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Research Article

Degree-Magic Labellings on Graphs Generalizing the Double Graph of the Disjoint Union of a Graph

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Abstract. A graph G is called supermagic if it admits a labelling of the edges by pairwise different consecutive positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. A graph G is called degree-magic if it admits a labelling of the edges by integers $1,2,\ldots,|E(G)|$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1+|E(G)|)\deg(v)/2$. In this paper, some constructions of degree-magic labellings of some graphs obtained by generalizing the double graph of the disjoint union of a graph are presented. As a result, some supermagic graphs are obtained.

Keywords. Double graphs; Supermagic graphs; Degree-magic graphs

Mathematics Subject Classification (2020). 05C78

1. Introduction

The finite graphs without loops and isolated vertices are considered. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G. The subgraph of a graph G induced by a set $Z \subseteq E(G)$ is denoted by $G[Z]$. For integers p and q, the set of all integers z satisfying $p \le z \le q$ is indicated by $[p,q]$.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e\in E(G)} \eta(v,e)f(e), \quad \text{for every } v \in V(G), \tag{1.1}$$

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where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies:

$$f^*(v) = \lambda, \quad \text{for all } v \in V(G). \tag{1.2}$$

A magic labelling f of G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. A graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

A bijection f from $E(G)$ onto $[1,|E(G)|]$ is called a *degree-magic labelling* (or only d*-magic labelling*) of a graph G if its index-mapping f^* satisfies:

$$f^*(v) = \frac{1+|E(G)|}{2}\deg(v), \quad \text{for all } v \in V(G). \tag{1.3}$$

A graph G is said to be *degree-magic* (or only d*-magic*) when a d-magic labelling of G exists.

The concept of magic graphs was put forward by Sedláček [9]. Later, supermagic graphs were introduced by Stewart [10]. Currently, numerous papers are published on magic and supermagic graphs (see [1, 3–7] for more comprehensive references). The thought of degree-magic graphs was then introduced by Bezegová and Ivančo [2]. Degree-magic graphs extend supermagic regular graphs because the following result holds.

Theorem 1.1 ([2]). *Let G be a regular graph. Then G is supermagic if and only if it is degree-magic.*

Suppose that $q \geq 2$ is an integer. A spanning subgraph H of a graph G is called a $\frac{1}{q}$*-factor* of G whenever $\deg_H(v) = \deg_G(v)/q$ for every vertex $v \in V(G)$. A bijection f from $E(G)$ onto $[1,|E(G)|]$ is called q*-gradual* if the set

$$F_q(f;i) := \{e \in E(G) : (i-1)|E(G)|/q < f(e) \leq i|E(G)|/q\}$$

induces a $\frac{1}{q}$-factor of G for each $i \in [1,q]$. A graph G is said to be *balanced degree-magic* if a 2-gradual d-magic labelling of G exists. A notion of a q-gradual bijection of a graph G was recommended by Ivančo [8]. Some properties of balanced d-magic graphs were described in [2] and [3]. However, the concept of a q-gradual labelling seems to be useful as well for $q > 2$.

The graph obtained by replacing each edge uv of a graph G with two edges joining u and v is denoted by ${}^2G$. Hence, $V({}^2G) = V(G)$ and $E({}^2G) = \bigcup_{e \in E(G)} \{(e,1),(e,2)\}$, where an edge (e,i), $i \in \{1,2\}$, is incident with a vertex v in ${}^2G$ whenever e is incident with v in G. In this case, $E_i({}^2G) := \bigcup_{e \in E(G)} \{(e,i)\}$, $i = 1,2$. Evidently, the subgraph of ${}^2G$ induced by $E_i({}^2G)$, $i = 1,2$, is isomorphic to G.

Let G be a graph. Suppose that $U \subseteq V(G)$ and $Z \subseteq E(G)$. A graph $D = D(G;Z,U)$ is defined by

$$V(D) = \bigcup_{v \in V(G)} \{v^0, v^1\}$$

and

$$E(D) = \bigcup_{vu \in Z} \{v^0u^0, v^1u^1\} \cup \bigcup_{vu \in E(G)-Z} \{v^0u^1, v^1u^0\} \cup \bigcup_{u \in U} \{u^0u^1\}.$$

The graph $D(G;Z,U)$ is called a *generalized double graph* because these cases hold. (i) The graph $D(G;E(G),\emptyset)$ consists of two disjoint copies of G, i.e., it is isomorphic to $2G$. (ii) The graph $D(G;E(G),V(G))$ is the Cartesian product of G and K_2. (iii) The graph $D(G;\emptyset,\emptyset)$ is the categorical product of G and K_2, also called the *bipartite double graph* of a graph G. (iv) The graph $D(^2G;E_1(^2G),\emptyset)$ is the lexicographic product (or composition) of G and $\overline{K}_2$, also called the *double graph* of a graph G.

