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Research Article

Analysis of Probability Distribution Due to Aspiration Level in Fuzzy Matrix Game

Sapana P. Dubey *[1] and G. D. Kedar [2]

[1] *Department of Applied Mathematics, P.C.E., Nagpur, India*
[2] *Department of Mathematics, RTMNU, Nagpur, India*

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Abstract. In this paper, we studied zero-sum matrix game. These games have been represented in form of fuzzy linear programming problems which consists the aspiration level and tolerance level. The aspiration levels are determined on the basis of payoff matrix instead of using probability distribution over the strategies. For a particular tolerance level, the aspiration level is studied and the result are illustrated graphically.

Keywords. Matrix game; Fuzzy Linear programming; Fuzzy membership function

Mathematics Subject Classification (2020). 91-XX, 90C05, 90C70

1. Introduction

Game theory has potential to handle the situation where the rational decision makers have some conflict. So game theory is applicable in the field of decision theory. The decisions are used to finalizing the strategies and optimizing the payoff.

Game theory is first introduced by economist Oskar Morgenstern and mathematician John Von Neumann [5] when they have published the book entitled "*The Theory of Games and Economic Behavior*". Game theory is applicable in various area like economics, political science, computer science and biology etc.

The game may have uncertain objective functions. Due to lack of information sometimes players are not able to evaluate the game. The imprecision of game can be modeled by fuzzy

*Corresponding author: sapana.dubey10@gmail.com

set. Fuzzy sets first introduced by Zadeh [9]. The fuzzy set includes the degree of belongingness with the element. The combination of game with fuzzy provides the new type of game called fuzzy game.

Butnariu [3], was presented a heuristical discussion of fuzzy game and its relationship with the classical game. Raman [7] discussed fuzzy view of prisoner's dilemma. Aristidou and Sarangi [1] have developed a static model of a fuzzy game. They have identified conditions that guarantee the existence of equilibrium. Selvakumari and Lavanya [8] have used ranking of fuzzy numbers to solve fuzzy game problems. In this paper, authors have considered all imprecise values as a triangular or trapezoidal fuzzy numbers.

Chakeriet al. [4] used a least deviation method to set the priority of every payoff. Fuzzy satisfaction function and satisfaction degree was introduced for each fuzzy payoff. Nishizaki and Sakawa [6] introduced the Fuzzy goal in the matrix game model. Bector and Chandra [2] presented application of fuzzy sets in the mathematical programming and matrix game theory.

In this paper we have discussed two player zero sum game with fuzzy goal, Linear programming problem of this matrix game and its fuzzy membership function. The condition for aspiration level is derived to get optimal solution of the game. The range of aspiration level and tolerance level have been discussed to get desired value of membership function.

This paper consist six sections. Section 2 includes some basic concepts used in the paper. Section 3 describe the model of matrix game with fuzzy goal introduced by Bector and Chandra [2]. In the next section, the value of Aspiration levels are determined for different cases of fuzzy goal. This section is followed by numerical illustration of the matrix game with fuzzy goal. The last section includes concluding remark.

2. Preliminaries

In this section, we present some basic concepts used in the paper.

Matrix Game. A crisp zero sum game are represented by the triplet $G = \{S^m, S^n, A\}$ where S^m and S^n are strategy space for player I and player II, respectively, whose element shows probability distribution on the set of pure strategies and A is payoff matrix. Symbolically it is expressed as

$$S^m = \left\{x \in R^m : x_i \geq 0,\, i = 1,2,\dots m,\, \sum_{i=0}^{m} x_i = 1\right\},$$

$$S^n = \left\{y \in R^n : y_j \geq 0,\, j = 1,2,\dots n,\, \sum_{j=0}^{n} y_j = 1\right\}$$

and

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the payoff of player I if player I chooses ith pure strategy and player II chooses jth pure strategy. At the same time payoff of player II is $-a_{ij}$ for zero sum game.

Expected Payoff Function. A function E from $S^m \times S^n$ to R is called expected payoff function. Using this function expected payoff is given by $E(x,y) = x^T Ay$.

Maxmin Principle. When the maximizing player chooses the strategy which corresponds to the maximum of minimum gain from his different strategies, is known as the maxmin principle.

Minmax Principle. When the minimizing player chooses the strategy which corresponds to the minimum of maximum losses from his different strategies, is known as the minmax principle.

Saddle Point. The Expected payoff at the point (x^*, y^*) is said to be saddle point if

$$E(x^*, y) \geq E(X^*, y^*), \quad \forall\ y \in S^n \quad \text{and} \quad E(x, y^*) \leq E(x^*, y^*), \quad \forall\ x \in S^m.$$

Fuzzy Set. Let X be universe of discourse then a fuzzy set A in X is defined as

$$A = \{(x, \mu_A(x)) : x \in X\}, \quad \text{where } \mu_A : X \to [0,1].$$

For $x \in X$, $\mu_A(x)$ is called the degree or grade of membership.

