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Research Article

A SEIS Criss-Cross Model for Online Social Networks Communities

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Abstract. Social Network has become important part of our daily life. An average number of people spend a lot of time of their daily life on social network. People belonging to different classes, communities or groups share their opinion or thoughts over a particular issue through social network. These ideologies supporting the arguments either join or divide people among different group on social network. The thoughts or opinions of different people belonging to different classes or communities create a negative environment among them and this resulting in social network attack from one group over the other. Consequently, it creates a criss-cross like environment over the social network and raises an idea of developing criss-cross epidemic model in order to minimize or restrict the epidemic outbreak. In this paper we have proposed a criss-cross epidemic model for attacks on online social networks. We drive the expression for Reproduction Number for given model and analyzed stability of equilibrium point in term of reproduction number. Also establish the Global stability of the model at endemic equilibrium. Finally, numerical simulation of given model using matlab has been done.

Keywords. Online social network; SEIS criss-cross model; Stability analysis; Numerical analysis

Mathematics Subject Classification (2020). 34D99; 34K99

1. Introduction

Right through the beginning people are establishing communication with one another from different means and modes. The means and modes for communication have kept changing over

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the period of time. As we are in the 21st century now, internet has become the most popular source of establishing communication. People are interacting over internet in different ways. Social network has become one of the most popular and convenient means for this. Now days these are using not only for establishing communication but also has become a platform from where we can receive information of different sphere. A social network can be broadly defined as a set of actors and the set of ties representing some relationship or lack of relationship [1]. The users of the social network get connected for different purpose. People get connected through social network for friendship, affiliation, fincial exchanges, trading relationship, information exchange, gathering knowledge, to know the environment etc. It includes various online techniques such as twitter, facebook, watsapp, messenger, zoom, blog etc. All these technologies are use for different purposes and have different sets of users. Some of the technology use for gathering information, making relationship thou some are use for commercial and official purposes. It is worth in saying that in the web age; the social network has occupied a significant place and has become integral part of our day to day life. People are having great assistance in their life coming from the network side. It is worth mentioning that a social network technology has made the world virtually coincide. People can find there ones across the globe and share their information, feeling, knowledge, practices through these.

Of course the user of the social network belongs to different communities, region and group having different idea or opinion over the particular issues. Here off this situation divides the users in several sets of distinct view and it creates division among them. The dissent or the dissatisfaction of one community over the other can be feeling easily on these plate forms and consequently they started to harm one another by attacking on their social network sites. As by attacking on the social network of particular community or group can result in damage to vulnerable data or stealing of valuable information. It becomes important to guard the integrity and security of such data and information. It evolves the idea of developing mathematical model to reduces or minimize the social network attacks which results in an ideal and harmonious social network environment.

Related Work

The spread of a biological virus in biological network and worm propagation in social network encourages researchers to adopt an epidemic model for social network attacks [5]. The study of epidemic models for social network helps us to know about dynamics of epidemics on social networks and in decision making when condition of epidemic outbreaks arises [11]. Recently modeling of the epidemics on social contact networks has been studied. The Kermack and McKendrick SIR classical epidemic model is widely studied and used [6–8]. The two population model of human and mosquito for malaria helps us to understand the spreading of infection between social communities [10]. The various proposed dynamical model providing that behavior and various characteristics of network depends upon parameters [9,12]. A important parameter is the basic reproductive number, which defined as the expected number of secondary infection caused by a single infective [3]. The rest of this paper is organized as follows: Section 2 Model formulation, Existence and positivity of solution. Section 3 explains the basic reproduction number and stability of Infection free equilibrium. Section 4 and 5 describe the global stability of the endemic equilibrium point and numerical simulation of theoretical result. Finally, Section 6 contains conclusion of the paper.

