r)ds]\phi_i(x)dx, \quad 1 \le i \le N_0, \quad (3.1)$$

with basis function

$$\phi_i(x) = e^{-\int_x^{x_i} a(t)dt}, \quad x_{i-1} \le x \le x_i \tag{3.2}$$

which is the solution of the following problem

$$-\phi_i'(x) + a(x)\phi_i(x) = 0, \quad x_{i-1} < x \le x_i, \quad \phi_i(x_i) = 1.$$

The relation (3.1) is rewritten as

$$\begin{aligned} h^{-1}\int_{x_{i-1}}^{x_i} u'(x)\phi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_i} a(x)u(x)\phi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_i} b(x)u(x-r)\phi_i(x)dx \\ = h^{-1}\int_{x_{i-1}}^{x_i} f(x)\phi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_i}\left(\lambda\int_0^l K(x,s)u(s-r)ds\right)\phi_i(x)dx. \end{aligned} \tag{3.3}$$

First, using the formulas (2.1) and (2.2) from [3] on interval (x_{i-1}, x_i) taking into account the left hand side equation (3.3), we have

$$\begin{aligned} h^{-1}\int_{x_{i-1}}^{x_i} u'(x)\phi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_i} a(x)u(x)\phi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_i} b(x)u(x-r)\phi_i(x)dx \\ = A_i u_{\bar{x},i} + B_i u_{\bar{x},i-N} + C_i u_i + D_i u_{i-N} + R_i^{(1)}, \end{aligned}$$

where

$$A_i = h^{-1}\int_{x_{i-1}}^{x_i} \phi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_i}(x-x_i)a(x)\phi_i(x)dx, \tag{3.4}$$

$$B_i = h^{-1}\int_{x_{i-1}}^{x_i}(x-x_i)b(x)\phi_i(x)dx, \quad C_i = h^{-1}\int_{x_{i-1}}^{x_i} a(x)\phi_i(x)dx, \tag{3.5}$$

$$D_i = h^{-1}\int_{x_{i-1}}^{x_i} b(x)\phi_i(x)dx, \tag{3.6}$$

$$R_i^{(1)} = h^{-1}\int_{x_{i-1}}^{x_i} b(x)\phi_i(x)dx\int_{x_{i-1}}^{x_i} u'(\xi-r)K_0(x,\xi)d\xi, \tag{3.7}$$

$$K_0(x,\xi) = T_0(x-\xi) - h^{-1}(x-x_{i-1}), \quad T_0(t) = 1,\ t \ge 0;\ T_0(t) = 0,\ t < 0.$$

Second, for the integral term from the right hand side eqution (3.3), we have

$$\begin{aligned} &h^{-1}\int_{x_{i-1}}^{x_i}\left[\lambda\int_0^l K(x,s)u(s-r)ds\right]\phi_i(x)dx \\ &= h^{-1}\lambda\int_{x_{i-1}}^{x_i} dx\phi_i(x)\int_0^l K(x_i,s)u(s-r)ds, \\ &\quad + h^{-1}\lambda\int_{x_{i-1}}^{x_i} dx\phi_i(x)\int_0^l [K(x,s)-K(x_i,s)]u(s-r)ds. \end{aligned}$$

Further using the composite right side rectangle rule, we obtain

$$\int_0^l K(x_i,s)u(s-r)ds = h\sum_{j=1}^{N_0} K_{i,j}u_{j-N} - \sum_{j=1}^{N_0}\int_{x_{j-1}}^{x_j}(s-x_{j-1})\frac{\partial}{\partial s}[K(x_i,s)u(s-r)]ds.$$

Therefore, we get

$$h^{-1}\int_{x_{i-1}}^{x_i}\left[\lambda\int_0^l K(x,s)u(s-r)ds\right]\phi_i(x)dx = h^{-1}\lambda\int_{x_{i-1}}^{x_i} dx\phi_i(x)h\sum_{j=1}^{N_0} K_{i,j}u_{j-N} + R_i^{(2)}$$

with remainder term

$$R_i^{(2)} = h^{-1}\lambda\int_{x_{i-1}}^{x_i} dx\phi_i(x)\int_0^l [K(x,s)-K(x_i,s)]u(s-r)ds$$
$$+\lambda h^{-1}\int_{x_{i-1}}^{x_i}\phi_i(x)dx\sum_{j=1}^{N_0}\int_{x_{j-1}}^{x_j}(s-x_{j-1})\frac{\partial}{\partial s}[K(x_i,s)u(s-r)]ds. \tag{3.8}$$

