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Weierstrass Representation for Minimal Surfaces into BCV-Spaces

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Abstract Bianchi-Cartan-Vranceanu spaces (BCV-spaces) are some 3-dimensional homogeneous manifolds equiped with a metric depending on 2 parameters κ and τ, and whose isometries groups are of dimension four. In this paper, we describe a Weierstrass-type representation formula for simply connected minimal surfaces immersed into BCV-spaces.

1. Introduction

The topic of Weierstrass representations for minimal surfaces has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [1] and Enneper [2] in the nineteenth century on systems inducing minimal surfaces in $\mathbb{R}^3$. There exist a great number of applications of Weierstrass representations for minimal surfaces in various domains of Mathematics, Physics, Chemistry and Biology [10].

By using the standard harmonic maps equation, Mercuri, Montaldo and Piu gave in [3] a Weierstrass-type representation formula for simply connected minimal surfaces into Riemannian manifolds and they applied the obtained general structure to the case of 3-dimensional Lie groups endowed with left invariant metrics. From this setting, they discussed then some examples of minimal surfaces both in 3-dimensional Heisenberg group $\mathbb{H}_3$ and in $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{H}^2$ is the 2-dimensional hyperbolic space.

Let κ and τ be two real numbers and $D_{\kappa,\tau}$ be the domain of $\mathbb{R}^3$ defined by

$$D_{\kappa,\tau} = \left\{(x,y,z) \in \mathbb{R}^3 \Big/ 1 + \frac{\kappa}{4}(x^2+y^2) > 0\right\}.$$

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By considering on $D_{\kappa,\tau}$ the 2-parameters family of homogeneous Riemannian metrics:

$$ds^2_{\kappa,\tau} = \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \left(dz + \tau \frac{ydx - xdy}{1 + \frac{\kappa}{4}(x^2 + y^2)}\right)^2, \quad \tau, \kappa \in \mathbb{R},$$

we obtain a 2-parameters family of 3-dimensional Riemannian manifolds $(D_{\kappa,\tau}, ds^2_{\kappa,\tau})$, also denoted by $M^3(\kappa,\tau)$, called *Bianchi-Cartan-Vranceanu spaces* (BCV-spaces, in short).

The class of BCV-spaces contains all the Riemannian manifolds with 4-dimensional or 6-dimensional isometries groups except the hyperbolic space forms. The BCV-spaces provide model spaces of Thurston's 3-dimensional geometries (see [12]). In theoretical cosmology, the metrics on BCV-spaces are known as the Bianchi-Kantowski-Sachs type metrics used to construct some homogeneous space-times (see [11]). In these last fifteen years, many differential geometers investigate curves and surfaces with some special properties in BCV-spaces [15, 16]. Surfaces with parallel fundamental forms in BCV-spaces are classified by Belkhelfa, Dillen and Inoguchi in [13], and more generally surfaces with higher order parallel second fundamental forms in BCV-spaces have been classified by J. Van der Veken [14]. In [17] and [18], the authors studied biharmonic curves in BCV-spaces and they obtained interesting classification results. A Weierstrass representation is a description of the surface by some holomorphic functions. D.A. Berdinski and I.A. Taimanov obtained in [9] a Weierstrass type representation for minimal surfaces into BCV-spaces in terms of spinors and Dirac operators.

In this paper, we describe a Weierstrass-type representation formula for minimal surfaces into BCV-spaces in terms of two complex-functions satisfying some integral conditions and we extend thus the results obtained in [3] and [4].

2. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold and $f : \Sigma \subset M \to M$ be a minimal conformal immersion, where Σ is a Riemann surface. The pull-back bundle $f^\star(TM)$ has a metric and compatible connection, the pull-back connection induced by the Riemannian metric and the Levi-Civita connection of M. Consider the complexified bundle $\mathbb{E} = f^\star(TM) \otimes \mathbb{C}$.