An idea of generalized double graphs was presented by Ivančo [8]. Some essential results are proved and some constructions of supermagic and degree-magic labellings of some graphs generalizing double graphs are also introduced in [8].

In this paper, some constructions of degree-magic and supermagic labellings on some graphs obtained by generalizing the double graph of the disjoint union of n copies of a graph are shown.

2. A Generalization of the Double Graph of the Disjoint Union of a Graph

Let G be a graph. Suppose that $U \subseteq V(G)$ and $Z \subseteq E(G)$. For any integer $n \geq 2$ and $t \in [1,n]$, let G^t, U^t and Z^t be the $t^{\underline{th}}$ copies of G, U and Z, respectively. Let $e^t \in E(G^t)(v^t \in V(G^t))$ be an edge (vertex) of G^t corresponding to $e \in E(G)\big(v \in V(G)\big)$. The disjoint unions of n copies of G, U and Z are denoted by $nG = G^1 \cup G^2 \cup \cdots \cup G^n$, $nU = U^1 \cup U^2 \cup \cdots \cup U^n$ and $nZ = Z^1 \cup Z^2 \cup \cdots \cup Z^n$, respectively. A graph $D = D(nG;nZ,nU)$ is defined by

$$V(D) = \bigcup_{v \in V(G), t \in [1,n]} \{v^{t0}, v^{t1}\}$$

and

$$E(D) = \bigcup_{vu \in Z, t \in [1,n]} \{v^{t0}u^{t0}, v^{t1}u^{t1}\} \cup \bigcup_{vu \in E(G)-Z, t \in [1,n]} \{v^{t0}u^{t1}, v^{t1}u^{t0}\} \cup \bigcup_{u \in U, t \in [1,n]} \{u^{t0}u^{t1}\}.$$

Therefore, the graph $D(nG;nZ,nU)$ is a generalization of the double graph of a graph nG.

Now, some vital findings are presented in this paper.

Lemma 2.1. *Let G be a graph such that* $\deg(v) \equiv 0 \pmod 2$ *for every vertex $v \in V(G)$. Suppose that the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. Then for any bijection $f: E(G) \longrightarrow [1,|E(G)|]$, there exists a 2-gradual bijection $g: E(D(nG;nZ,\emptyset)) \longrightarrow [1,2|E(nG)|]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^{t0}) = g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2n-1)|E(G)|\deg(v).$$

Proof. The subgraph $G[Z]$ of a graph G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. Then, there is a set $Z_1 \subseteq Z$ such that the subgraph of $G[Z]$ induced by Z_1 is a $\frac{1}{2}$-factor of $G[Z]$. Obviously, the subgraph of $G[Z]$ induced by $Z_2 = Z - Z_1$ is also a $\frac{1}{2}$-factor of $G[Z]$. Moreover, the degree of each vertex of $G[Z]$ is even as well as the degree of each vertex of $H = G[E(G)-Z]$ is even. It means that every component of H is Eulerian. Therefore, there is a digraph $\vec{H}$ gotten from H by an orientation of its edges such that the outdegree of every vertex of $\vec{H}$ is equal to its indegree. Let $[u,v]$ be an arc of $\vec{H}$ and let $A(\vec{H})$ be the set of all arcs of $\vec{H}$. For any integer $n \geq 2$ and $t \in [1,n]$, put $m := |E(G)|$ and $D := D(nG; nZ, \emptyset)$. Consider the bijection $g : E(D) \longrightarrow [1, 2mn]$ given by

$$g(u^{ti}v^{tj}) = \begin{cases} f(uv) + (t-1)m & \text{if } i = 0,\ j = 1,\ [u,v] \in A(\vec{H}), \\ f(uv) + (2n-t)m & \text{if } i = 1,\ j = 0,\ [u,v] \in A(\vec{H}), \\ f(uv) + (t-1)m & \text{if } i = j = 0,\ uv \in Z_1, \\ f(uv) + (2n-t)m & \text{if } i = j = 1,\ uv \in Z_1, \\ f(uv) + (t-1)m & \text{if } i = j = 1,\ uv \in Z_2, \\ f(uv) + (2n-t)m & \text{if } i = j = 0,\ uv \in Z_2. \end{cases}$$