2. Model Formulation

We consider the transmission of infection in two communities of social network. We subdivide the users in each social network communities, into Susceptible — $S_I(t)$, Exposed — $E_I(t)$, Infectious — $I_I(t)$ classes, such that $S_I(t)+E_I(t)+I_I(t)=1$. Any users belong to particular communities is considered as susceptible. The followers or friends of susceptible users suppose to be exposed users. The user who posts a new message or reacts on that post is considered as infectious users in online social networks. When these exposed users react on a particular message become infectious. After some time, the infectious users do not react further on any particular post or comment and again become susceptible and moves to the susceptible class. Every class of users can leave the network. The infected users can be blocked due to their infectious behavior. The compartmental model which shows the mode of transmission infection between the two communities is depicted in Fgure 1

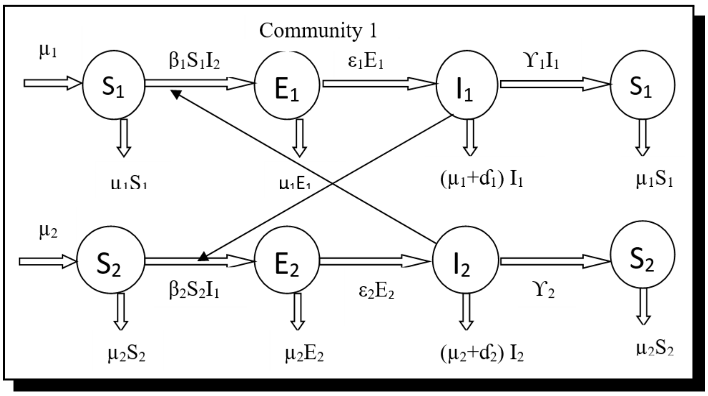

Figure 1. SEIS criss-cross model for social network

The differential equation based on transmission of infection in communities is given below:

$$\frac{dS_1}{dt}=\mu_1-\beta_1 S_1 I_2-\mu_1 S_1+\gamma_1 I_1, \tag{2.1}$$

$$\frac{dE_1}{dt}=\beta_1 S_1 I_2-(\mu_1+\varepsilon_1)E_1, \tag{2.2}$$

$$\frac{dI_1}{dt}=\varepsilon_1 E_1-(\mu_1+\gamma_1+d_1)E_1, \tag{2.3}$$

$$\frac{dS_2}{dt}=\mu_2-\beta_2 S_2 I_1-\mu_2 S_2+\gamma_2 I_2, \tag{2.4}$$

$$\frac{dE_2}{dt}=\beta_2 S_2 I_1-(\mu_2+\varepsilon_2)E_2, \tag{2.5}$$

$$\frac{dI_2}{dt}=\varepsilon_2 E_2-(\mu_2+\gamma_2+d_2)E_2, \tag{2.6}$$

where $S_1+E_1+I_1=S_2+E_2+I_2=1$, with initial condition

$$S_1(0)=S_{01},\ E_1(0)=E_{01},\ I_1(0)=I_{01};\ S_2(0)=S_{02},\ E_2(0)=E_{02},\ I_2(0)=I_{02}. \tag{2.7}$$

The state variables and parameters used for the model are described below:

μ_i: Rate of users Joing and leaving the network.

β_i: Rate of infection ith community.

δ_i: Rate of blocking due to infection.

ε_i: Per capital rate of progression from the exposed state to the infectious state.

γ_I: Per capital recovery rate from the infectious state to the susceptible state.

Existence and Positivity of Solution

Theorem 2.1. *There exists domain D in which the solution set $(S_1,E_1,I_1,S_2,E_2,I_2)$ is contained and bounded and given by $D=(d_1\times d_2)$ is subset of $(R_+^3\times R_+^3)$, where $d_1=\{(S_1,E_1,I_1)\in R_+^3:(S_1,E_1,I_1)\le 1\}$; $D_2=\{(S_2,E_2,I_2)\in R_+^3:(S_2,E_2,I_2)\le 1\}$.*

