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Research Article

Apply the Sturm-Liouville Problem With Green's Function to Linear System

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Abstract. In this paper, we will study the Sturm-Liouville problem, we find the eigenvalues are the solution of the Sturm-Liouville problem and the eigenfunctions are corresponding solutions. Thus, we study construction of the Green's function to solving the first order differential n-dimensional linear system, and application for Fourier series, and we show that the Green's function solution to the two- and three-dimensional Laplace and Poisson equations.

Keywords. Sturm-Liouville problem; Green's function; Differential equation; Integral operator; Laplace equation; Poisson equation; Fourier series

Mathematics Subject Classification (2020). 34B24; 34B27; 34B60

1. Introduction

In the recent literature, there is a growing interest to solve Sturm-Liouville problem to find eigenvalues and eigenfunctions corresponding linear system. The reader is referred to [3, 8, 9, 13, 15–17] for an overview of the recent work in this area. In the beginning of the 1980s, [11, 12, 14, 20] proposed a new and fruitful method (hereafter called the Green's function) for solving linear (algebraic, differential, partial differential, integral, etc.) equations. We shown that this method yields a rapid convergence of the solutions series to linear and nonlinear deterministic and stochastic equations. The main objective of this paper is to apply the Sturm-Liouville problem with Green's function to linear system. The Green's function is powerful tool of mathematical method which used in solving linear non-homogenous differential equation (ordinary and partial).

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In this paper, we introduction to ordinary and partial differential equations acquaints with equations describing the more important theories of classical physics [5], and introduce some of the standard ways for solving those differential equations which have been derived: Eigenfunctions, Fourier series and integrals [25], Green's theorem, particular solutions in coordinates, asymptotic expansions, change of variables, conformal mapping, singularities, and transition to integral equations [10].

Sturm-Liouville Problems

In this paper, we will study the Sturm-Liouville problem, a differential equation of the form

$$-\frac{d}{dx}(p(x)du\,dx)+q(x)u=\lambda u \quad \text{with } u(a)=u(b)=0,$$

where p and q are given functions on the interval $[a,b]$. The values of λ for which the problem has a non-trivial solution are called eigenvalues of the Sturm-Liouville problem and the corresponding solutions u are called eigenfunctions. An eigenvalue is called simple, if the corresponding eigenspace is one-dimensional. The main conclusion of this paper is the following theorem:

Theorem 1.1. *If $p\in C^1[a,b]$, $q\in C^0[a,b]$, $p(x)>0$ and $q(x)\geq 0$ for all $x\in[a,b]$, then*

(i) *eigenvalues of the Sturm-Liouville problem are all simple,*

(ii) *they form an unbounded monotone sequence,*

(iii) *eigenfunctions of the Sturm-Liouville problem form an orthonormal basis in $L^2(a,b)$.*

Proof. Since $A:L^2(a,b)\to L^2(a,b)$ is compact and self-adjoint. If u is an eigenfunction of A, then Lemma 3.1 implies that $u\in C^2[a,b]$ and $u(a)=u(b)=0$. Moreover, Lemma 3.3 and $Au=\mu u$ with $\mu\neq 0$ imply that $Lu=\lambda u$ with $\lambda=\mu^{-1}$. Consequently, u is also an eigenfunction of the Sturm-Liouville problem.

Then, suppose that u is an eigenfunction of the Sturm-Liouville problem and $\widetilde{u}$ is another eigenfunction which corresponds to the same eigenvalue. Both eigenfunctions satisfy the linear ordinary differential equation $L(u)=\lambda u$ and $u(a)=\widetilde{u}(a)=0$. Then $\widetilde{u}(x)=u(x)\widetilde{u}'(a)/u'(a)$. Thus, the eigenspace is one-dimensional.

For a function $u\in C^2[a,b]$ we define

$$L(u)=-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u\,.$$

Let $L_0:D_0\to C^0[a,b]$ be the restriction of L onto the space

$$D_0:=\{u\in C^2[a,b]:u(a)=u(b)=0\}\,.$$

We can equip both D_0 and $C^0[a,b]$ with the L^2 norm. Integrating by parts, we can check that L_0 is symmetric, i.e., $(u,L_0v)=(L_0u,v)$ for all $u,v\in D_0$. On the other hand, considering L_0 on the sequence $u_n=n^{-1}\sin(\pi n(x-a)/(b-a))$, we can check that L_0 is not bounded. We have not studied unbounded operators.