3. The Model of Matrix Game With Fuzzy Goal

This model is introduced by Bector *et al*. [2]. Two person zero sum crisp game $G = \{S^m, S^n, A\}$ is represented by $FG = \{S^m, S^n, A, v_0, \gtrsim, w_0, \lesssim\}$ which is two person zero sum matrix game with fuzzy goals. Terms used in the structure of FG are as follows:

S^m and S^n are strategy space for player I and player II and A is payoff matrix as discussed in Section 2.

v_0 and w_0 are aspiration level of player I and player II, defined as

$v_0 = \max\limits_{x \in S^m} \min\limits_{y \in S^n} x^T Ay$ and $w_0 = \min\limits_{y \in S^n} \max\limits_{x \in S^m} x^T Ay$.

$\gtrsim$ and $\lesssim$ are fuzzy version of $\geq$ and $\leq$. To get the solution of FG game, we choose specific membership function to satisfy the relation $\gtrsim$ and $\lesssim$.

If t be a real variable and $a \in R$. Let $p > 0$, then the fuzzy set F defining the fuzzy statement $t \succcurlyeq_p a$, which states "t essentially greater than or equal to a with tolerance error p", is defined by the following membership function.

$$\mu_F(t) = \begin{cases} 1, & t \geq a, \\ 1 - \frac{a-t}{p}, & (a-p) \leq t < a, \\ 0, & t < (a-p). \end{cases}$$

Inclusion of tolerance p_0 and q_0 for the player I and Player II produces the need to redefine the FG. So the game FG can be defined as

$$FG = \{S^m, S^n, A, v_0, \gtrsim, p_0, w_0, \lesssim, q_0\}.$$

Solution of the fuzzy matrix game FG:
A point (x^*, y^*) is said to be solution of the fuzzy matrix game FG if it satisfies following conditions:

$$x^{*^T} Ay \gtrsim_{p_0} v_0, \quad \forall\ y \in S^n,$$

$$x^T A y^* \precsim_{q_0} w_0, \quad \forall\, x \in S^m.$$

It is very difficult to find the solution of matrix game in present situation. Introducing fuzzy sense in the matrix game can provide flexibility to take decisions for the players. Modeling matrix game with fuzzy sense in to fuzzy linear programming is desirable in this situation.

Therefore the fuzzy matrix game FG is converted into two fuzzy linear programming problems (FLPP1) and (FLPP2) for player I and player II respectively

(FLPP1) Find $x = (x_1, x_2, \dots, x_m)$ such that

$$\sum_{i=1}^{m} a_{ij} x_i \succsim_{p_0} v_0 \Longrightarrow A_j^T x \succsim_{p_0} v_0 \quad (j = 1, 2, \dots, n) \tag{3.1}$$

where A_j represent jth column of payoff matrix A

$$\sum_{i=1}^{m} x_i = 1,$$

$$x \geq 0$$

and

(FLPP2) Find $y = (y_1, y_2, \dots, y_n)$ such that

$$\sum_{j=1}^{n} a_{ij} y_j \precsim_{q_0} w_0 \Longrightarrow A_i y \precsim_{q_0} w_0 \quad (i = 1, 2, \dots, m), \tag{3.2}$$

$$\sum_{j=1}^{n} y_j = 1,$$

$$y \geq 0.$$

The membership function which gives the degree to which x satisfies the fuzzy constraint (3.1) as follows:

$$\mu_j(A_j^T x) = \begin{cases} 1, & A_j^T x \geq v_0, \\ 1 - \frac{v_0 - A_j^T x}{p_0}, & (v_0 - p_0) \leq A_j^T x < v_0, \\ 0, & A_j^T x \leq (v_0 - p_0). \end{cases} \tag{3.3}$$

Fuzzy linear programming for membership function for the first player can be represented as

$$\max \lambda$$

$$\text{subject to } \lambda \leq 1 - \frac{v_0 - A_j^T x}{p_0}, \quad (j = 1, 2, 3, \dots, n) \tag{3.4}$$

$$e^T x = 1,\ \lambda \leq 1,\ x, \lambda \geq 0.$$

Similarly, membership function which gives the degree to which y satisfies the fuzzy constraint (3.2) as follows:

$$\mu_i(A_i y) = \begin{cases} 1, & A_i y \geq w_0, \\ 1 - \frac{A_i y - w_0}{q_0}, & w_0 < A_i y \leq w_0 + q_0, \\ 0, & A_i y > (w_0 + q_0). \end{cases} \tag{3.5}$$