For its index-mapping, one then has

$$\begin{aligned} g^*(v^{t0}) &= \sum_{[v,w] \in A(\vec{H})} g(v^{t0}w^{t1}) + \sum_{[w,v] \in A(\vec{H})} g(w^{t1}v^{t0}) + \sum_{vw \in Z_1} g(v^{t0}w^{t0}) + \sum_{vw \in Z_2} g(v^{t0}w^{t0}) \\ &= \sum_{[v,w] \in A(\vec{H})} (f(vw) + (t-1)m) + \sum_{[w,v] \in A(\vec{H})} (f(wv) + (2n-t)m) \\ &\quad + \sum_{vw \in Z_1} (f(vw) + (t-1)m) + \sum_{vw \in Z_2} (f(vw) + (2n-t)m) \\ &= \sum_{vw \in E(G)} f(vw) + (2n-1)m \frac{\deg(v)}{2} = f^*(v) + \frac{1}{2}(2n-1)m \deg(v) \end{aligned}$$

for every vertex $v^{t0} \in V(D)$. Similarly, $g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2n-1)m \deg(v)$ is obtained for every vertex $v^{t1} \in V(D)$. Since the outdegree of every vertex of $\vec{H}$ is equal to its indegree and the sets Z_1 and Z_2 induce $\frac{1}{2}$-factors of $G[H]$, the sets

$$F_2(g;1) = \{u^{t0}v^{t1}; [u,v] \in A(\vec{H}), t \in [1,n]\} \cup \{u^{t0}v^{t0}; uv \in Z_1, t \in [1,n]\} \cup \{u^{t1}v^{t1}; uv \in Z_2, t \in [1,n]\}$$

and

$$F_2(g;2) = \{u^{t1}v^{t0}; [u,v] \in A(\vec{H}), t \in [1,n]\} \cup \{u^{t1}v^{t1}; uv \in Z_1, t \in [1,n]\} \cup \{u^{t0}v^{t0}; uv \in Z_2, t \in [1,n]\}$$

induce $\frac{1}{2}$-factors of D. □

Lemma 2.2. *Let $q \geq 2$ be a positive integer and let G be a graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. Then for any q-gradual bijection $f : E(G) \longrightarrow [1, |E(G)|]$, there exists a 2-gradual bijection g from $E(D(nG; \emptyset, \emptyset))$ onto $\left[1, 2|E(nG)|\right]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^{t0}) = g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2n-1)|E(G)| \deg(v).$$

Proof. Because $\deg(v) \equiv 0 \pmod{2q}$ for every vertex $v \in V(G)$, the degree of each vertex of $H_i = G[F_q(f;i)]$, $i \in [1,q]$, is even. Therefore, there is a digraph $\vec{H}_i$ which it is obtained from H_i by an orientation of its edges such that the outdegree of every vertex of $\vec{H}_i$ is equal to its indegree. Let $\vec{H}$ be an orientation of G such that the set $A(\vec{H})$ of all arcs of $\vec{H}$ is equal to $\bigcup_{i=1}^{q} A(\vec{H}_i)$. For any integer $n \geq 2$ and $t \in [1,n]$, put $m := |E(G)|$ and $D := D(nG; \emptyset, \emptyset)$. Consider the bijection $g : E(D) \longrightarrow [1, 2mn]$ given by

$$g(u^{tj}v^{tk}) = \begin{cases} f(uv) + (t-1)m & \text{if } j = 0,\ k = 1,\ [u,v] \in A(\vec{H}), \\ f(uv) + (2n-t)m & \text{if } j = 1,\ k = 0,\ [u,v] \in A(\vec{H}). \end{cases}$$

For its index-mapping, one obtains

$$\begin{aligned} g^*(v^{t0}) &= \sum_{[v,w]\in A(\vec{H})} g(v^{t0}w^{t1}) + \sum_{[w,v]\in A(\vec{H})} g(w^{t1}v^{t0}) \\ &= \sum_{[v,w]\in A(\vec{H})} (f(vw)+(t-1)m) + \sum_{[w,v]\in A(\vec{H})} (f(wv)+(2n-t)m) \\ &= \sum_{vw\in E(G)} f(vw) + (2n-1)m\frac{\deg(v)}{2} \\ &= f^*(v) + \frac{1}{2}(2n-1)m\deg(v) \end{aligned}$$

for every vertex $v^{t0} \in V(D)$. Likewise, $g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2n-1)m\deg(v)$ is gotten for every vertex $v^{t1} \in V(D)$. Furthermore, the outdegree of every vertex of $\vec{H}_i$ is equal to its indegree, and so the sets

$$F_2(g;1) = \{u^{t0}v^{t1}; [u,v] \in A(\vec{H}), t \in [1,n]\} \quad \text{and}$$
$$F_2(g;2) = \{u^{t1}v^{t0}; [u,v] \in A(\vec{H}), t \in [1,n]\}$$