Let (u, v) be a local coordinates on Σ, $z = u + iv$ the local conformal complex parameter and $(x_1, \ldots, x_n)$ be a system of local coordinates in a neighborhood U of M such that $U \cap f(\Sigma) \neq \varnothing$. The pull-back connection extends to a complex connection on $\mathbb{E}$ and it is well known that $\mathbb{E}$ has a unique holomorphic structure such that a section $\phi : \Sigma \to \mathbb{E}$ is holomorphic if and only if

$$\widetilde{\nabla}_{\frac{\partial}{\partial \bar{z}}} \phi = 0, \tag{2.1}$$

where $\widetilde{\nabla}$ is the pull-back connection on Σ.

The induced metric on Σ is

$$ds^2 = \lambda^2(du^2 + dv^2) = \lambda^2|dz|^2,$$

and the beltrami-Laplace operator on Σ, with respect to the induced metric ds^2 is given by

$$\Delta = \lambda^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

We recall that $f : \Sigma \to M$ is harmonic if and only if its tension field $\tau(f) = trace\,\nabla df$ vanishes and for conformal immersions, harmonicity and minimality are equivalent.
Let us consider

$$\phi = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right).$$

By putting

$$\phi = \sum_{j=1}^{n} \phi_j \frac{\partial}{\partial x_j}$$

where ϕ_j are some complex-valued functions defined on Σ, the tension field $\tau(f)$ of f can be written as:

$$\tau(f) = 4\lambda^{-2}\sum_i \left\{\frac{\partial \phi_i}{\partial \bar{z}} + \Gamma^i_{jk}\bar{\phi}_j\phi_k\right\}\frac{\partial}{\partial x_i}$$

where Γ^i_{jk} are the Christoffel symbols of M.
The section ϕ is then holomorphic if and only if

$$\widetilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\left(\sum_{i=1}^{n}\phi_i\frac{\partial}{\partial x_i}\right) = \sum_j\left\{\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{k,j}\Gamma^i_{jk}\bar{\phi}_j\phi_k\right\}\frac{\partial}{\partial x_i} = 0\ ;$$

or equivalently if and only if

$$\frac{\partial \phi_i}{\partial \bar{z}} + \sum_{k,j}\Gamma^i_{jk}\bar{\phi}_j\phi_k = 0, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

We have then

$$4\lambda^{-2}(\widetilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\phi) = \tau(f).$$

Thus $f : \Sigma \to M$ is harmonic if and only if $\phi = \frac{\partial f}{\partial z}$ is a holomorphic section of $\mathbb{E}$.
Relation (2.2) is a system of first order differential equations in the ϕ_i, it can be written as:

$$\frac{\partial \phi_i}{\partial \bar{z}} + 2\sum_{j>k}\Gamma^i_{jk}Re(\bar{\phi}_j\phi_k) + \sum_j \Gamma^i_{jj}|\phi_j|^2 = 0, \quad i = 1, \ldots, n.$$

This implies that $\frac{\partial \phi_i}{\partial \bar{z}} \in \mathbb{R}$, and ensures that (locally) the 1-forms $\phi_i dz$ do not have real periods as it has been mentioned in [3]. Therefore we have the following:

Proposition 2.1 ([4]). *Let (M, g) be a Riemannian manifold and $(x_1, \dots, x_n)$ local coordinates. Let ϕ_j, $j = 1, \dots, n$, be complex-valued functions in an open simply connected domain $\Omega \subset \mathbb{C}$ which are solutions of* (2.2). *Then the map*

$$f_j(u,v) = 2\,\mathrm{Re}\left(\int_{z_0}^{z} \phi_j dz\right) \tag{2.3}$$

is well defined and determines a minimal conformal immersion if and only if the following conditions are satisfied:

(i) $\sum_{j,k=1}^{n} g_{ij}\phi_j\bar{\phi}_k \neq 0,$

(ii) $\sum_{j,k=1}^{n} g_{ij}\phi_j\phi_k = 0.$

In [3], the authors proved that if M is a Lie group then the system (2.2) has a solution. In the next section we describe a Weierstrass representation for minimal surfaces into 3-dimensional manifold.