Fuzzy linear programming for membership function for the second player can be represented as

$$\max \eta$$
$$\text{subject to } \eta \le 1 - \frac{A_i y - w_0}{q_0}, \quad (i = 1,2,3,\ldots,m) \tag{3.6}$$
$$e^T y = 1, \;\; \eta \le 1, \;\; y,\eta \ge 0.$$

4. Specifying the Value of Aspiration Level

While using the model describe in Section 3, we have faced the problem related to selection of aspiration level. The selection of aspiration level is very important to get optimal solution to the Fuzzy linear programming problem and hence for the fuzzy game.

In this section, we derived the condition for the aspiration levels v_0 and w_0 for player I and player II, respectively.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the payoff matrix for two person zero sum game. $x^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y^* = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be the probability distributions of strategies of player I an player II.

First of all, we define the aspiration level for player I using payoff matrix A.

Using equation (3.4) we get

$$\lambda \le 1 - \frac{v_0 - (ax_1 + cx_2)}{p_0} \Longrightarrow v_0 - (ax_1 + cx_2) \le (1-\lambda)p_0, \tag{4.1}$$
$$\lambda \le 1 - \frac{v_0 - (bx_1 + dx_2)}{p_0} \Longrightarrow v_0 - (bx_1 + dx_2) \le (1-\lambda)p_0. \tag{4.2}$$

Case I: When $\lambda = 1$ (i.e., satisfaction of an aspiration level as best as possible)

$$v_0 \le (ax_1 + cx_2), \tag{4.3}$$
$$v_0 \le (bx_1 + dx_2). \tag{4.4}$$

Also, we know that the sum of probability distribution is 1. Therefore

$$x_1 + x_2 = 1. \tag{4.5}$$

On solving (4.3), (4.4) and (4.5) we get

$$v_0 \le \frac{(ad - cb)}{(a-c)-(b-d)}. \tag{4.6}$$

Case II: When $\lambda < 1$ (i.e., satisfaction of an aspiration level is as desired by the player)

From equations (4.1) and (4.2)

$$v_0 \le (1-\lambda)p_0 + (ax_1 + cx_2), \tag{4.7}$$
$$v_0 \le (1-\lambda)p_0 + (bx_1 + dx_2), \tag{4.8}$$

where p_0 is tolerance level for player I.

On solving (4.5), (4.7) and (4.8) we get

$$v_0 \le (1-\lambda)p_0 + \frac{(ad - cb)}{(a-c)-(b-d)}. \tag{4.9}$$

Similarly, we define the aspiration level for player II using payoff matrix A.

Using equation (3.6) we get

$$\eta \le 1 - \frac{(ay_1 + by_2) - w_0}{q_0} \Longrightarrow (ay_1 + by_2) - w_0 \le (1-\eta)q_0, \tag{4.10}$$

$$\eta \le 1 - \frac{(cy_1 + dy_2) - w_0}{q_0} \Longrightarrow (cy_1 + dy_2) - w_0 \le (1-\eta)q_0, \tag{4.11}$$

$$y_1 + y_2 = 1. \tag{4.12}$$

For $\eta = 1$ (i.e. satisfaction of an aspiration level as best as possible) we get

$$w_0 \ge \frac{(ad - cb)}{(a-b)-(c-d)}. \tag{4.13}$$

For $\eta < 1$ (i.e. satisfaction of an aspiration level is as desired by the player).
From equations (4.10) and (4.11)

$$w_0 \ge \frac{(ad - cb)}{(a-b)-(c-d)} - (1-\eta)q_0, \tag{4.14}$$

where q_0 is tolerance level for player II.

5. Numerical Example

In this section, we present an example to illustrate the fuzzy implementation of two person zero sum game and effect of aspiration level on probability distribution of strategies.

Consider the two player zero sum matrix game G with payoff matrix $A = \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$.

This game is expressed as Fuzzy linear programming (FLP) for each player.

For player I, with $p_0 = 1$ this FLP is as follows:

Maximize λ

$$-\lambda + 2x_1 + 6x_2 \ge v_0 - 1$$
$$-\lambda + 4x_1 + x_2 \ge v_0 - 1$$
$$\lambda \le 1;\ x_1 + x_2 = 1$$
$$\lambda \ge 0;\ x_1 \ge 0;\ x_2 \ge 0$$

For player II with $q_0 = 1$ this FLP is as follows

Maximize η

$$\eta + 2y_1 + 4y_2 \le w_0 + 1$$
$$\eta + 6y_1 + y_2 \le w_0 + 1$$
$$\eta \le 1;\ y_1 + y_2 = 1$$
$$\eta \ge 0;\ y_1 \ge 0;\ y_2 \ge 0$$

Now, the main question arises for the players that "*What should be the value of aspiration for which they can start the game*?"

