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Research Article

Studies on Coefficient Estimates and Fekete-Szegö Problem for a Class of Bi-Univalent Functions Associated With (*p*,*q*)-Chebyshev Polynomial

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Abstract. In this present work, authors studied and investigated the concept of (p,q)-Chebyshev polynomial of second kind for the subclass of analytic bi-univalent function with respect to the subordination. We give an elementary proof to estimate the coefficient bounds for the bi-univalent functions defined in the open unit disk. Also, we included the result of Fekete-Szegö theorem.

Keywords. Analytic functions; Univalent and bi-univalent functions; Coefficient bounds; Fekete-Szegö problem; (p,q)-Chebyshev polynomials

Mathematics Subject Classification (2020). 30C45; 30C50

1. Introduction

Let $\mathscr{A}$ denote the class of all functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k=z+a_2z^2+a_3z^3+\dots \qquad (z\in\mathbb{U}). \tag{1.1}$$

Further, we shall denote the subclass of all functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$ is denoted by $\mathscr{S}$.

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The well known example for the class $\mathscr{S}$ is the Koebe function

$$K(z)=\frac{z}{(1-z)^2}=\sum_{n=1}^{\infty} nz^n = z+2z^2+3z^3+\cdots \qquad (z\in\mathbb{U})$$

which maps $\mathbb{U}$ onto the complex plane except for a slit along the negative real axis from $w=-\infty$ to $w=-\frac{1}{4}$. The Koebe one-quater theorem [4] states that the image of $\mathbb{U}$ under every function $f(z)$ from $\mathscr{S}$ contains a disk of radius 1/4. Hence every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z))=z$, $(z\in\mathbb{U})$ and

$$f^{-1}(f(w))=w, \quad (|w|<r_0(f), r_0(f)\geq 1/4)$$

where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots. \tag{1.2}$$

Historically speaks, the interest on estimates the initial coefficients $|a_2|$ and $|a_3|$ on bi-univalent functions for the different subclasses are keep on by the researchers in the filed of geometric function theory. The functions in the class Σ, Brannan and Clunie [8], and Srivastava *et al*. [13] proved some results within these coefficient for different classes. Moreover, Brannan and Taha [2] introduced the bi-univalent function class Σ for certain subclasses of $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$. Several authors worked on Chebyshev polynomial expansion to find coefficient estimates for bi-univalent functions defined in the open unit diskṠtill it attracts more attention on researches in this field.

Most recent studies of Kizilates *et al*. [9] and Altinkaya *et al*. [1] motivated us to define the new class of Sakaguchi type function subordinate to $(p,q)-$ Chebyshev polynomials.

Let $f(z)$ and $g(z)$ be two analytic functions, we say that $f(z)$ is subordinate to $g(z)$ written

$$f(z)\prec g(z) \qquad (z\in\mathbb{U})$$

if there exist an analytic function ω such that

$$\omega(0)=0, \qquad |\omega(z)|<1 \text{ and } f(z)=g(\omega(z)).$$

Chebyshev polynomial of the second kind is the natural generalization of Chebyshev polynomial of first kind. It can be used in different areas of mathematics like theory of approximation, linear algebra, discrete analysis, representation theory etc., and also in physics [3, 5–7, 9–12, 14].

For $n\geq 2$ and $0<q<p\leq 1$, the recurrence relation for the (p,q)-Chebyshev polynomials of second kind is defined by:

$$U_n(x,\mathfrak{s},p,q)=(p^n+q^n)xU_{n-1}(x,\mathfrak{s},p,q)+(pq)^{n-1}\mathfrak{s}U_{n-2}(x,\mathfrak{s},p,q),$$

with the initial values $U_0(x,\mathfrak{s},p,q)=1$ and $U_1(x,\mathfrak{s},p,q)=(p+q)x$ and $\mathfrak{s}$ is a variable.

In slight view of this recurrence relation, we will give the list of some special cases of the (p,q)-Chebyshev polynomials of second kind as follows:

(i) $U_n(x/2,\mathfrak{s},p,q)=F_n(x,\mathfrak{s},p,q)$ (p,q)-Fibonacci polynomial,

(ii) $U_n(x,-1,1,1)=U_n(x)$ second kind of Chebyshev polynomials,

(iii) $U_n(x/2,1,1)=F_{n+1}(x)$ Fibonacci polynomials,

(iv) $U_n(1/2,1,1)=F_{n+1}$ Fibonacci numbers,

(v) $U_n(x,1,1,1) = P_{n+1}(x)$ Pell polynomials,

(vi) $U_n(1,1,1,1) = P_{n+1}$ Pell numbers,

(vii) $U_n(1/2,2y,1,1) = J_{n+1}(y)$ Jacobsthal polynomials,

(viii) $U_n(1/2,2,1,1) = J_{n+1}$ Jacobsthal numbers.