3. Weierstrass Representation in 3-dimensional Manifolds

Let M^3 be a 3-dimensional manifold, endowed with an analytic Riemannian metric g. We consider M^3 as a single chart and (x^1, x^2, x^3) a system of coordinates on M^3. By the Gram-Schmidt orthonormalization, we have a basis of vector fields E_i, $i = 1, 2, 3$, defined by

$$\begin{aligned} E_1 &= \frac{1}{A}\left\{\frac{\partial}{\partial x^1} - \frac{1}{B^2}(g_{12} - g^{33}g_{23}g_{13})\frac{\partial}{\partial x^2}\right. \\ &\qquad \left. + g^{33}\left(\frac{1}{B^2}(g_{12} - g^{33}g_{23}g_{13})g_{23} - g_{13}\right)\frac{\partial}{\partial x^3}\right\}, \\ E_2 &= \frac{1}{B}\left\{\frac{\partial}{\partial x^2} - g^{33}g_{23}\frac{\partial}{\partial x^3}\right\}, \\ E_3 &= \frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^3}. \end{aligned} \tag{3.1}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}; \frac{\partial}{\partial x^j}\right), \quad g^{ij} = [g_{ij}]^{-1},$$

$$A = \sqrt{g_{11} - \frac{1}{B^2}(g_{12} - g^{33}g_{13}g_{23})^2 - g^{33}(g_{13})^2},$$

$$B = \sqrt{g_{22} - g^{33}(g_{23})^2}.$$

In some open set $\Omega \subset \Sigma$ the section $\phi = \frac{\partial f}{\partial z} \in \Gamma(f^*(TM) \otimes \mathbb{C})$ can be decomposed with respect to the coordinates vector fields as

$$\phi = \sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x^i} = \sum_{i=1}^{3}\psi_i E_i, \tag{3.2}$$

for some open complex functions ϕ_i, $\psi_i : \Omega \to \mathbb{C}$. Moreover, there exists an invertible matrix $Mat = (m_{ij})_{i,j=1,2,3}$ with the functions entries $m_{ij} : f(\Omega) \to \mathbb{R}$, $i,j = 1,2,3$, satisfying

$$\phi_i = \sum_j m_{ij}\psi_j\,,$$

where

$$Mat = \begin{bmatrix} A^{-1} & 0 & 0 \\ -(g_{12} - g^{33}g_{23}g_{13})(AB^2)^{-1} & B^{-1} & 0 \\ (A)^{-1}[g^{33}((B^2)^{-1}(g_{12} - g^{33}g_{23}g_{13})g_{23} - g_{13})] & -g^{33}g_{23}B^{-1} & \sqrt{g^{33}} \end{bmatrix}. \tag{3.3}$$

By (3.2), we have

$$\widetilde{\nabla}_{\frac{\partial}{\partial \bar z}}\left(\sum_i \psi_i E_i\right) = \sum_i \left\{\frac{\partial \psi_i}{\partial \bar z}E_i + \sum_{j,k}\psi_k\bar{\psi}_j g(\nabla_{E_j}E_k, E_i)E_i\right\}.$$

This means that the section ϕ is holomorphic if and only if

$$\frac{\partial \psi_i}{\partial \bar z} + \sum_{j,k}\psi_k\bar{\psi}_j g(\nabla_{E_j}E_k, E_i) = 0, \quad i = 1,2,3. \tag{3.4}$$

Theorem 3.1. *Let ψ_i, $i = 1,2,3$ be complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$ such as the following conditions are satisfied*:

(i) $\sum_{i=1}^{n} \psi_i\bar{\psi}_i \neq 0$,

(ii) $\sum_{i=1}^{n} \psi_i^2 = 0$,

(iii) ψ_i *are solutions of* (3.4).