induce $\frac{1}{2}$-factors of D. □

Lemma 2.3. *Let $q \geq 3$ be an odd positive integer. Then for any q-gradual bijection $f : E(G) \longrightarrow [1,|E(G)|]$, there exists a q-gradual bijection*

$$g : E(qG) \longrightarrow [1,|E(qG)|]$$

such that for every vertex $v \in V(G)$ it holds

$$g^*(v^t) = f^*(v) + \frac{1}{2}(q-1)|E(G)|\deg(v).$$

Proof. For any $t \in [1,q]$, put $m := |E(G)|$. Consider the bijection g from $E(qG)$ onto $[1,|E(qG)|]$ given by

$$g(u^tv^t) = \begin{cases} f(uv)+(i-1)m & \text{if } uv \in F_q(f,i),\ i \in [1,q],\ t=1, \\ f(uv)+(i)m & \text{if } uv \in F_q(f,i),\ i \in [1,q-1],\ t=2, \\ f(uv)+(i-q)m & \text{if } uv \in F_q(f,i),\ i=q,\ t=2, \\ f(uv)+(i+1)m & \text{if } uv \in F_q(f,i),\ i \in [1,q-2],\ t=3, \\ f(uv)+(i+1-q)m & \text{if } uv \in F_q(f,i),\ i \in [q-1,q],\ t=3, \\ \vdots & \\ f(uv)+(i-3+q)m & \text{if } uv \in F_q(f,i),\ i \in [1,2],\ t=q-1, \\ f(uv)+(i-3)m & \text{if } uv \in F_q(f,i),\ i \in [3,q],\ t=q-1, \\ f(uv)+(i-2+q)m & \text{if } uv \in F_q(f,i),\ i=1,\ t=q, \\ f(uv)+(i-2)m & \text{if } uv \in F_q(f,i),\ i \in [2,q],\ t=q. \end{cases}$$

Consider $t = 1$, for its index-mapping one receives

$$\begin{aligned} g^*(v^1) &= \sum_{i=1}^{q} \sum_{vw\in F_q(f;i)} g(v^1w^1) \\ &= \sum_{i=1}^{q} \sum_{vw\in F_q(f;i)} f(vw) + \sum_{i=1}^{q} (i-1)m\frac{\deg(v)}{q} \end{aligned}$$

$$= \sum_{vw \in E(G)} f(vw) + \left(\frac{1}{2}q(q+1) - q\right) m \frac{\deg(v)}{q}$$
$$= f^*(v) + \frac{1}{2}(q-1)m\deg(v).$$

Likewise, $g^*(v^t) = f^*(v) + \frac{1}{2}(q-1)m\deg(v)$ is obtained for $t \in [2,q]$. Moreover, for $i \in [1,q]$ the sets

$$F_q(g;1) = \{u^1v^1 : uv \in F_q(f;1)\} \cup \{u^{q+2-i}v^{q+2-i} : uv \in F_q(f;i), i \in [2,q]\},$$
$$F_q(g;2) = \{u^{3-i}v^{3-i} : uv \in F_q(f;i), i \in [1,2]\} \cup \{u^{q+3-i}v^{q+3-i} : uv \in F_q(f;i), i \in [3,q]\}, \ldots,$$
$$F_q(g;q-1) = \{u^{q-i}v^{q-i} : uv \in F_q(f;i), i \in [1,q-1]\} \cup \{u^qv^q : uv \in F_q(f;q)\}$$

and

$$F_q(g;q) = \{u^{q+1-i}v^{q+1-i} : uv \in F_q(f;i), i \in [1,q]\}$$

induce $\frac{1}{q}$-factors of qG. □

Lemma 2.4. *Let $q \geq 3$ be an odd positive integer and let G be a graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. Suppose that the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. Then for any q-gradual bijection $f : E(G) \longrightarrow [1,|E(G)|]$, there exists a bijection $g : E(D(qG;qZ,\emptyset)) \longrightarrow [1,2|E(qG)|]$ such that for every vertex $v \in V(G)$ it holds*

$$g^*(v^{t0}) = g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2q-1)|E(G)|\deg(v).$$

Proof. The subgraph $G[Z]$ of a graph G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. Then, there is a set $Z_1 \subseteq Z$ such that the subgraph of $G[Z]$ induced by Z_1 is a $\frac{1}{2}$-factor of $G[Z]$. Obviously, the subgraph of $G[Z]$ induced by $Z_2 = Z - Z_1$ is also a $\frac{1}{2}$-factor of $G[Z]$. Moreover, the degree of each vertex of $G[Z]$ as well as the degree of each vertex of $H = G[E(G) - Z]$ is even. Therefore, there is a digraph $\vec{H}$ obtained from H by an orientation of its edges such that the outdegree of every vertex of $\vec{H}$ is equal to its indegree. Let $[u,v]$ be an arc of $\vec{H}$ and let $A(\vec{H})$ be the set of all arcs of $\vec{H}$.