Hereby, we write the exact relation for $u(x_i)$:

$$\ell u_i \equiv A_i u_{\bar{x},i} + B_i u_{\bar{x},i-N} + C_i u_i + D_i u_{i-N}$$
$$= F_i + E_i\lambda\sum_{j=1}^{N_0} K_{i,j}u_{j-N} + R_i, \quad 1\le i\le N_0, \tag{3.9}$$

with

$$E_i = \int_{x_{i-1}}^{x_i}\phi_i(x)dx,$$
$$F_i = h^{-1}\int_{x_{i-1}}^{x_i} f(x)\phi_i(x)dx,$$
$$R_i = R_i^{(1)} + R_i^{(2)} \tag{3.10}$$

where A_i, B_i, C_i, D_i and $R_i^{(k)}$ $(k=1,2)$ are determined by (3.4)-(3.6) and (3.7), (3.8), respectively. By virtue of (3.9) we suggest the following difference scheme for approximating (1.1)-(1.2):

$$\ell y_i \equiv A_i y_{\bar{x},i} + B_i y_{\bar{x},i-N} + C_i y_i + D_i y_{i-N} = F_i + E_i\lambda\sum_{j=1}^{N_0} K_{i,j}y_{j-N}, \quad 1\le i\le N_0, \tag{3.11}$$

$$y_i = \varphi_i, \quad -N\le i\le 0. \tag{3.12}$$

We also propose another difference scheme that can be easily obtained using the implicit Euler method which is known as the classical method and appropriate quadrature rules:

$$y_{\bar{x},i} + a_i y_i + b_i y_{i-N} = f_i + h\lambda\sum_{j=1}^{N_0} K_{i,j}y_{j-N}, \quad 1\le i\le N_0, \tag{3.13}$$

$$y_i = \varphi_i, \quad -N\le i\le 0. \tag{3.14}$$

4. Error Analysis

To analyze the convergence of the method, we define the error function $z_i = y_i - u_i$, $1\le i\le N_0$ is the solution of the following discrete problem

$$\ell z_i = E_i\lambda\sum_{j=1}^{N_0} K_{i,j}z_{j-N} - R_i, \quad 1\le i\le N_0, \tag{4.1}$$

$$z_i = 0, \quad -N\le i\le 0, \tag{4.2}$$

where the truncation error R_i is given by (3.10).

Lemma 4.1. *If $a, b, f\in C(\bar{\Omega})$, $\varphi\in C^1(\Omega_0)$, $K\in C^1(\bar{\Omega}\times\bar{\Omega})$ and then for the truncation error R_i we have*

$$\|R\|_\infty \le Ch. \tag{4.3}$$

Proof. From (3.7), we write

$$|R_i^{(1)}| \le h^{-1}\int_{x_{i-1}}^{x_i} dx|b(x)|\phi_i(x)\int_{x_{i-1}}^{x_i} |u'(\xi-r)|d\xi.$$

Due to Lemma 2.1 and $0<\phi_i(t)\le 1$, we have

$$\begin{aligned}|R_i^{(1)}| &\le Ch^{-1}\|b\|_\infty\int_{x_{i-1}}^{x_i} dx\int_{x_{i-1}}^{x_i} |u'(\xi-r)|d\xi\\ &\le C\int_{x_{i-1}}^{x_i} |u'(\xi-r)|d\xi \le Ch.\end{aligned}$$