Proof. Here the solution set is $(S_1,E_1,I_1,S_2,E_2,I_2)$ with initial condition (2.7).
Let $U_1(S_1,E_1,I_1)=S_1(t)+E_1(t)+I_1(t)$ and $U_2(S_2,E_2,I_2)=S_2(t)+E_2(t)+I_2(t)$ then

$$\frac{dU_1}{dt}=\frac{\partial U_1}{\partial S_1}\frac{dS_1}{dt}+\frac{\partial U_1}{\partial E_1}\frac{dE_1}{dt}+\frac{\partial U_1}{\partial I_1}\frac{dI_1}{dt},$$
$$\frac{dU_2}{dt}=\frac{\partial U_2}{\partial S_2}\frac{dS_2}{dt}+\frac{\partial U_2}{\partial E_2}\frac{dE_2}{dt}+\frac{\partial U_2}{\partial I_2}\frac{dI_2}{dt}.$$

It follows from (2.1)-(2.6) that

$$\frac{dU_1}{dt}\le \mu_1-\mu_1U_1 \quad\text{and}\quad \frac{dU_2}{dt}\le \mu_2-\mu_2U_2.$$

On solving above differential equation yields,

$$U_1\le(1-E^{-\mu_1t})+U_1E^{-\mu_1t} \quad\text{and}\quad U_2\le(1-E^{-\mu_2t})+U_2E^{-\mu_2t}$$

as $t\to\infty$ gives $U_1\le 1$ and $U_2\le 1$.
Thus all solution of community 1 and community 2 exists in feasible region D_1 and D_2.
Thus feasible region for model (2.1)-(2.6) exists and given by

$$D=\{(S_1,E_1,I_1,S_2,E_2,I_2)\in R_+^6:(S_1+E_1+I_1)\le 1 \text{ and } (S_2+E_2+I_2)\le 1\}.$$ □

Theorem 2.2. *The solution $(S_1,E_1,I_1,S_2,E_2,I_2)$ of the model (2.1)-(2.6) with non-negative initial condition (2.7) in the feasible region D, remains non-negative in D for all $t\ge 1$.*

Proof. From (2.1), $S_1(t)>0$ for all $t\ge 0$. If not let $t^*>0$, such that $S_1(t^*)=0$ and $S_1'(t*)\le 0$ and

$$(S_1,E_1,I_1,S_2,E_2,I_2)>0, \quad \text{for all } 0<t<t^*.$$

Then from (2.1), we have

$$\begin{aligned}S_1'(t^*)&=\mu_1-\beta_1S_1(t^*)I_2(t^*)-\mu_1S_1(t^*)+\gamma_1I_1(t^*)\\&=\mu_1+\gamma_1I_1(t^*)>0.\end{aligned} \tag{2.8}$$

Therefore, $S_1(t^*)>0$, which is contradictioin. Hence, $S_1(t)>0$, for all $t\ge 0$.

Again, assume that

$$t^* = \sup\{t > 0 : S_1, E_1, I_1, S_2, E_2, I_2\} > 0,$$

then from system (2.2) we have

$$\frac{d\{E_1 E^{(\mu_1+\varepsilon_1)t}\}}{dt} = \beta_1 S_1 I_2 E^{(\mu_1+\varepsilon_1)t}. \tag{2.9}$$

On integrating from 0 to t^*,

$$E_1(t^*)E^{(\mu_1+\varepsilon_1)t^*} - E_1(0) = \int_0^{t^*} \beta_1 S_1 I_2 E^{(\mu_1+\varepsilon_1)t} dt. \tag{2.10}$$

Multiplying by $E^{-(\mu_1+\varepsilon_1)t^*}$ both sides

$$E_1(t^*) = E_1(0)E^{-(\mu_1+\varepsilon_1)t^*} + E^{-(\mu_1+\varepsilon_1)t^*} \int_0^{t^*} \beta_1 S_1 I_2 E^{(\mu_1+\varepsilon_1)t} dt \geq 0.$$

Hence $E_1(t) \geq 0$ for all $t \geq 0$.
Similarly for I_1, from (2.3), we have

$$\frac{dI_1 E^{(\mu_1+\gamma_1+\delta_1)t}}{dt} = \varepsilon_1 E_1 E^{(\mu_1+\gamma_1+\delta_1)t}. \tag{2.11}$$