For any integer $t \in [1,q]$, put $m := |E(G)|$ and $D := D(qG;qZ,\emptyset)$. Since f is a q-gradual bijection from $E(G)$ onto $[1,|E(G)|]$, according to Lemma 2.3 there exists a q-gradual bijection $f_1 : E(qG) \longrightarrow \left[1,|E(qG)|\right]$ such that

$$f_1^*(v^t) = f^*(v) + \frac{1}{2}(q-1)m\deg(v)$$

for every vertex $v \in V(G)$. Consider the bijection $g : E(D) \longrightarrow [1,2qm]$ given by

$$g(u^{ti}v^{tj}) = \begin{cases} f_1(u^tv^t) & \text{if } i=0,\ j=1,\ [u,v] \in A(\vec{H}), \\ f_1(u^tv^t) + qm & \text{if } i=1,\ j=0,\ [u,v] \in A(\vec{H}), \\ f_1(u^tv^t) & \text{if } i=j=0,\ uv \in Z_1, \\ f_1(u^tv^t) + qm & \text{if } i=j=1,\ uv \in Z_1, \\ f_1(u^tv^t) & \text{if } i=j=1,\ uv \in Z_2, \\ f_1(u^tv^t) + qm & \text{if } i=j=0,\ uv \in Z_2. \end{cases}$$

For its index-mapping, one then has

$$\begin{aligned}
g^*(v^{t0}) &= \sum_{[v,w]\in A(\vec{H})} g(v^{t0}w^{t1}) + \sum_{[w,v]\in A(\vec{H})} g(w^{t1}v^{t0}) + \sum_{vw\in Z_1} g(v^{t0}w^{t0}) + \sum_{vw\in Z_2} g(v^{t0}w^{t0}) \\
&= \sum_{[v,w]\in A(\vec{H})} f_1(v^tw^t) + \sum_{[w,v]\in A(\vec{H})} \big(f_1(w^tv^t)+qm\big) \\
&\quad + \sum_{vw\in Z_1} f_1(v^tw^t) + \sum_{vw\in Z_2} \big(f_1(v^tw^t)+qm\big) \\
&= \sum_{vw\in E(G)} f_1(v^tw^t) + qm\frac{\deg(v)}{2} \\
&= f_1^*(v^t) + \frac{1}{2}qm\deg(v) \\
&= f^*(v) + \frac{1}{2}(q-1)m\deg(v) + \frac{1}{2}qm\deg(v) \\
&= f^*(v) + \frac{1}{2}(2q-1)m\deg(v)
\end{aligned}$$

for every vertex $v^{t0}\in V(D)$. Similarly, $g^*(v^{t1}) = f^*(v)+\frac{1}{2}(2q-1)m\deg(v)$ is obtained for every vertex $v^{t1}\in V(D)$. □

3. Degree-Magic and Supermagic Graphs

In this section, some sufficient conditions of some graphs obtained by generalizing the double graph of the disjoint union of a graph $D(nG;nZ,\emptyset)$ to be degree-magic are presented.

Theorem 3.1. *Let G be a degree-magic graph such that $\deg(v)\equiv 0 \pmod 2$ for every vertex $v\in V(G)$. If the subgraph of G induced by a set $Z\subseteq E(G)$ has a $\frac{1}{2}$-factor, then the graph $D(nG;nZ,\emptyset)$ of a graph G is balanced degree-magic.*

Proof. Since G is a d-magic graph, there is a d-magic labelling f from $E(G)$ onto $[1,|E(G)|]$. According to Lemma 2.1, there exists a 2-gradual bijection $g: E(D(nG;nZ,\emptyset)) \longrightarrow [1,2|E(nG)|]$ satisfying

$$g^*(v^{t0}) = g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2n-1)|E(G)|\deg(v)$$

for every vertex $v\in V(G)$. As f is a d-magic labelling, $f^*(v)=\frac{1}{2}(1+|E(G)|)\deg(v)$. Hence,

$$\begin{aligned}
g^*(v^{t0}) = g^*(v^{t1}) &= \frac{1}{2}(1+|E(G)|)\deg(v) + \frac{1}{2}(2n-1)|E(G)|\deg(v) \\
&= \frac{1}{2}(1+2n|E(G)|)\deg(v) = \frac{1}{2}(1+|E(D(nG;nZ,\emptyset))|)\deg(v).
\end{aligned}$$