Eigenfunctions of the Sturm-Liouville problem are eigenvectors of L_0. The Sturm-Liouville theorem implies that eigenfunctions of A form an orthonormal basis in $L^2(a,b)$. Moreover, we will see that all eigenfunctions of A belong to D_0 and, consequently, L_0 have the same eigenfunctions as A.

2. Construction of the Green's Function

Thus, we are interested in solving the following first order differential n-dimensional linear system

$$x'(t) = A(t)x(t) + f(t), \quad t \in J = [a,b] \tag{2.1}$$

together with the two-point boundary value conditions

$$Bx(a) + Cx(b) = h. \tag{2.2}$$

Here, n is a positive integer, $a,b \in \mathbb{R}$, $a < b$, $A : J \to M_{n\times n}$ is a $L^1(J,M_{n\times n})$ function, $f : J \to \mathbb{R}_n$ belongs to $L^1(J,\mathbb{R}_n)$, $B,C \in M_{n\times n}$ and $h \in \mathbb{R}_n$ have constants coefficients, and $x : J \to \mathbb{R}_n$ belongs to the set $\wp(J,\mathbb{R}_n)$. As usual, we denote by $L^1(J,\mathbb{R}_n)$ and $L^1(J,M_{n\times n})$ the set of all Lebesgue integrable functions on J and by $\wp(J,\mathbb{R}_n)$ the set of absolutely continuous functions on J.

Now, we study the structure of the set of solutions of the homogeneous problem ($f \equiv 0, h \equiv 0$).

$$x'(t) = A(t)x(t),\ t \in J,\ Bx(a) + Cx(b) = 0. \tag{2.3}$$

Let $W = \{x \in \wp(J,\mathbb{R}_n); Bx(a) + Cx(b) = 0\}$ and define the linear operator

$$L : x \in W \to Lx = x' - Ax \in L^1(J,\mathbb{R}_n). \tag{2.4}$$

As a consequence, the set of solutions of equation (2.3) coincides with the kernel of operator L. So, we have that the set of solutions of equation (2.3) is a linear space of dimension $k \le n$.

Theorem 2.1. *x is a solution of equation* (2.1)-(2.2) *if and only if $x = y + p$, where y is a solution of the homogeneous equation* (2.3) *and p is a solution of* (2.1)-(2.2).

Proof. Let y be a solution of (2.3) and p be a solution of equation (2.1)-(2.2). As a consequence

$$y'(t) + p'(t) = A(t)y(t) + A(t)p(t) + f(t) = A(t)(y(t) + p(t)) + f(t)$$

and $x \equiv y + p$ fulfills equation (2.1) on J.

On the other hand,

$$B(y+p)(a) + C(y+p)(b) = Bp(a) + Cp(b) = h$$

and x is a solution of (2.1)-(2.2).

Consider, now x_1 and x_2, two solutions of equation (2.1)-(2.2). As a consequence

$$x_1'(t) - x_2'(t) = A(t)(x_1(t) - x_2(t)), \quad \text{for all } t \in J$$

and

$$B(x_1(a) - x_2(a)) + C(x_2(b) - x_1(b)) = h - h = 0.$$

That is, the difference of two solutions of equation (2.1)-(2.2) is a solution of the homogeneous equation (2.3).

Let $\varphi : J \to M_{n\times n}$ be a fundamental matrix related to equation (2.3), then the solution of the linear equation:

$$\varphi'(t) = A(t)\varphi(t), \quad t \in J. \tag{2.5}$$

We arrive at the following existence and uniqueness result for equation (2.1)-(2.2).

Theorem 2.2. *Equations* (2.1)-(2.2) *have a unique solution $x \in \wp(J,\mathbb{R}_n)$ if and only if*

$$\det(M_\varphi) \ne 0 \tag{2.6}$$

with φ *any fundamental matrix of system* (2.3) *and* $M_\varphi \equiv B\varphi(a)+C\varphi(b)$.