Generating functions are very powerful tool in geometric function theory. Let us turn to the concept of defining the generating function of (p,q)-Chebyshev polynomial of second kind.

$$\mathscr{G}_{p,q}(z) = \frac{1}{1 - xpz\eta_p - xqz\eta_q - \mathfrak{s}pqz^2\eta_{p,q}} = \sum_{n=0}^{\infty} U_n(x,\mathfrak{s},p,q)z^n \qquad (z \in \mathbb{U}), \tag{1.3}$$

where $\eta_q f(z) = f(qz)$, known as Fibonacci operator introduced and studied by [11]. Similarly, the operator $\eta_{p,q} f(z) = f(pqz)$.

2. The Class $\mathrm{S}_\Sigma(\mathscr{G},s,t)$ and the Fekete- Szegö Problem

From the above brief introduction, in this section we defining the new class $\mathrm{S}_\Sigma(\mathscr{G},s,t)$ by combining the concept of Sakaguchi type functions and subordination with (p,q)-Chebyshev polynomial.

Definition 1. The function $f \in \Sigma$ is said to be in the class $\mathrm{S}_\Sigma(\mathscr{G},s,t)$ if it holds the following subordination:

$$\frac{(s-t)zf(z)}{f(sz)-f(tz)} \prec \mathscr{G}_{p,q}(z) \qquad (z \in \mathbb{U}), \tag{2.1}$$

and

$$\frac{(s-t)\omega g(\omega)}{g(s\omega)-g(t\omega)} \prec \mathscr{G}_{p,q}(\omega) \qquad (\omega \in \mathbb{U}), \tag{2.2}$$

where $g(\omega) = f^{-1}(\omega)$, $s,t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$.

Remark 1. For $p = q = 1$ and $\mathfrak{s} = -1$, the above class can be reduced to the subclass $\mathrm{S}_\Sigma(\mathscr{H},z,x)$ and defined as

$$\frac{(s-t)zf(z)}{f(sz)-f(tz)} \prec \mathscr{H}(z,x) \quad (z \in \mathbb{U})$$

and

$$\frac{(s-t)\omega g(\omega)}{g(s\omega)-g(t\omega)} \prec \mathscr{H}(\omega,x) \quad (\omega \in \mathbb{U}). \tag{2.3}$$

Let ϕ and φ be two analytic functions defined in $\mathbb{U}$ such that $\phi(0) = \varphi(0) = 0$ and $|\phi(z)| < 1$, $|\varphi(\omega)| < 1$, that is

$$|\phi(z)| = |c_1 z + c_2 z^2 + c_3 z^3 + \ldots| < 1, \qquad (z \in \mathbb{U}), \tag{2.4}$$

$$|\varphi(\omega)| = |d_1 \omega + d_2 \omega^2 + d_3 \omega^3 + \ldots| < 1, \quad (\omega \in \mathbb{U}) \tag{2.5}$$

and it is well known that

$$|c_n| \leq 1, \qquad |d_n| \leq 1,\ n \in \mathbb{N}. \tag{2.6}$$

Theorem 1. *Let $f \in \mathscr{A}$ be in the class $\mathrm{S}_\Sigma(\mathscr{G}, s, t)$. Then*

$$|a_2| \le \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{|(3-2s-2t+st)(p+q)^2x^2-(2-s-t)^2\left((p^2+q^2)(p+q)x^2+pq\mathfrak{s}\right)|}},$$

$$|a_3| \le (p+q)x\left[\frac{1}{(3-s^2-t^2-st)}+\frac{(p+q)x}{(2-s-t)^2}\right]$$

and for any real μ,

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{(p+q)x}{(3-s^2-t^2-st)} & \text{if } |\mu-1| \le \sigma_1 \\ \dfrac{(p+q)^3x^3|\mu-1|}{(3-2s-2t+st)(p^2+q^2)x^2-(2-s-t)^2\left[(p^2+q^2)x^2(p+q)+pq\mathfrak{s}\right]} & \text{if } |\mu-1| \ge \sigma_1 \end{cases}$$

where

$$\sigma_1 = \frac{(3-2s-2t+st)-(2-s-t)^2\left(\frac{(p^2+q^2)}{(p+q)}+\frac{pq\mathfrak{s}}{(p+q)^2x^2}\right)}{(3-s^2-t^2-st)}.$$

Proof. Let $f \in \mathrm{S}_\Sigma(\mathscr{G}, s, t)$, then there exist an analytic functions ϕ and φ defined in (2.4) and (2.5) such that

$$\frac{(s-t)zf(z)}{f(sz)-f(tz)} = \mathscr{G}_{p,q}(\phi(z)) \tag{2.7}$$

and

$$\frac{(s-t)\omega g(\omega)}{g(s\omega)-g(t\omega)} = \mathscr{G}_{p,q}(\varphi(\omega)). \quad \square \tag{2.8}$$