Then the map $f := (x^1, x^2, x^3) : \Omega \to (M^3, g)$, defined by

$$x^1(p) = 2\,\mathrm{Re}\int_{p_0}^{p}\left(\sqrt{(g_{11} - (g_{22} - g^{33}(g_{23})^2)^{-1}(g_{12} - g^{33}g_{13}g_{23})^2 - g^{33}(g_{13})^2}\right)^{-1}\psi_1 d\rho\,,$$

$$x^2(p) = 2\,\mathrm{Re}\int_{p_0}^{p} -(g_{22} - g^{33}(g_{23})^2)^{-1}(g_{12} - g_{13}g_{23}g^{33})\psi_1 + \left(\sqrt{g_{22} - g^{33}(g_{23})^2}\right)^{-1}\psi_2 d\rho\,,$$

$$x^3(p) = 2\,\mathrm{Re}\int_{p_0}^{p} [(g_{22} - g^{33}(g_{23})^2)^{-1}(g_{23}g^{33}(g_{12} - g_{13}g_{23}g^{33}) - g^{33}g_{13}]\psi_1 - \left(\sqrt{g_{22} - g^{33}(g_{23})^2}\right)^{-1} g_{23}g^{33}\psi_2 + \sqrt{g^{33}}\psi_3 d\rho\,, \tag{3.5}$$

is a conformal minimal immersion.

Proof. Using (3.1) and (3.2), we get

$$\phi_1 = \left(\sqrt{(g_{11} - (g_{22} - g^{33}(g_{23})^2)^{-1}(g_{12} - g^{33} g_{13} g_{23})^2 - g^{33}(g_{13})^2}\right)^{-1} \psi_1,$$

$$\phi_2 = -(g_{22} - g^{33}(g_{23})^2)^{-1}(g_{12} - g_{13} g_{23} g^{33})\psi_1 + i\left(\sqrt{g_{22} - g^{33}(g_{23})^2}\right)^{-1} \psi_2,$$

$$\phi_3 = (g_{22} - g^{33}(g_{23})^2)^{-1}(g_{23} g^{33}(g_{12} - g_{13} g_{23} g^{33}) - g^{33} g_{13}]\psi_1$$
$$- \left(\sqrt{g_{22} - g^{33}(g_{23})^2}\right)^{-1} g_{23} g^{33} \psi_2 + 2\sqrt{g^{33}}\psi_3.$$

From proposition 2.1, the theorem is proved. □

Remark 3.2. If $M = \mathbb{R}^3$ and g the flat metric on M, we have a Weierstrass representation for minimal surfaces in $\mathbb{R}^3$, see [4].

Since the parameter z is conformal, we have

$$\psi_1^2 + \psi_2^2 + \psi_3^2 = 0. \tag{3.6}$$

From (3.6) we have

$$(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = -\psi_3^2,$$

which suggests the definition of two new complex functions

$$G := \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)}\ , \qquad H := \sqrt{-\frac{1}{2}(\psi_1 + i\psi_2)}\ . \tag{3.7}$$

The functions G and H are single-valued complex functions which satisfy

$$\psi_1 = G^2 - H^2, \qquad \psi_2 = i(G^2 + H^2), \qquad \psi_3 = 2GH. \tag{3.8}$$

In the following, we give a Weierstrass representation for minimal surfaces into BCV-spaces.

4. Weierstrass Representation in BCV-space $M^3(\kappa, \tau)$

Let κ and τ be two real numbers, with $\tau \geq 0$. Bianchi-Cartan-Vranceanu space (BCV-space) $M^3(\kappa, \tau)$ is defined as the set

$$D_{\kappa,\tau} = \left\{(x, y, z) \in \mathbb{R}^3 \big/ 1 + \frac{\kappa}{4}(x^2 + y^2) > 0\right\}$$

endowed with the metric

$$ds^2_{\kappa,\tau} = \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \left(dz + \tau \frac{ydx - xdy}{1 + \frac{\kappa}{4}(x^2 + y^2)}\right)^2. \tag{4.1}$$

It was Cartan [8] who obtained this family of spaces by classifying of three-dimensional Riemannian manifolds with four-dimensional isometry group. They also appeared in the work L. Bianchi [5, 6], and G. Vranceanu [7]. The complete classification of BCV-spaces is as follows:

- if $\kappa = \tau = 0$, then $M^3(\kappa, \tau) \cong \mathbb{R}^3$;
- if $\kappa = 4\tau^2 \neq 0$, then $M^3(\kappa, \tau) \cong \mathbb{S}^3(\frac{\kappa}{4}) \backslash \{\infty\}$;

- if $\kappa > 0$ and $\tau = 0$, then $M^3(\kappa,\tau) \cong (\mathbb{S}^2(\kappa)\backslash\{\infty\}) \times \mathbb{R}$;
- if $\kappa < 0$ and $\tau = 0$, then $M^3(\kappa,\tau) \cong \mathbb{H}^2(\kappa) \times \mathbb{R}$;
- if $\kappa > 0$ and $\tau \neq 0$, then $M^3(\kappa,\tau) \cong SU(2)\backslash\{\infty\}$;
- if $\kappa < 0$ and $\tau \neq 0$, then $M^3(\kappa,\tau) \cong \widetilde{SL}(2,\mathbb{R})$;
- if $\kappa = 0$ and $\tau \neq 0$, then $M^3(\kappa,\tau) \cong Nil_3$.

By the Gram-Schmidt orthonormalization the following vectors fields form an orthonormal frame on $M^3(\kappa,\tau)$:

$$E_1 = \left(1 + \frac{\kappa}{4}(x^2+y^2)\frac{\partial}{\partial x} - \tau y\frac{\partial}{\partial z}\right);$$
$$E_2 = \left(1 + \frac{\kappa}{4}(x^2+y^2)\frac{\partial}{\partial y} + \tau x\frac{\partial}{\partial z}\right); \quad E_3 = \frac{\partial}{\partial z}. \tag{4.2}$$

The corresponding Lie Bracket are

$$[E_1;E_2] = -\frac{\kappa}{2}yE_1 + \frac{\kappa}{2}xE_2 + 2\tau E3; \quad [E_1;E_3] = 0; \quad [E_2;E_3] = 0. \tag{4.3}$$

With respect to this orthonormal basis, the Levi-Civita connection can be computed as:

$$\nabla_{E_1}E_1 = \tfrac{\kappa}{2}yE_2 \qquad \nabla_{E_1}E_2 = -\tfrac{\kappa}{2}yE_1 + \tau E_3, \quad \nabla_{E_1}E_3 = -\tau E_2,$$
$$\nabla_{E_2}E_1 = -\tfrac{\kappa}{2}xE_2 - \tau E_3, \qquad \nabla_{E_2}E_2 = \tfrac{\kappa}{2}xE_1, \qquad \nabla_{E_2}E_3 = \tau E_1,$$
$$\nabla_{E_3}E_1 = -\tau E_2, \quad \nabla_{E_3}E_2 = \tau E_1, \quad \nabla_{E_3}E_3 = 0.$$

We have by Kozul's formula

$$g(\nabla_{E_1}E_1,E_2) = \tfrac{\kappa}{4}y, \quad g(\nabla_{E_1}E_2,E_1) = -\tfrac{\kappa}{4}y, g(\nabla_{E_1}E_2,E_3) = \tfrac{\tau}{2}, \quad g(\nabla_{E_1}E_3,E_2) = -\tfrac{\tau}{2},$$
$$g(\nabla_{E_2}E_1,E_2) = -\tfrac{\kappa}{4}x, g(\nabla_{E_2}E_1,E_3) = -\tfrac{\tau}{2}, \quad g(\nabla_{E_2}E_2,E_1) = \tfrac{\kappa}{4}x, g(\nabla_{E_2}E_3,E_1) = \tfrac{\tau}{2},$$
$$g(\nabla_{E_3}E_1,E_2) = -\tfrac{\tau}{2}, \quad g(\nabla_{E_3}E_2,E_1) = \tfrac{\tau}{2}.$$

The matrice (3.3) is then given by

$$\begin{bmatrix} 1+\frac{\kappa}{4}(x^2+y^2) & 0 & 0 \\ 0 & 1+\frac{\kappa}{4}(x^2+y^2) & 0 \\ -\tau y & \tau x & 1 \end{bmatrix}.$$