On integrating from 0 to t^*

$$I_1(t^*) = I_1(0)E^{-(\mu_1+\gamma_1+\delta_1)t^*} + E^{-(\mu_1+\gamma_1+\delta_1)t^*} \int_0^{t^*} \varepsilon_1 E_1 E^{(\mu_1+\gamma_1+\delta_1)t} dt \geq 0.$$

Similarly, we easily prove that $S_2 > 0$, $E_2 \geq 0$ and $I_2 \geq 0$.
Thus the solution $(S_1, E_1, I_1, S_2, E_2, I_2)$ of model (2.1)-(2.6) remain non-negative in D for all $t \geq 0$. □

3. Existence and Stability of Infection Free Equilibrium

Infection-free equilibrium is that state where system has no rumors this implies that

$$E_1 = 0,\ I_1 = 0,\ E_2 = 0,\ I_2 = 0.$$

Let infection-free equilibrium point, E_0, for model (2.1)-(2.6). Then solving model (2.1)-(2.6) we get $S_1 = 1$, $S_2 = 1$. Hence $E_0 = (1,0,0,1,0,0)$.

3.1 Basic Reproduction Number

The basic reproductive ratio, R_0, is defined as the expected number of secondary users arising from an infected user during his or her entire infectious period, in population of susceptible users [4]. Let basic reproduction number for community 1 is R_{01} and for community 2 is R_{02}.
For community 1,

$$F(x) = \begin{bmatrix} \beta_1 S_1 I_2 \\ 0 \\ 0 \end{bmatrix},$$

$$V(x) = \begin{bmatrix} (\mu_1+\varepsilon_1)E_1 \\ (\mu_1+\gamma_1+d_1)E_1 - \varepsilon_1 E_1 \\ \mu_1 S_1 - \mu_1 - \gamma_1 I_1 \end{bmatrix},$$

$$\bar{F}(x) = \begin{bmatrix} 0 & \beta_1 S_1 \\ 0 & 0 \end{bmatrix},$$

$$\bar{V}(x) = \begin{bmatrix} (\mu_1+\varepsilon_1) & 0 \\ -\varepsilon_1 & (\mu_1+\gamma_1+d_1) \end{bmatrix},$$

$$\bar{F}\bar{V}^{-1} = \frac{1}{(\mu_1+\varepsilon_1)(\mu_1+\gamma_1+d_1)} \begin{bmatrix} \beta_1 S_1 \varepsilon_1 & \beta_1 S_1(\mu_1+\varepsilon_1) \\ 0 & 0 \end{bmatrix}.$$

Now, we know that

$$R_0 = \rho(\bar{F}\bar{V}^{-1}).$$

Thus,

$$R_{01} = \frac{\beta_1 \varepsilon_1}{(\mu_1+\varepsilon_1)(\mu_1+\gamma_1+d_1)}.$$

Similarly, reproduction number for community 2

$$R_{02} = \frac{\beta_2 \varepsilon_2}{(\mu_2+\varepsilon_2)(\mu_2+\gamma_2+d_2)}.$$

Hence reproduction number for whole system

$$R_0 = \sqrt{\frac{\beta_1\beta_2\varepsilon_1\varepsilon_2}{(\mu_1+\varepsilon_1)(\mu_2+\varepsilon_2)(\mu_1+\gamma_1+d_1)(\mu_2+\gamma_2+d_2)}}.$$

Theorem 3.1. *The system* (2.1)-(2.7) *is asymptotically stable at IFE when* $R_0 < 1$ *and unstable when* $R_0 > 1$.

Proof. Analyze the stability of system with the help of reproduction number [7], Jacobean of system (2.1)-(2.7) at IFE is given by

$$J(E_0) = \begin{pmatrix} -\mu_1 & 0 & \gamma_1 & 0 & 0 & -\beta_1 \\ 0 & -(\mu_1+\varepsilon_1) & 0 & 0 & 0 & \beta_1 \\ 0 & \varepsilon_1 & -(\mu_1+\gamma_1+d_1) & 0 & 0 & 0 \\ 0 & 0 & -\beta_2 & -\mu_2 & 0 & \gamma_2 \\ 0 & 0 & \beta_2 & 0 & -(\mu_2+\varepsilon_2) & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 & -(\mu_2+\gamma_2+\delta_2) \end{pmatrix}$$