From (3.8), we write

$$\begin{aligned}|R_i^{(2)}| &\le h^{-1}|\lambda|\int_{x_{i-1}}^{x_i} dx\phi_i(x)\int_0^l |K(x,s)-K(x_i,s)||u(s-r)|ds\\
&\quad +|\lambda|h^{-1}\int_{x_{i-1}}^{x_i}\phi_i(x)dx\sum_{j=1}^{N_0}\int_{x_{j-1}}^{x_j}(s-x_{j-1})\left|\frac{\partial}{\partial s}[K(x_i,s)u(s-r)]\right|ds\\
&\le h^{-1}|\lambda|\int_{x_{i-1}}^{x_i} dx\phi_i(x)\int_0^l (x-x_i)|\frac{\partial}{\partial x}K(x,s)||u(s-r)|ds\\
&\quad +|\lambda|h^{-1}\int_{x_{i-1}}^{x_i}\phi_i(x)dxh\int_0^l\left|\frac{\partial}{\partial s}[K(x_i,s)u(s-r)]\right|ds\\
&\le |\lambda|\int_{x_{i-1}}^{x_i} dx\phi_i(x)\int_0^l\left|\frac{\partial}{\partial x}K(x,s)\right||u(s-r)|ds\\
&\quad +|\lambda|\int_{x_{i-1}}^{x_i}\phi_i(x)dx\int_0^l\left[\left|\frac{\partial}{\partial s}K(x_i,s)\right||u(s-r)|+|K(x_i,s)||u'(s-r)|\right]ds.\end{aligned}$$

By virtue of Lemma 2.1, $\left|\frac{\partial}{\partial x}K(x,t)\right|\le C$, $\left|\frac{\partial}{\partial s}K(x,s)\right|\le C$ and $0<\phi_i(t)\le 1$, we have

$$|R_i^{(2)}| \le |\lambda|\int_{x_{i-1}}^{x_i} dx\int_0^l\left[|u(s-r)|+|u'(s-r)|\right]ds \le Ch,$$

and so we arrive at (4.3). □

Lemma 4.2. *Let $|G_i|\le\bar{G}_i$ and $\bar{G}_i$ be nondecreasing function, we consider the following problem*

$$lv_i = A_i v_{\bar{t},i}+C_i v_i = G_i,\quad 1\le i\le N, \tag{4.4}$$

$$v_0=\beta. \tag{4.5}$$

Then the solution of difference problem (4.4)-(4.5) *hold:*

$$|v_i|\le|\beta|+\alpha^{-1}\bar{G}_i,\quad 1\le i\le N.$$

Proof. The proof is almost identical that of [4]. □

Lemma 4.3. *Let z_i be the solution of* (4.1)-(4.2) *and*

$$2\alpha^{-1}\|b\|_\infty+\alpha^{-1}|\lambda|\max_{1\le i\le N_0} h\sum_{j=1}^{N_0}|K_{i,j}|<1$$

holds true. Then

$$\|z\|_\infty\le C\|R\|_\infty.$$

Proof. From (4.1), we write

$$A_i z_{\bar{x},i} + B_i z_{\bar{x},i-N} + C_i z_i + D_i z_{i-N} = E_i h\lambda \sum_{j=1}^{N_0} K_{i,j} z_{j-N} - R_i\,.$$

Here, we rewrite as

$$A_i z_{\bar{x},i} + C_i z_i = G_i\,,$$

$$G_i = -B_i z_{\bar{x},i-N} - D_i z_{i-N} + E_i h\lambda \sum_{j=1}^{N_0} K_{i,j} z_{j-N} - R_i$$

and here, we have

$$|G_i| \le \|b\|_\infty |z_{i-N}| + \|b\|_\infty |z_{i-N-1}| + |\lambda| h \sum_{j=1}^{N_0} |K_{i,j}||z_{j-N}| + \|R\|_\infty\,.$$

Considering Lemma 4.2, we can write

$$\|z\|_\infty \le 2\alpha^{-1}\|b\|_\infty \|z\|_\infty + \alpha^{-1}|\lambda|\|z\|_\infty \max_{1\le i\le N_0} h\sum_{j=1}^{N_0} |K_{i,j}| + \alpha^{-1}\|R\|_\infty,$$

which completes the proof. □

Now, we give the main convergence result.

Theorem 4.4. *Let u be the solution of* (1.1)-(1.2) *and y be the solution* (3.11)-(3.12). *Then*

$$\|y-u\|_{\infty,\omega_{N_0}} \le Ch\,.$$

Proof. This follows immediately by combining the previous lemmas. □

5. Numerical Results

In this section, we present numerical results obtained by applying the new numerical scheme (3.11)-(3.12) to the particular problem. Also, we present numerical results obtained by using implicit Euler method in (3.13)-(3.14).