According to (3.4) the section $\phi = \psi_1E_1 + \psi_2E_2 + \psi_3E_3$ is holomorphic if and only if

$$\frac{\partial\psi_1}{\partial\bar z} - \frac{\kappa}{4}y\psi_2\bar\psi_1 + \frac{\kappa}{4}x\psi_2\bar\psi_2 + \frac{\tau}{2}\psi_2\bar\psi_3 + \frac{\tau}{2}\psi_3\bar\psi_2 = 0,$$
$$\frac{\partial\psi_2}{\partial\bar z} + \frac{\kappa}{4}y\psi_1\bar\psi_1 - \frac{\tau}{2}\psi_1\bar\psi_3 - \frac{\kappa}{4}x\psi_1\bar\psi_2 - \frac{\tau}{2}\psi_3\bar\psi_1 = 0, \tag{4.4}$$
$$\frac{\partial\psi_3}{\partial\bar z} - \frac{\tau}{2}\psi_1\bar\psi_2 + \frac{\tau}{2}\psi_2\bar\psi_1 = 0.$$

Let us now write equations (4.4), which ensures that ϕ is holomorphic section, in term of the functions G and H:

If ψ satisfies (3.8) then

$$G\frac{\partial G}{\partial \bar{z}} = \frac{\kappa}{8}yi(|G|^4 - G^2\bar{H}^2) - \frac{\kappa}{8}x(|G|^4 + G^2\bar{H}^2) - \frac{i\tau}{2}G\bar{H}(|G|^2-|H|^2), \tag{4.5}$$

$$H\frac{\partial H}{\partial \bar{z}} = \frac{\kappa}{8}yi(|H|^4 - H^2\bar{G}^2) + \frac{\kappa}{8}x(|H|^4 + H^2\bar{G}^2) - \frac{i\tau}{2}H\bar{G}(|G|^2-|H|^2), \tag{4.6}$$

$$H\frac{\partial G}{\partial \bar{z}} + G\frac{\partial H}{\partial \bar{z}} = -\frac{i\tau}{2}(|G|^4-|H|^4). \tag{4.7}$$

Therefore, Theorem 3.1 can be written as:

Theorem 4.1. *Let G and H be complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$ such that:*

(i) *G and H are not identically zeros.*
(ii) *G and H are solutions of* (4.5)-(4.7).

Then the map $f := (x, y, z) : \Omega \to M^3(\kappa, \tau)$, defined by

$$x(p) = 2\,\mathrm{Re}\int_{p_0}^{p} (1 + \frac{\kappa}{4}(x^2 + y^2))(G^2 - H^2)d\rho\,, \tag{4.8}$$

$$y(p) = 2\,\mathrm{Re}\int_{p_0}^{p} (1 + \frac{\kappa}{4}(x^2 + y^2))i(G^2 + H^2)d\rho\,, \tag{4.9}$$

$$z(p) = 2\,\mathrm{Re}\int_{p_0}^{p} -y\tau(G^2 - H^2) + ix\tau(G^2 + H^2) + 2GHd\rho\,, \tag{4.10}$$

is a conformal minimal immersion.

Proof. Using (4.2), we get

$$\phi_1 = \left(1+\frac{\kappa}{4}(x^2+y^2)\right)\psi_1\,, \quad \phi_2 = \left(1+\frac{\kappa}{4}(x^2+y^2)\right)\psi_2\,, \quad \phi_3 = -\tau y\psi_1 + \tau x\psi_2 + \psi_3\,.$$

From Theorem 3.1 and (3.8), we have the result. □

Remark 4.2. Equations (4.5) and (4.6) are non-linear partial differential equations with non-constant coefficients and it is more complicated to find explicitly solutions ϕ_i, $i = 1, 2, 3$. By replacing κ by $\delta = 1 + \frac{\kappa}{4}(x^2 + y^2)$, with $|\delta| > 2$ and δ is constant, we obtain the new metric

$$ds^2_{\delta,\tau} = \frac{dx^2 + dy^2}{\delta^2} + \left(dz + \tau\frac{ydx - xdy}{\delta}\right)^2, \tag{4.11}$$

which looks like a Heisenberg metric but is not isometric to a Heisenberg metric.