Characteristics equation of above matrix is given by

$$(\lambda+\mu_1)(\lambda+\mu_2)\{(\lambda+\mu_1+\varepsilon_1)(\lambda+\mu_1+\gamma_1+d_1)(\lambda+\mu_2+\varepsilon_2)(\lambda+\mu_2+\gamma_2+\delta_2)-\varepsilon_1\varepsilon_2\} = 0.$$

From first two factors, Eigen values are $-\mu_1$ and $-\mu_2$. Then stability of system depends on the Eigen value of rest of equation.

$$\{(\lambda+\mu_1+\varepsilon_1)(\lambda+\mu_1+\gamma_1+d_1)(\lambda+\mu_2+\varepsilon_2)(\lambda+\mu_2+\gamma_2+\delta_2)-\beta_1\beta_2\varepsilon_1\varepsilon_2\} = 0 \tag{3.1}$$

or

$$\{(\lambda+B_1)(\lambda+B_2)(\lambda+B_3)(\lambda+B_4)-\beta_1\beta_2\varepsilon_1\varepsilon_2\} = 0$$

where

$$B_1 = \mu_1+\varepsilon_1;\ B_2 = \mu_1+\gamma_1+d_1;\ B_3 = \mu_2+\varepsilon_2;\ B_4 = \mu_2+\gamma_2+\delta_2. \tag{3.2}$$

Then above equation becomes

$$A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0, \tag{3.3}$$

where

$$A_4 = 1;$$
$$A_3 = B_1 + B_2 + B_3 + B_4;$$
$$A_2 = (B_1 + B_2)(B_3 + B_4) + B_1B_2 + B_3B_4;$$
$$A_1 = (B_1 + B_2)B_3B_4 + (B_3 + B_4)B_1B_2;$$
$$A_0 = B_1B_2B_3B_4 - \beta_1\beta_2\varepsilon_1\varepsilon_2; \tag{3.4}$$
$$A_0 = B_1B_2B_3B_4(1 - R_0^2). \tag{3.5}$$

From Routh-Hurwitz criterion that all roots of a polynomial have negative real part if and only if the coefficient A_I are positive and matrices $H_I > 0$ for all $i = 0, 1, 2, 3, 4$.
From (3.2) and (3.4) A_1, A_2, A_3, A_4 are positive since B_I's are positive. Also, from (3.5) that if $R_0 < 1$ then $A_0 > 0$.
Now, the Hurwitz matrices are

$$H_1 = A_3 > 0,$$
$$H_2 = \begin{bmatrix} A_3 & A_4 \\ A_1 & A_2 \end{bmatrix} > 0,$$
$$H_3 = \begin{bmatrix} A_3 & A_4 & 0 \\ A_1 & A_2 & A_3 \\ 0 & A_0 & A_1 \end{bmatrix} > 0,$$
$$H_4 = \begin{bmatrix} A_3 & A_4 & 0 & 0 \\ A_1 & A_2 & A_3 & A_4 \\ 0 & A_0 & A_1 & A_2 \\ 0 & 0 & 0 & A_0 \end{bmatrix} > 0.$$

Therefore, all eigen value of $J(E_0)$ have negative real part when $R_0 < 1$. Thus IFE is asymptotically stable. Again if $R_0 > 1$ then $A_0 < 0$ then there exist a sign change in A_I's. So, at least one root has positive real part. Hence IFE is unstable when $R_0 > 1$.