Therefore, g is a 2-gradual d-magic labelling of $D(nG;nZ,\emptyset)$. □

Combining Theorem 1.1 and Theorem 3.1, one certainly has

Corollary 3.1. *Let G be a supermagic regular graph of even degree. If the subgraph of G induced by a set $Z\subseteq E(G)$ has a $\frac{1}{2}$-factor, then the graph $D(nG;nZ,\emptyset)$ of a graph G is supermagic.*

A totally disconnected graph has a $\frac{1}{2}$-factor and one then obtains

Corollary 3.2. *Let G be a supermagic regular graph of even degree. Then the graph $D(nG;\emptyset,\emptyset)$ of a graph G is supermagic.*

In the next result, a sufficient condition for a graph $D(n^2G;nE_1(^2G),\emptyset)$ to be balanced degree-magic is proved.

Corollary 3.3. *Let G be a graph having a $\frac{1}{2}$-factor. Then the graph $D(n^2G;nE_1(^2G),\emptyset)$ of a graph G is balanced degree-magic.*

Proof. Let g be a bijection from $E(G)$ onto $[1,|E(G)|]$. Consider a mapping $f: E(^2G) \longrightarrow [1,2|E(G)|]$ given by

$$f((e,j)) = \begin{cases} g(e) & \text{if } j=1, \\ 1+2|E(G)|-g(e) & \text{if } j=2. \end{cases}$$

Evidently, f is a bijection. Moreover, $f((e,1))+f((e,2)) = 1+2|E(G)|$ for every edge $e \in E(G)$. Thus,

$$f^*(v) = (1+2|E(G)|)\deg_G(v) = \frac{1}{2}(1+|E(^2G)|)\deg_{^2G}(v).$$

Hence, f is a degree-magic labelling of ^2G. Because the subgraph of ^2G induced by $E_1(^2G)$ is isomorphic to G, it contains a $\frac{1}{2}$-factor. By Theorem 3.1, $D(n^2G;nE_1(^2G),\emptyset)$ is a balanced d-magic graph. □

Corollary 3.4. *Let G be a graph such that* $\deg(v) \equiv 0 \pmod 2$ *for every vertex $v \in V(G)$ and let $q \geq 2$ be an even positive integer. If G can be decomposed into q pairwise edge-disjoint $\frac{1}{q}$-factors, then the graph $D(n^2G;nE_1(^2G),\emptyset)$ of a graph G is balanced degree-magic.*

Proof. Since the union of $q/2$ edge-disjoint $\frac{1}{q}$-factors induces a $\frac{1}{2}$-factor of G, According to Corollary 3.3, $D(n^2G;nE_1(^2G),\emptyset)$ is a balanced d-magic graph. □

Since any regular graph of even degree d is decomposable into $d/2$ pairwise edge-disjoint 2-factors (i.e., $\frac{1}{d/2}$-factors), one suddenly gets

Corollary 3.5. *Let G be a regular graph of degree d, where* $4 \leq d \equiv 0 \pmod 4$. *Then the graph $D(n^2G;nE_1(^2G),\emptyset)$ of a graph G is supermagic.*

Combining Theorem 1.1 and Corollary 3.3, one immediately has

Corollary 3.6. *Let G be a regular graph having a $\frac{1}{2}$-factor. Then the graph $D(n^2G;nE_1(^2G),\emptyset)$ of a graph G is supermagic.*

Now, a sufficient condition for a generalization of the disjoint union of a graph $D(nG;\emptyset,\emptyset)$ to be degree-magic is shown.

Theorem 3.2. *Let $q \geq 2$ be a positive integer and let G be a graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. If G admits a q-gradual degree-magic labelling, then the graph $D(nG;\emptyset,\emptyset)$ of a graph G is balanced degree-magic.*

Proof. Suppose that f is a q-gradual d-magic labelling of G. According to Lemma 2.2, there exists a 2-gradual bijection $g : E(D(nG;\emptyset,\emptyset)) \longrightarrow [1, 2|E(nG)|]$ satisfying

$$g^*(v^{t0}) = g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2n-1)|E(G)|\deg(v)$$

for every vertex $v \in V(G)$. Since f is a d-magic labelling, $f^*(v) = \frac{1}{2}(1+|E(G)|)\deg(v)$. Thus,

$$\begin{aligned} g^*(v^{t0}) = g^*(v^{t1}) &= \frac{1}{2}(1+|E(G)|)\deg(v) + \frac{1}{2}(2n-1)|E(G)|\deg(v) \\ &= \frac{1}{2}(1+2n|E(G)|)\deg(v) = \frac{1}{2}(1+|E(D(nG;\emptyset,\emptyset))|)\deg(v). \end{aligned}$$