As we have discussed in Section 4, using payoff matrix we have defined the value v_0 and w_0 as follow: $v_0 \le 3.1$ and $w_0 > 3.1$.

We get maximum satisfaction ($\lambda = 1$) of various aspiration level of player I. Figure 1 shows the graph of aspiration level (with $\lambda = 1$) and probability distribution for player I.

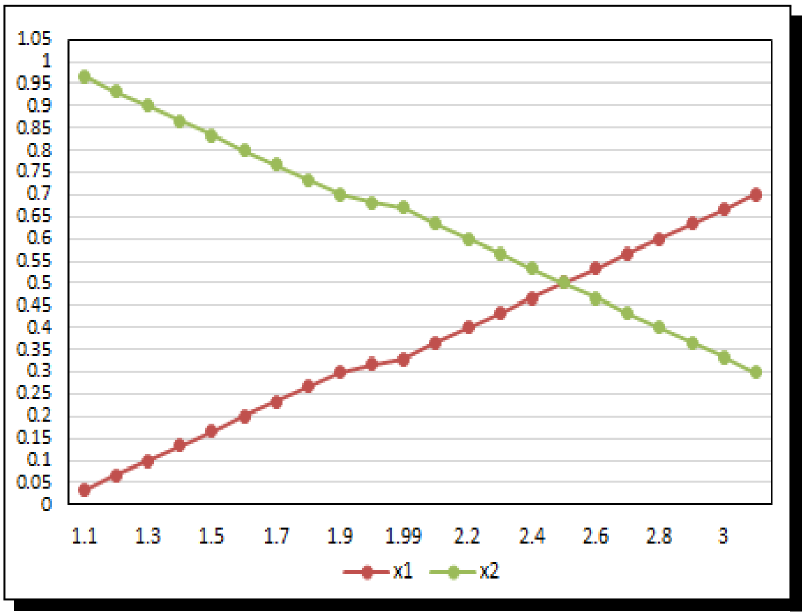

Figure 1. Probability distribution of strategies of player I and aspiration level

From Figure 1, we conclude that probability of selecting first strategy is increases and probability of selecting second strategy is decreases as aspiration level increases. At $v_0 = 2.5$, probability of both strategies are same.

Similarly, we get the maximum satisfaction ($\eta = 1$) of various aspiration level of player II. Figure 2 shows the graph of aspiration level (with $\eta = 1$) and probability distribution for player II.

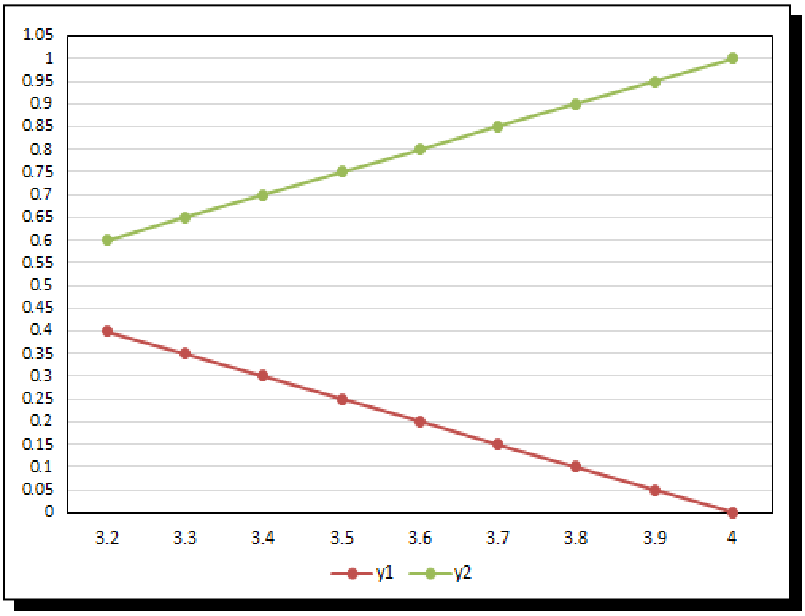

Figure 2. Probability distribution of strategies of player II and aspiration level

From Figure 2 we conclude that probability of selecting first strategy is decreases and probability of selecting second strategy is increases as aspiration level increases.

6. Conclusion

In this paper, we have determined the aspiration levels using payoff matrix. We have not only evaluated the probability distribution of the strategies, but the effect of aspiration level on the probability distribution for maximum satisfaction level is also analyzed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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