4. Existence and Stability of Endemic Equilibrium

Let system (2.1)-(2.6) with (2.7) has endemic equilibrium $(S_1^*, E_1^*, I_1^*, S_2^*, E_2^*, I_2^*)$. On solving (2.1)-(2.6) and express S_1^*, E_1^*, S_2^*, E_2^*, I_2^* in term of I_1^*, then

$$S_1^* = \frac{\mu_1\varepsilon_1 - (\mu_1(\mu_1 + \gamma_1 + \delta_1) + \varepsilon_1(\mu_1 + \delta_1))I_1^*}{\mu_1\varepsilon_1},$$
$$S_2^* = \frac{\mu_2 + \gamma_2 I_2^*}{\mu_2 + \beta_2 I_1^*},$$
$$E_1^* = \frac{(\mu_1 + \gamma_1 + \delta_1)I_1^*}{\varepsilon_1},$$

$$E_2^* = \frac{(\mu_1+\gamma_1+\delta_1)(\mu_2+\gamma_2+\delta_2)(\mu_1+\varepsilon_1)}{\beta_1\varepsilon_1\varepsilon_2 S_1^*},$$
$$I_2^* = \frac{(\mu_1+\gamma_1+\delta_1)(\mu_1+\varepsilon_1)}{\beta_1\varepsilon_1 S_1^*}.$$

Theorem 4.1. *The endemic equilibrium* $(S_1^*, E_1^*, I_1^*, S_2^*, E_2^*, I_2^*)$ *of system* (2.1)-(2.6) *is globally stable.*

Proof. Firstly, reduces the system (2.1)-(2.6) and then use Lapunov's function [2] for global stability

$$\sum_{i=1}^{2}\frac{dS_I}{dt} = \sum_{i=1}^{2}\mu_I - \beta_I S_I I_{2/i} - \mu_I S_I + \gamma_I I_I, \tag{4.1}$$
$$\sum_{i=1}^{2}\frac{dE_I}{dt} = \sum_{i=1}^{2}\beta_I S_I I_{2/i} - (\mu_I+\varepsilon_I)E_I, \tag{4.2}$$
$$\sum_{i=1}^{2}\frac{dI_I}{dt} = \sum_{i=1}^{2}\varepsilon_I E_I - (\mu_I+\gamma_I+\delta_I)I_i. \tag{4.3}$$

For global stability we consider Lapunov's function for above reduced system

$$L = (S_I - S_1^*\ln S_I) + A(E_I - E_1^*\ln E_I) + B(I_I - I_1^*\ln I_I), \tag{4.4}$$
$$\frac{dL}{dt} = \left(1-\frac{S_I}{S_I^*}\right)S_1' + A\left(1-\frac{E_I}{E_I^*}\right)E_I' + B\left(1-\frac{{I_I}^*}{I_I}\right)I_i'. \tag{4.5}$$

Putting the value of S_1', E_I', I_I' from above reduced system

$$\frac{dL}{dt} = \left(1-\frac{S_I}{S_I^*}\right)(\mu_I - \beta_I S_I I_{2/i} - \mu_I S_I + \gamma_I I_I) + A\left(1-\frac{E_I}{E_I^*}\right)(\beta_I S_I I_{2/i} - (\mu_I+\varepsilon_I)E_I)$$
$$+B\left(1-\frac{I_I}{I_I^*}\right)(\varepsilon_I E_I - (\mu_I+\gamma_I+\delta_I)I_I).$$

Again from above reduced system

$$\frac{dL}{dt} = \left(1-\frac{S_I}{S_I^*}\right)(\beta_I S_I^* I_{2/i}^* - \beta_I S_I I_{2/i} - \mu_I(S_I - S_I^*))$$
$$+A\left(1-\frac{E_I}{E_I^*}\right)\left((\mu_I+\varepsilon_I)E_I^*\frac{S_I I_{2/i}}{S_I^* I_{2/i}^*} - (\mu_I+\varepsilon_I)E_I\right) + B\left(1-\frac{I_I}{I_I^*}\right)\left(\varepsilon_I E_I - \varepsilon_I E_I^*\frac{I_I}{I_I^*}\right),$$
$$\frac{dL}{dt} = \frac{-\mu_I(S_I - S_I^*)^2}{S_I} + (\mu_I+\varepsilon_I)E_I^*\left(1-\frac{S_I I_{2/i}}{S_I^* I_{2/i}^*}\right)\left(1-\frac{S_I}{S_I^*}\right)$$
$$+A(\mu_I+\varepsilon_I)E_I^*\left(1-\frac{E_I}{E_I^*}\right)\left(\frac{S_I I_{2/i}}{S_I^* I_{2/i}^*} - \frac{E_I}{E_I^*}\right) + B\varepsilon_I E_I^*\left(1-\frac{I_I^*}{I_I}\right)\left(\frac{E_I}{E_I^*} - \frac{I_I}{I_I^*}\right),$$
$$\frac{dL}{dt} = \frac{-\mu_I(S_I - S_I^*)^2}{S_I} + (\mu_I+\varepsilon_I)E_I^* F(x,y,z,w). \tag{4.6}$$