Proof. From the variation of constants formula [18, Corollary 2.1], we have that $x \in \wp(J,\mathbb{R}_n)$ is a solution of equation (2.1) if and only if there exists $\lambda \in \mathbb{R}_n$ such that

$$x(t)=\varphi(t)\lambda+\varphi(t)\int_a^t \varphi^{-1}(s)f(s)ds, \quad t \in J. \tag{2.7}$$

Obviously, function x satisfies the boundary value condition (2.2) if and only if λ solves the following algebraic equation

$$M_\varphi\lambda \equiv (B\varphi(a)+C\varphi(b))\lambda = h-C\varphi(b)\int_a^b \varphi^{-1}(s)f(s)ds. \tag{2.8}$$

It is clear that this equation has a unique solution if and only if matrix M_φ is invertible. □

Remark 2.1. Notice that when condition (2.6) holds, the expression of the unique solution of equation (2.1)-(2.2) is given by:

$$x(t)=\varphi(t)M_\varphi^{-1}\left(h-C\varphi(b)\int_a^b \varphi^{-1}(s)f(x)ds\right)+\varphi(t)\int_a^t \varphi^{-1}(s)f(s)ds.$$

Remark 2.2. It is important to remark that, to ensure the uniqueness of solution of equation (2.1)-(2.2) for any f in $L^1(J,\mathbb{R}_n)$ and $h \in \mathbb{R}_n$, the involved boundary conditions (2.2) must define n linearly independent conditions. Thus, we obtain the following necessary condition

$$\operatorname{rank}(B \mid C)=n. \tag{2.9}$$

Having in mind the previous remark, we are interested in obtaining a characterization of the uniqueness of solutions for equation (2.1)-(2.2) that involves condition (2.9). To this end, we must take into account that the general solution of the differential equation (2.1) is given by (2.7), or, alternatively, by

$$x(t)=\varphi(t)\lambda+\varphi(t)\int_{t_0}^t \varphi^{-1}(s)f(s)ds, \quad \lambda \in \mathbb{R}_n, \tag{2.10}$$

where $t_0 \in J$ can be chosen as we please.

For later purposes, it will be convenient to fix $t_0 \in (a,b)$, and then the solution x given by (10) is a solution of (2.1)-(2.2) if and only if $\lambda \in \mathbb{R}_n$ solves the algebraic system

$$M_\varphi\lambda = h-B\varphi(a)\int_{t_0}^a \varphi^{-1}(s)f(s)ds-C\varphi(b)\int_{t_0}^b \varphi^{-1}(s)f(s)ds \tag{2.11}$$

where M_φ is given in Theorem 2.2.

Next, we present the following characterization of the uniqueness of solutions of equations (2.1)-(2.2) by means of the condition (2.9).

Notice that, considering $\chi_{(0,t)}$, the indicator function in $(0,t)$, equation (2.8) can be rewritten as follows:

$$x(t)=\varphi(t)M_\varphi^{-1}\left(h-C\varphi(b)\int_a^b \varphi^{-1}(s)f(s)ds\right)+\varphi(t)\int_a^b \varphi^{-1}(s)\chi(0,t)(s)f(s)ds,$$

or, which is the same,

$$x(t)=\int_a^b G(t,s)f(s)ds+\varphi(t)M_\varphi^{-1}h \tag{2.12}$$

with

$$G(t,s)=\begin{cases}-\varphi(t)M_{\varphi}^{-1}C\varphi(b)\varphi^{-1}(s)+,\varphi(t)\varphi^{-1}(s) & \text{if } a\le s<t\le b,\\ -\varphi(t)M_{\varphi}^{-1}C\varphi(b)\varphi^{-1}(s), & \text{if } a\le s<t\le b.\end{cases} \tag{2.13}$$

The function $G:(J\times J)\backslash\{(t,t),t\in J\}\to M_{n\times n}$ is called the Green's function related to problem (2.3). □

3. Differential Equation $Lu=f$

Lemma 3.1. *If both u_1 and u_2 satisfy the equation $Lu=0$, i.e.*

$$-(pu')^t+qu=0 \tag{3.1}$$

then

$$W_p(u_1,u_2)=p(u_1'u_2-u_1u_2')$$

is constant. Moreover, if $W_p(u_1,u_2)\neq 0$ then u_1 and u_2 are linearly independent.

Proof. Differentiating W_p with respect to x and using $pu''=-p'u'+qu$ we obtain

$$\begin{aligned}W_p'&=p'(u_1'u_2-u_1u_2')+p(u_1''u_2-u_1u_2'')\\&=p'(u_1'u_2-u_1u_2')+((-p'u_1'+qu_1)u_2-(-p'u_{t2}+qu_2)u_1)=0.\end{aligned}$$

Therefore W_p is constant.