First, we suggest the following iterative technique for solving problem (3.11)-(3.12). So, if we reformulate (3.11) then we can write

$$A_i y_{\bar{x},i}^{(n)} + B_i y_{\bar{x},i-N}^{(n)} + C_i y_i^{(n)} + D_i y_{i-N}^{(n)} = F_i + E_i\lambda \sum_{j=1}^{N_0} K_{ij} y_{j-N}^{(n-1)}, \quad 1 \le i \le N_0 - 1, \tag{5.1}$$

$$y_i^{(n)} = \varphi_i, \quad -N \le i \le 0, \tag{5.2}$$

$n = 1, 2, \ldots,$ $y_i^{(0)}$ $(1 \le i \le N_0 - 1)$ are given and stopping criterion is

$$\max_i |y_i^{(n)} - y_i^{(n-1)}| \le 10^{-5}.$$

Latter, for the iterative error $z_i^{(n)} = y_i^{(n)} - y_i$ from (3.11)-(3.12) and (5.1)-(5.2), we have

$$A_i z_{\bar{x},i}^{(n)} + B_i z_{\bar{x},i-N}^{(n)} + C_i z_i^{(n)} + D_i z_{i-N}^{(n)} = E_i\lambda \sum_{j=1}^{N_0} K_{ij} z_{j-N}^{(n-1)}, \quad 1 \le i \le N_0 - 1,$$

$$z_i^{(n)} = 0, \quad -N \le i \le 0.$$

According to maximum principle

$$\|z^{(n)}\|_\infty \le |\lambda| h \sum_{j=1}^{N_0} |K_{ij}||z_j^{(n-1)}|$$
$$\le q\|z^{(n-1)}\|_\infty$$

with

$$q = |\lambda| h \max_{0\le i\le N_0} \sum_{j=1}^{N_0} |K_{ij}|.$$

For $|\lambda| < 1/\left(h \max_{1\le i\le N_0} \sum_{j=1}^{N_0} |K_{ij}|\right)$ the iterative process obviously convergent.

Example 5.1. Now, we consider the test problem:

$$u'(x) + 4u(x) + u(x-1) = \frac{3}{4}e^{(x-1)} + \frac{1}{4}\int_0^2 e^{x-t}u(t-1)dt, \quad 0 < x \le 2,$$
$$u(x) = e^x, \quad -1 \le x \le 0$$

the exact solution of the problem is given by

$$u(x) = \begin{cases} d_1 e^{-4x} + d_2 e^{x-1}, & 0 < x \le 1, \\ [d_3 + (1-x)d_1]e^{-4(x-1)} + d_4 e^{x-1}, & 1 < x \le 2, \end{cases}$$

where

$$d_1 = 1 - \frac{e^{-1} - e^{-6}}{100 - 4e^{-1} - e^{-6}},$$
$$d_2 = \frac{1 - e^{-5}}{100 - 5e^{-1}} d_1,$$
$$d_3 = \left[e^{-4} - \frac{1}{100}(1 - e^{-5})\right] d_1 + \left(1 + \frac{3e^{-1}}{20}\right) d_2 - \frac{1}{5},$$
$$d_4 = \frac{1}{100}(1 - e^{-5}) d_1 - \frac{3e^{-1}}{20} d_2 + \frac{1}{5}.$$

We define the exact error e_i^N and the computed maximum pointwise error e^N for any N as follows:

$$e_i^N = |y_i - u_i|, \quad e^N = \max_{0\le i\le N} e_i^N,$$

where y_i is the numerical approximation to exact value u_i for the nodes x_i. The computational results of the test problem obtained by using both methods are presented in Tables 1-3. Furthermore, the maximum pointwise errors by means of the corresponding numbers e^N obtained by taking $y_i^{(0)} = 1 - e^{-x_i}$ for the test problem are listed in Table 3.