Here $x = \frac{S_I}{S_I^*}$; $y = \frac{E_I}{E_I^*}$; $z = \frac{I_I^*}{I_I}$; $w = \frac{I_{2/i}}{I_{2/i}^*}$.

Now,

$$F(x,y,z,w) = (1-xw)\left(1-\frac{1}{x}\right) + A\left(1-\frac{1}{y}\right)(xw-y) + B\frac{\varepsilon_I}{(\mu_I+\varepsilon_I)}\left(1-\frac{1}{x}\right)(y-x)$$
$$= \left(1-\frac{1}{x}-xw+w\right) + A\left(xw-y-\frac{xw}{y}\right) + B\frac{\varepsilon_I}{(\mu_I+\varepsilon_I)}\left(y-z-\frac{y}{z}+1\right).$$

Comparing coefficient of w and y, we get $A=1$; $B=(\mu_I+\varepsilon_I)/\varepsilon_I$

$$F(x,y,z,w) = 3-\frac{1}{x}-z+w-\frac{xw}{y}-\frac{y}{z}.$$

At endemic equilibrium $I_{2/i} = I^*_{2/i}$ or $w=1$

$$F(x,y,z) = 4-\frac{1}{x}-z-\frac{x}{y}-\frac{y}{z}.$$

Thus, $F(x,y,z) \le 0$ because *Arithemetic mean* $\ge$ *Geometric mean*. Hence from (4.6) $\frac{dL}{dt} \le 0$ if $S_I = S_I^*$. From LaSalle's Extension the solutions which intersect the interior of $R_3 > 0$ limit to an invariant set contained in $\{(S_I,E_I,I_I) \in R_3 > 0 : S_I = S_I^*\}$. The only invariant set in $\{(S_I,E_I,I_I) \in R_3 > 0 : S_I = S_I^*\}$ is the set consisting of the endemic equilibrium $(S_1^*,E_1^*,I_1^*,S_2^*,E_2^*,I_2^*)$. Thus, all solutions to equation (4.1)-(4.3) which intersect the interior of $R_3 > 0$ limit to $(S_1^*,E_1^*,I_1^*,S_2^*,{}_2^*,I_2^*)$. In fact, $(S_1^*,E_1^*,I_1^*,S_2^*,E_2^*,I_2^*)$ is globally asymptotically stable for all non-negative initial conditions for which $E_I+I_I>0$. □

5. Numerical Simulation

Example 5.1. In this example we are considering the situation $R_0 < 1$ where both communities become infection free after some time. This means that both communities do not spread the infection for each other. Such a situation is an effective functioning for Social Networks (shown in Figure 2).