Therefore, g is a 2-gradual d-magic labelling of $D(nG;\emptyset,\emptyset)$. □

Combining Theorem 1.1 and Theorem 3.2, one immediately has

Corollary 3.7. *Let $q \geq 2$ be a positive integer and let G be a regular graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. If G admits a q-gradual supermagic labelling, then the graph $D(nG;\emptyset,\emptyset)$ of a graph G is supermagic.*

In the next result, for any odd positive integer $q \geq 3$ a sufficient condition for the generalization of the double graph of the disjoint union of a graph $D(qG;qZ,\emptyset)$ to be degree-magic is presented.

Theorem 3.3. *Let $q \geq 3$ be an odd positive integer and let G be a graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. Let the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. If G admits a q-gradual degree-magic labelling, then the graph $D(qG;qZ,\emptyset)$ is degree-magic.*

Proof. Suppose that f is a q-gradual d-magic labelling of G. According to Lemma 2.4, there exists a bijection $g : E(D(qG;qZ,\emptyset)) \longrightarrow [1, 2|E(qG)|]$ satisfying

$$g^*(v^{t0}) = g^*(v^{t1}) = f^*(v) + \frac{1}{2}(2q-1)|E(G)|\deg(v)$$

for every vertex $v \in V(G)$. Since f is a d-magic labelling, $f^*(v) = \frac{1}{2}(1+|E(G)|)\deg(v)$. Thus,

$$\begin{aligned} g^*(v^{t0}) &= g^*(v^{t1}) \\ &= \frac{1}{2}(1+|E(G)|)\deg(v) + \frac{1}{2}(2q-1)|E(G)|\deg(v) \\ &= \frac{1}{2}(1+2q|E(G)|)\deg(v) \\ &= \frac{1}{2}(1+|E(D(qG;qZ,\emptyset))|)\deg(v). \end{aligned}$$

Therefore, g is a d-magic labelling of $D(qG;qZ,\emptyset)$. □

Combining Theorem 1.1 and Theorem 3.3, one certainly has

Corollary 3.8. *Let $q \geq 3$ be an odd positive integer and let G be a regular graph such that* $\deg(v) \equiv 0 \pmod{2q}$ *for every vertex $v \in V(G)$. Let the subgraph of G induced by a set $Z \subseteq E(G)$ has a $\frac{1}{2}$-factor. If G admits a q-gradual supermagic labelling, then the graph $D(qG;qZ,\emptyset)$ is supermagic.*

Now, a sufficient condition for the graph $D(q^2G;qE_1(^2G),\emptyset)$ to be degree-magic is indicated.

Corollary 3.9. *Let $q \ge 3$ be an odd positive integer and let G be a graph having a $\frac{1}{2}$-factor such that $\deg(v) \equiv 0 \pmod 2$ for every vertex $v \in V(G)$. If G can be decomposed into q pairwise edge-disjoint $\frac{1}{q}$-factors, then the graph $D(q^2G;qE_1(^2G),\emptyset)$ of a graph G is degree-magic.*

Proof. Evidently, $\deg(v) \equiv 0 \pmod{2q}$ for every vertex $v \in V(G)$. Let $H_1,H_2,\dots,H_q$ be pairwise edge-disjoint $\frac{1}{q}$-factors of a graph G. Put $m := |E(G)|/q$. Clearly, the subgraph H_i, $i \in [1,q]$, has m edges. Suppose that h_i is a bijection from $E(H_i)$ onto $[1,m]$, for $i \in [1,q]$. Consider a mapping $f : E(^2G) \longrightarrow [1,2qm]$ given by

$$f((e,j)) = \begin{cases} h_i(e)+(i-1)m & \text{if } j=1 \text{ and } e \in E(H_i), \\ 1+(1+2q-i)m-h_i(e) & \text{if } j=2 \text{ and } e \in E(H_i). \end{cases}$$