Table 1. The numerical results for the test problem on $(0,2]$ (PM)

x_i	u_i	y_i $(N=64)$	e_i^{64}	y_i $(N=128)$	e_i^{128}
0.125	0.6084837	0.6085000	1.628E-5	0.6084920	8.282E-6
0.250	0.3712771	0.3713054	2.832E-5	0.3712915	1.441E-5
0.375	0.2276987	0.2277368	3.808E-5	0.2277181	1.937E-5
0.500	0.1409479	0.1409947	4.678E-5	0.1409717	2.380E-5
0.625	0.0887093	0.0887645	5.521E-5	0.0887374	2.809E-5
0.750	0.0574537	0.0575176	6.390E-5	0.0574862	3.251E-5
0.875	0.0389820	0.0390553	7.322E-5	0.0390193	3.725E-5
1.000	0.0283289	0.0284124	8.346E-5	0.0283714	4.246E-5
1.125	0.0518891	0.0519606	7.150E-5	0.0519313	4.219E-5
1.250	0.1105784	0.1106571	7.874E-5	0.1106252	4.682E-5
1.375	0.1808347	0.1809297	9.495E-5	0.1808888	5.404E-5
1.500	0.2532287	0.2533440	1.154E-4	0.2532915	6.280E-5
1.625	0.3251265	0.3252646	1.381E-4	0.3251992	7.269E-5
1.750	0.3969579	0.3971204	1.625E-4	0.3970415	8.364E-5
1.875	0.4703886	0.4705773	1.888E-4	0.4704843	9.577E-5
2.000	0.5474807	0.5476979	2.173E-4	0.5475899	1.093E-4

Table 2. The numerical results for the test problem on $(0,2]$ (EM)

x_i	u_i	y_i $(N=64)$	e_i^{64}	y_i $(N=128)$	e_i^{128}
0.125	0.6084837	0.6175932	9.109E-3	0.6131136	4.630E-3
0.250	0.3712771	0.3823978	1.112E-2	0.3769083	5.631E-3
0.375	0.2276987	0.2378741	1.018E-2	0.2328320	5.133E-3
0.500	0.1409479	0.1492148	8.267E-3	0.1451027	4.155E-3
0.625	0.0887093	0.0949942	6.285E-3	0.0918559	3.147E-3
0.750	0.0574537	0.0620265	4.573E-3	0.0597342	2.281E-3
0.875	0.0389820	0.0421995	3.217E-3	0.0405800	1.599E-3
1.000	0.0283289	0.0305259	2.197E-3	0.0294153	1.086E-3
1.125	0.0518891	0.0550450	3.156E-3	0.0534830	1.594E-3
1.250	0.1105784	0.1123125	1.734E-3	0.1114465	8.681E-4
1.375	0.1808347	0.1810065	1.718E-4	0.1809065	7.179E-5
1.500	0.2532287	0.2523515	8.772E-4	0.2527695	4.592E-4
1.625	0.3251265	0.3237562	1.370E-3	0.3244212	7.053E-4
1.750	0.3969579	0.3955066	1.451E-3	0.3962162	7.417E-4
1.875	0.4703886	0.4691072	1.281E-3	0.4697369	6.517E-4
2.000	0.5474807	0.5464950	9.857E-4	0.5469813	4.994E-4

Table 3. Comparison of E^N both methods on $(0,2]$

N	E^N (EM)	E^N (PM)	N	E^N (EM)	E^N (PM)
32	2.170E-2	4.195E-4	256	2.834E-3	5.350E-5
64	1.112E-2	2.173E-4	512	1.421E-3	2.519E-5
128	5.631E-3	1.093E-4	1024	7.115E-4	1.092E-5

6. Conclusion

In this paper, we proposed a difference scheme using the finite difference method for solving a linear first order Fredholm delay integro-differential equation. This method was based on a fitted difference scheme on an equidistant mesh on each time subinterval. By applying the method, first order convergence in the discrete maximum norm was obtained. A numerical example was solved not only by the presented method but also using the implicit Euler method. Moreover, the computational results for $N = 64, 128$ were displayed in the Tables 1-2 and the maximum pointwise errors for various N obtained by both methods were shown in the Table 3. From the results in these Tables, we conclude that the proposed approach more effective than the other approach. The ideas presented approach here can be used for the study of initial or boundary value problems for Fredholm integro-differential equations with delay as well as neutral type.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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