Suppose u_1 and u_2 are linearly dependent, then there are constants α_1, α_2 such that $\alpha_1u_1+\alpha_2u_2=0$ and at least one of the constants does not vanish. Suppose $\alpha_2\neq 0$ (otherwise swap u_1 and u_2). Then $u_2=-\alpha_1u_1/\alpha_2$ and $u_2'=-\alpha_1u_1'/\alpha_2$. Substituting these equalities into $W_p(u_1,u_2)$ we see that $W_p(u_1,u_2)=0$. Therefore, $W_p(u_1,u_2)\neq 0$ implies that u_1,u_2 are linearly independent. □

Lemma 3.2. *The equation* (3.1) *has two linearly independent solutions, $u_1,u_2\in C^2[a,b]$ such that $u_1(a)=u_2(b)=0$.*

Proof. Let u_1, u_2 be solutions of the Cauchy problems

$$\begin{aligned}&-(pu_1')'+qu_1=0,\quad u_1(a)=0,\ u_1'(a)=1,\\&-(pu_2')'+qu_2=0,\quad u_2(b)=0,\ u_2'(b)=1.\end{aligned}$$

According to the theory of linear ordinary differential equations u_1 and u_2 exist, belong to $C^2[a,b]$ and are unique.

Moreover, u_1 and u_2 are linearly independent. Indeed, suppose $Lu=0$ for some $u\in C^2[a,b]$ and $u(a)=u(b)=0$. Then

$$\begin{aligned}0=(Lu,u)&=-\int_a^b(pu')'u+qu^2dx\quad\text{(using definition of } L)\\&=p(x)u'(x)u(x)\big|_a^b+\int_a^b p(u')^2+qu^2dx\quad\text{(using integration by parts)}\\&=\int_a^b p(u')^2+qu^2dx.\end{aligned}$$

Since $p>0$ on $[a,b]$, we conclude that $u'\equiv 0$. Then $u(a)=u(b)=0$ implies $u(x)=0$ for all $x\in[a,b]$.

As $u_2(b)=0$ and u_2 is not identically zero, $u_2(a)\neq 0$ and thus

$$W_p(u_1,u_2)=p(a)\big(u_1'(a)u_2(a)-u_1(a)u_2'(a)\big)=p(a)u_1'(a)u_2(a)\neq 0.$$

Therefore, u_1,u_2 are linearly independent by Lemma 3.1. □

Lemma 3.3. *If u_1 and u_2 are linearly independent solutions of the equation $Lu=0$ such that $u_1(a)=u_2(b)=0$ and*

$$G(x,y)=\frac{1}{W_p(u_1,u_2)}\begin{cases}u_1(x)u_2(y), & a\le x<y\le b,\\ u_1(y)u_2(x), & a\le y\le x\le b,\end{cases}$$

then for any $f\in C^0[a,b]$ the function

$$u(x)=\int_a^b G(x,y)f(y)dy$$

belongs to $C^2[a,b]$ satisfies the equation $Lu=f$ and the boundary conditions $u(a)=u(b)=0$.

Proof. The statement is proved by a direct substitution of

$$u(x)=\frac{u_2(x)}{W_p(u_1,u_2)}\int_a^x u_1(y)f(y)dy+\frac{u_1(x)}{W_p(u_1,u_2)}\int_x^b u_2(y)f(y)dy$$

into the differential equation. Moreover, $u_1(a)=u_2(b)=0$ implies $u(a)=u(b)=0$. □

4. Integral Operator

Lemma 4.1. *The operator $A:L^2(a,b)\to L^2(a,b)$ defined by*

$$(Af)(x)=\int_a^b G(x,y)f(y)dy$$

is compact and self-adjoint. Moreover, Range(A) *is dense in $L^2(a,b)$,* $\ker A=\{0\}$*, and all eigenfunctions, $Au=\mu u$, belong to $C^2[a,b]$ and satisfy $u(a)=u(b)=0$.*

Proof. Since the kernel G is continuous, the operator A is compact.

Moreover, G is real and symmetric and so A is self-adjoint. Lemma 3.3 implies the range of A contains all functions from $C^2[a,b]$ such that $u(a)=u(b)=0$. This set is dense in $L^2(a,b)$. Now, suppose $Au=0$ for some $u\in L^2[a,b]$. Then for any $v\in L^2$

$$0=(Au,v)=(u,Av),$$

which implies $u=0$ because u is orthogonal to a dense set (the range of A). Thus $\ker(A)=\{0\}$.