By the Gram-Schmidt orthonormalization the following vectors fields form an orthonormal frame on $M^3(\delta, \tau)$:

$$E_1 = \delta\frac{\partial}{\partial x} - \tau y\frac{\partial}{\partial z}\,; \qquad E_2 = \delta\frac{\partial}{\partial y} + \tau x\frac{\partial}{\partial z}\,; \qquad E_3 = \frac{\partial}{\partial z}\,. \tag{4.12}$$

The corresponding Lie Bracket are

$$[E_1;E_2]=2\delta\tau E_3\ ;\quad [E_1;E_3]=0\ ;\quad [E_2;E_3]=0\ . \tag{4.13}$$

With respect to this orthonormal basis, the Levi-Civita connection can be computed as:

$$\begin{aligned}
&\nabla_{E_1}E_1=0, && \nabla_{E_1}E_2=\delta\tau E_3, && \nabla_{E_1}E_3=-\delta\tau E_2,\\
&\nabla_{E_2}E_1=-\delta\tau E_3, && \nabla_{E_2}E_2=0, && \nabla_{E_2}E_3=\delta\tau E_1,\\
&\nabla_{E_3}E_1=-\delta\tau E_2, && \nabla_{E_3}E_2=\delta\tau E_1, && \nabla_{E_3}E_3=0.
\end{aligned}$$

We have by Kozul's formula

$$\begin{aligned}
&g(\nabla_{E_1}E_2,E_3)=\delta\tau, \quad g(\nabla_{E_1}E_3,E_2)=-\delta\tau, \quad g(\nabla_{E_2}E_1,E_3)=-\delta\tau,\\
&g(\nabla_{E_2}E_3,E_1)=\delta\tau, \quad g(\nabla_{E_3}E_1,E_2)=-\delta\tau, \quad g(\nabla_{E_3}E_2,E_1)=\delta\tau.
\end{aligned}$$

According to (3.4), the section $\phi=\psi_1E_1+\psi_2E_2+\psi_3$ is holomorphic if and only if

$$\begin{aligned}
&\frac{\partial\psi_1}{\partial\bar{z}}+2\delta\tau\, Re(\psi_2\bar{\psi}_3)=0\ ;\\
&\frac{\partial\psi_2}{\partial\bar{z}}-2\delta\tau\, Re(\psi_1\bar{\psi}_3)=0\ ;\quad \frac{\partial\psi_3}{\partial\bar{z}}-2i\delta\tau\, Im(\psi_1\bar{\psi}_2)=0.
\end{aligned} \tag{4.14}$$

Equations (4.14) can be written in terms of the functions G and H defined by (3.7).

$$\frac{\partial G}{\partial\bar{z}}=-i\delta\tau\bar{H}(|G|^2-|H|^2), \tag{4.15}$$

$$\frac{\partial H}{\partial\bar{z}}=-i\delta\tau\bar{G}(|G|^2-|H|^2). \tag{4.16}$$

Therefore, Theorem 4.1 becomes:

Theorem 4.3. *Let G and H be complex-valued functions defined in a simply connected domain $\Omega\subset\mathbb{C}$ such that:*

(i) *G and H are not identically zeros.*

(ii) *G and H are solutions of* (4.15)-(4.16).

Then the map $f:=(x,y,z):\Omega\to M^3(\delta,\tau)$, defined by

$$\begin{cases}
x(p)=2\,\mathrm{Re}\int_{p_0}^{p}\delta(G^2-H^2)d\rho\,,\\
y(p)=2\,\mathrm{Re}\int_{p_0}^{p}i\delta(G^2+H^2)d\rho\,,\\
z(p)=2\,\mathrm{Re}\int_{p_0}^{p}-y\tau(G^2-H^2)+ix\tau(G^2+H^2)+2GH d\rho\,,
\end{cases}$$

is a conformal minimal immersion.

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