Evidently, f is a bijection. Moreover, $f((e,1))+f((e,2)) = 1+2qm$ for every edge $e \in E(G)$. Thus,

$$f^*(v) = (1+2qm)\deg_G(v) = \frac{1}{2}(1+|E(^2G)|)\deg_{^2G}(v).$$

Moreover, the sets

$$F_q(f;1) = \{(e,1) \in E(^2G) : e \in E(H_1)\cup E(H_2)\},$$
$$F_q(f;2) = \{(e,1) \in E(^2G) : e \in E(H_3)\cup E(H_4)\},\dots,$$
$$F_q\left(f;\frac{q+1}{2}\right) = \{(e,1),(e,2) \in E(^2G) : e \in E(H_q)\},\dots,$$
$$F_q(f;q-1) = \{(e,2) \in E(^2G) : e \in E(H_4)\cup E(H_3)\}$$

and

$$F_q(f;q) = \{(e,2) \in E(^2G) : e \in E(H_2)\cup E(H_1)\}$$

induce $\frac{1}{q}$-factors of ^2G. Hence, f is a q-gradual d-magic labelling of ^2G. Since the subgraph of ^2G induced by $E_1(^2G)$ is isomorphic to G, it contains a $\frac{1}{2}$-factor. By Theorem 3.3, $D(q^2G;qE_1(^2G),\emptyset)$ is a d-magic graph. □

As any regular graph of even degree d is decomposable into $d/2$ pairwise edge-disjoint 2-factors (i.e., $\frac{1}{d/2}$-factors), one also gets

Corollary 3.10. *Let G be a regular graph of degree d, where $6 \le d \equiv 2 \pmod 4$. Then the graph $D(\frac{d}{2}{}^2G;\frac{d}{2}E_1(^2G),\emptyset)$ of a graph G is supermagic.*

For any even positive interger $m \ge 4$, the graph obtained by replacing each edge uv of a graph G with m edges joining u and v is denoted by ^mG. Hence, $V(^mG) = V(G)$ and $E(^mG) = \bigcup_{e\in E(G)} \{(e,1),(e,2),\dots,(e,m)\}$, where an edge (e,i), $i \in \{1,2,\dots,m\}$, is incident with a vertex v in ^mG whenever e is incident with v in G. Also, in this case $E_i(^mG) := \bigcup_{e\in E(G)} \{(e,i)\}$, $i = 1,2,\dots,m$. Certainly, the subgraph of ^mG induced by $E_i(^mG)$, $i = 1,2,\dots,m$, is isomorphic to G.

This paper is concluded with proving a sufficient condition for a graph $D(n^mG;nE_1(^mG),\emptyset)$ to be balanced degree-magic for any even positive integer $m \ge 4$.

Corollary 3.11. *Let $m \geq 4$ be an even positive integer and let G be a graph having a $\frac{1}{2}$-factor. Then the graph $D(n^mG; nE_1(^mG), \emptyset)$ of a graph G is balanced degree-magic.*

Proof. Let g be a bijection from $E(G)$ onto $[1, |E(G)|]$. Consider a mapping $f : E(^mG) \longrightarrow [1, m|E(G)|]$ given by

$$f((e,j)) = \begin{cases} g(e) + (j-1)|G(E)| & \text{if } j = 1, 2, \ldots, m/2, \\ 1 + j|E(G)| - g(e) & \text{if } j = 1 + m/2, \ldots, m. \end{cases}$$

Evidently, f is a bijection. Moreover,

$$f((e,1)) + f((e,2)) + \ldots + f((e,m)) = \frac{m}{2} + \frac{m^2}{2}|E(G)|$$

for every edge $e \in E(G)$. Thus,

$$\begin{aligned} f^*(v) &= \left(\frac{m}{2} + \frac{m^2}{2}|E(G)|\right)\deg_G(v) \\ &= \left(\frac{m}{2} + \frac{m^2}{2}|E(G)|\right)\frac{\deg_{^mG}(v)}{m} \\ &= \frac{1}{2}\big(1 + m|E(G)|\big)\deg_{^mG}(v) \\ &= \frac{1}{2}\big(1 + |E(^mG)|\big)\deg_{^mG}(v). \end{aligned}$$

Therefore, f is a degree-magic labelling of ^mG. Since the subgraph of ^mG induced by $E_1(^mG)$ is isomorphic to G, it contains a $\frac{1}{2}$-factor. By Theorem 3.1, $D(n^mG; nE_1(^mG), \emptyset)$ is a balanced d-magic graph. □

Combining Theorem 1.1 and Corollary 3.11, one immediately has

Corollary 3.12. *Let $m \geq 4$ be an even positive integer and let G be a regular graph having a $\frac{1}{2}$-factor. Then the graph $D(n^mG; nE_1(^mG), \emptyset)$ of a graph G is supermagic.*

4. Conclusion

In this paper, some constructions of degree-magic and supermagic labellings on some graphs obtained by generalizing the double graph of the disjoint union of n copies of a graph are presented as well as some supermagic graphs are obtained. However, the labelling of discrete structures is an extensive field of study, so a further open area of research would be to investigate and derive similar results for different families in the context of varying graph-labelling problems.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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