Finally, let $u\in L^2[a,b]$ be an eigenfunction of A, i.e., $Au=\mu u$. Since $\ker(A)=\{0\}$, $\mu\neq 0$. So we can write $u=\mu^{-1}Au$, which takes the form of the following integral equation:

$$u(x)=\mu^{-1}\int_a^b G(x,y)u(y)dy.$$

Obviously,

$$|G(x,y)u(y)|\le\|G\|_\infty|u(y)|,\quad \text{for all } x,y\in[a,b].$$

Since G is continuous, the dominated convergence theorem implies that we can swap a limit $x\to x_0$ and the integration, and thus the integral in the right-hand-side is a continuous function of x. Consequently, u is continuous. For a continuous u the integral is in $C^2[a,b]$ and satisfies the boundary conditions $u(a)=u(b)=0$ due to Lemma 3.3. Thus $u\in D_0$. Therefore, the eigenfunctions of A belong to D_0. □

Example 4.1 (An application for Fourier series)**.** Consider the Strum-Liouville problem

$$-\frac{d^2u}{dx^2} = \lambda u, \quad u(0) = u(1) = 0$$

It corresponds to the choice $p = 1$, $q = 0$. Theorem 1.1 implies that the normalized eigenfunctions of this problem form an orthonormal basis in $L^2(0,1)$. In this example, the eigenfunctions are easy to find:

$$\left\{\frac{1}{\sqrt{2}}\sin k\pi x : k \in N\right\}.$$

Consequently, any function $f \in L^2(0,1)$ can be written in the form

$$f(x) = \sum_{k=1}^{\infty} \alpha_k \sin k\pi x,$$

where

$$\alpha_k = \frac{1}{2}\int_0^1 f(x)\sin k\pi x\, dx.$$

The series converges in the L^2 norm.

5. Green's Function Solution to the Laplace and Poisson Equations

Laplace's equation 5.1. *The two- and three-dimensional Laplace and Poisson equations are given by*

$$\nabla^2 u = 0,$$
$$\nabla^2 u = -f, \tag{5.1}$$

respectively. We consider the Poisson equation first [15,20]*. The general approach is identical to that used to derive a solution to the inhomogeneous Helmholtz equation. Thus, working in three dimensions and defining the Green's function to be the solution of*

$$\nabla^2 g(\vec{r} \mid \vec{r}_0) = -\delta^3(\vec{r} - \vec{r}_0)$$

from equation (5.1) *we obtain the following result:*

$$u = \oint_s (g\nabla u - u\nabla g)\cdot \hat{n} d^2\vec{r} + \int_v g f d^3\vec{r}$$

where we have used Green's theorem to obtain the surface integral on the right-hand side. The problem now is to find the Green's function for this problem. Clearly, since the solution to the equation

$$(\vec{r}^2 + k^2)g = -\delta^3(\vec{r} - \vec{r}_0)$$

is

$$g(\vec{r} \mid \vec{r}_0, k) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}\exp(ik|\vec{r} - \vec{r}_0|)$$

we should expect the Green's function for the three-dimensional Poisson equation (*and the Laplace equation*) *to be of the form*

$$g(\vec{r} \mid \vec{r}_0) = \frac{1}{4\pi\,|\vec{r} - \vec{r}_0|}. \tag{5.2}$$

Thus, we obtain the following fundamental result:

$$V^2\left(\frac{1}{4\pi R}\right) = -\delta^3(R).$$

With homogeneous boundary conditions, the solution to the Poisson equation is

$$u(\vec{r}_0) = \frac{1}{4\pi}\int_v \frac{f(\vec{r})}{|\vec{r}-\vec{r}_0|} d^3\vec{r}.$$

In two dimensions the solution is of the same form, but with a Green's function given by

$$g(\vec{r}\mid\vec{r}_0) = \frac{1}{2\pi}\ln(|\vec{r}-\vec{r}_0|).$$

The general solution to Laplace's equation is

$$u = \oint_s (g\nabla u - u\nabla g)\cdot\hat{n}d^2\vec{r}$$

with g *given by equation* (5.2)*.*

Poisson's equation 5.1. *Poisson's equation in ID* (*Infinite Domain*) *with homogeneous BCs serves to exemplify the general case. The operator in this example is* $L = -d^2/dx^2$. *For simplicity, we take* $x_1 = 0$, $x_2 = a$. *The homogeneous solutions* Ψ *defined by* (2.2) *can be identified by inspection:*

$$\Phi_1 = x \quad \textit{and} \quad \Phi_2 = (a-x).$$

Then

$$W \equiv \Phi_1\Phi_2 - \Phi_1\Phi_2 = x(-1) - 1(a-x) = -a \neq 0.$$

Consequently g *becomes*

$$g(x\mid\xi) = -\left\{\frac{H(\zeta-x)\cdot(a-x)\cdot x}{a} + \frac{H(x-\xi)\cdot(a-x)}{a}\right\} = -x_<(a-x)_>.$$