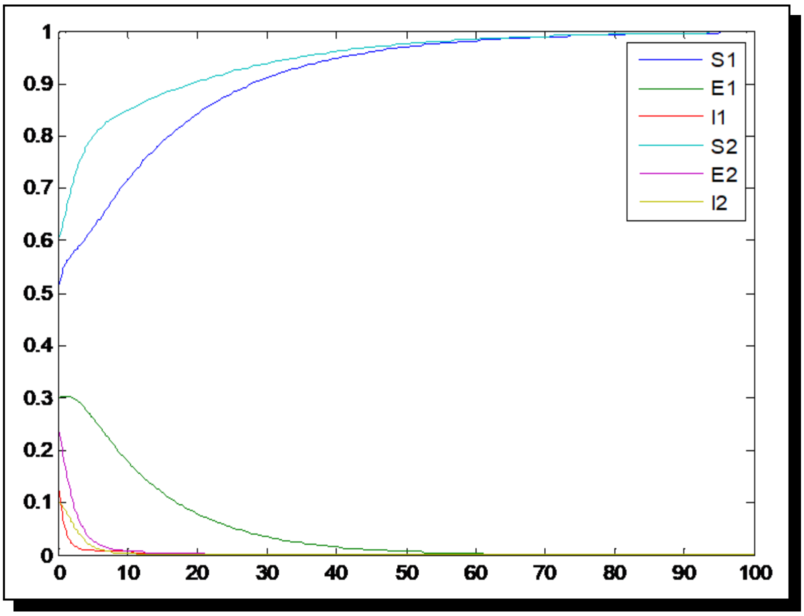

Figure 2. SEIS criss-cross model when $R < 1$

Parameters: $\mu_1 = 0.05$; $\beta_1 = 0.6$; $\delta_1 = 0.5$; $\gamma_1 = 0.7$; $\varepsilon_1 = .04$; $\mu_2 = 0.05$; $\beta_2 = 0.6$; $\delta_2 = 0.5$; $\gamma_1 = 0.7$; $\varepsilon_2 = 0.5$.

Example 5.2. In this example we are considering the opposite situation $R > 1$ of above example. In this situation susceptible population of both communities drastically decreased due to which both communities become highly infeasible against any rumor attacks. This is the condition for asymptotical stability of endemic equilibrium (shown in Figure 3).

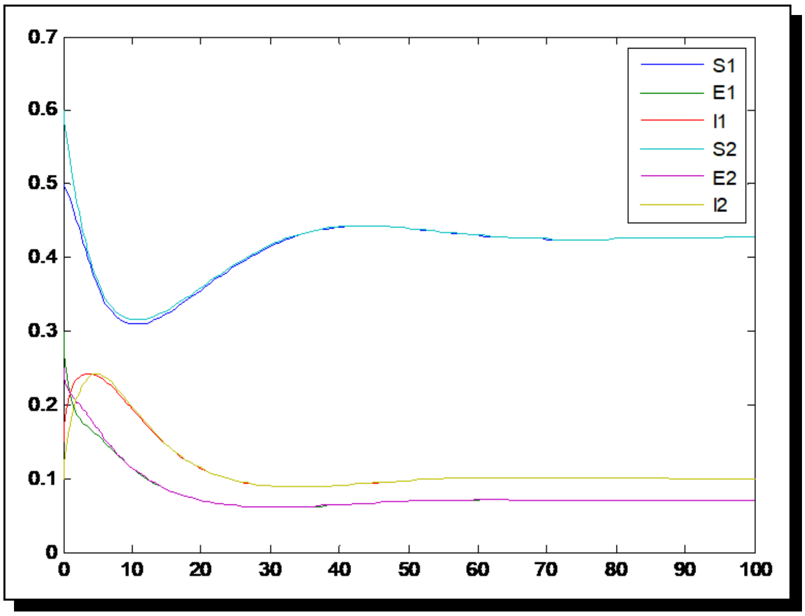

Figure 3. SEIS criss-cross model when $R > 1$

Parameters: $\mu_1 = 0.05$; $\beta_1 = 0.9$; $\delta_1 = 0.2$; $\gamma_1 = 0.1$; $\varepsilon_1 = .5$; $\mu_2 = 0.05$; $\beta_2 = 0.9$; $\delta_2 = 0.2$; $\gamma_1 = 0.1$; $\varepsilon_2 = 0.5$.

6. Conclusion

We see that how social network become an important part of our life. People belong from different community share their opinions or thoughts on any issue which divides them to several groups. The main purpose of this paper is to make people aware of community attacks and also give an idea about minimizing these risks for healthy social network environment. In this paper criss-cross epidemic model has been developed and analyzed for two different group of social network. It has been found that the solution of model exists which is non-negative and bounded in given domain. The expression of Basic Reproduction number has been developed. It is found that the system is stable at infection-free equilibrium when $R_0 < 1$ and unstable when $R_0 > 1$. The system is globally stable at endemic equilibrium. Numerical simulation validates the theoretical results for different conditions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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