The end-result, now reads

$$\Psi(x) = \frac{1}{a}\left\{x\int_x^a f(\xi)(a-\xi)d\xi + (a-x)\int_{x0}^x f(\xi)\zeta d\xi\right\}.$$

6. Green's Function Solution to the Laplace Equations and Fourier Series

Consider the equation

$$\frac{\delta u}{\delta t} - a^2\frac{\delta^2 u}{\delta x^2} = -f(x,t) \tag{6.1}$$

subject to boundary conditions $|u(x,t)| < \infty$ as $|x| < \infty$ and initial condition.
Suppose that $G(x,t,\xi,\tau)$ be the Green's function, then

$$\frac{\delta G}{\delta t} - a^2\frac{\delta^2 G}{\delta x^2} = \delta(x-\xi)\delta(t-\tau), \quad -\infty < x,\ \xi < \infty,\ 0 < t \tag{6.2}$$

subject to the boundary condition $G(x,t,\xi,\tau) < \infty$ as $|x| < \infty$, and the initial condition $G(x,0,\xi,\tau) = 0$. Let us find $G(x,t,\xi,\tau)$.

We begin by taking the Laplace transform of (6.1) with respect to t, we have

$$sg(x,s,\xi,\tau) - g(x,0,\xi,\tau) - a^2\frac{d^2g}{dx^2} = \delta(x-\xi)e^{-sz}.$$

So

$$\frac{d^2g}{dx^2} - \frac{s}{a^2}g = \delta - (x-\xi)e^{-s\tau}, \tag{6.3}$$

where $g(x,s,\xi,\tau)$ the Laplace transform of $G(x,t,\xi,\tau)$.

Now by taking the Fourier transform of (6.3) with respect to x, so that

$$(-ik)^2\bar{G}(k,s,\xi,\tau) - \frac{s}{a^2}\bar{G}(k,s,\xi,\tau) = -\frac{e^{-ik\xi-s\tau}}{a^2},$$

$$k^2\bar{G}(k,s,\xi,\tau) + \frac{s}{a^2}\bar{G}(k,s,\xi,\tau) = \frac{e^{-ik\xi-s\tau}}{a^2},$$

where $\bar{G}(k,s,\xi,\tau)$ is Fourier transform of $g(x,s,\xi,\tau)$, now let $\frac{s}{a^2} = b^2$

$$(k^2+b^2)\bar{G}(k,s,\xi,\tau) = \frac{e^{-ik\xi-s\tau}}{a^2}. \tag{6.4}$$

To find $g(x,s,\xi,\tau)$, we use the inversion integral

$$g(x,s,\xi,\tau) = \frac{e^{-s\tau}}{2\pi a^2}\int_{-\infty}^{\infty}\left(\frac{e^{(x-\xi)}}{k^2+b^2}\right)dk. \tag{6.5}$$

Transforming (6.4) into a closed contour, we evaluate it by the residue theorem and find that

$$g(x,s,\xi,\tau) = \frac{e^{-s\tau}}{2\pi a^2}\int_{-\infty}^{\infty}\left(\frac{e^{i(x-\xi)}}{(k+ib)(k-ib)}\right)dk,$$

$$g(x,s,\xi,\tau) = \frac{e^{-s\tau}}{2\tau a^2}\sum b_i$$

at $k = \pm ib$ then

$$\sum b_i = \frac{1}{2ib}(e^{-|x-\xi|b} - e^{(x-\xi)}) = \frac{1}{2ib}e^{-|x-\xi|b}$$

therefore

$$g(x,s,\xi,\tau) = \frac{e^{-s\tau}}{2a^2b}e^{-|x-\xi|} = \frac{1}{2ib}e^{-|x-\xi|}b - sr.$$

Now substituting for $b = \frac{\sqrt{s}}{a}$, we have

$$g(x,s,\xi,\tau) = \frac{\exp\left(-|x-\xi|\sqrt{\frac{s}{a}} - sr\right)}{2a\sqrt{s}}. \tag{6.6}$$

Taking Laplace transform of (6.7) we obtain

$$G(x,s,\xi,\tau) = \frac{H(t-\tau)}{\sqrt{a\pi a^2(t-\tau)}}\exp\left(\frac{-(x-\xi)^2}{aa^2(t-\tau)}\right). \tag{6.7}$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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