Using the above equations, we have

$$\frac{(s-t)zf(z)}{f(sz)-f(tz)} = U_0(x,\mathfrak{s},p,q)+U_1(x,\mathfrak{s},p,q)\phi(z)+U_2(x,\mathfrak{s},p,q)\phi^2(z)+\dots,$$

$$\frac{(s-t)\omega g(\omega)}{g(s\omega)-g(t\omega)} = U_0(x,\mathfrak{s},p,q)+U_1(x,\mathfrak{s},p,q)\varphi(\omega)+U_2(x,\mathfrak{s},p,q)\varphi^2(\omega)+\dots.$$

Now from the series expansion of $\phi(z)$ and $\varphi(\omega)$, we write

$$\frac{(s-t)zf(z)}{f(sz)-f(tz)} = 1+U_1(x,\mathfrak{s},p,q)c_1z+\left[U_1(x,\mathfrak{s},p,q)c_2+U_2(x,\mathfrak{s},p,q)c_1^2\right]z^2+\dots, \tag{2.9}$$

$$\frac{(s-t)\omega g(\omega)}{g(s\omega)-g(t\omega)} = 1+U_1(x,\mathfrak{s},p,q)d_1\omega+\left[U_1(x,\mathfrak{s},p,q)d_2+U_2(x,\mathfrak{s},p,q)d_1^2\right]\omega^2+\dots. \tag{2.10}$$

If we equate the coefficients of z and z^2 on both sides of (2.9) and (2.10), then we have

$$(2-s-t)a_2 = U_1(x,\mathfrak{s},p,q)c_1, \tag{2.11}$$

$$(3-s^2-t^2-st)a_3-(2s+2t-s^2-t^2-2st)a_2^2 = U_1(x,\mathfrak{s},p,q)c_2+U_2(x,\mathfrak{s},p,q)c_1^2 \tag{2.12}$$

and

$$-(2-s-t)a_2 = U_1(x,\mathfrak{s},p,q)d_1, \tag{2.13}$$

$$(6-s^2-t^2-2s-2t)a_2^2-(3-s^2-t^2-st)a_3 = U_1(x,\mathfrak{s},p,q)d_2+U_2(x,\mathfrak{s},p,q)d_1^2. \tag{2.14}$$

From (2.11) and (2.13), it is clear that

$$c_1 = -d_1. \tag{2.15}$$

Also squaring and adding of (2.11) and (2.13),

$$\frac{2(2-s-t)^2 a_2^2}{U_1^2(x,\mathfrak{s},p,q)} = c_1^2 + d_1^2. \tag{2.16}$$

Now, by adding (2.12) and (2.14) we get

$$2(3-2s-2t+st)a_2^2 = U_1(x,\mathfrak{s},p,q)(c_2+d_2) + U_2(x,\mathfrak{s},p,q)(c_1^2+d_1^2). \tag{2.17}$$

Making use of (2.16) in (2.17)

$$\left[2(3-2s-2t+st)U_1^2(x,\mathfrak{s},p,q) - 2(2-s-t)^2 U_2(x,\mathfrak{s},p,q)\right] a_2^2 = U_1^3(x,\mathfrak{s},p,q)(c_2+d_2), \tag{2.18}$$

$$a_2^2 = \frac{U_1^3(x,\mathfrak{s},p,q)(c_2+d_2)}{\left[2(3-2s-2t+st)U_1^2(x,\mathfrak{s},p,q) - 2(2-s-t)^2 U_2(x,\mathfrak{s},p,q)\right]}. \tag{2.19}$$

From (2.15) and (2.18) together with (2.6), we obtained that

$$|a_2| \le \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{|(3-2s-2t+st)(p+q)^2x^2 - (2-s-t)^2\left((p^2+q^2)(p+q)x^2 + pq\mathfrak{s}\right)|}}. \tag{2.20}$$

In order to estimates the bound on $|a_3|$, we subtract (2.15) from (2.13) and get

$$2(3-s^2-t^2-st)(a_3-a_2^2) = U_1(x,\mathfrak{s},p,q)(c_2-d_2) + U_2(x,\mathfrak{s},p,q)(c_1^2-d_1^2). \tag{2.21}$$

In view of (2.16) and (2.17), the above equation becomes

$$a_3 = \frac{U_1(x,\mathfrak{s},p,q)(c_2-d_2)}{2(3-s^2-t^2-st)} + \frac{U_1^2(x,\mathfrak{s},p,q)(c_1^2+d_1^2)}{2(2-s-t)^2}.$$

Finally, we find the desired inequality using (2.6),

$$|a_3| \le (p+q)x\left[\frac{1}{(3-s^2-t^2-st)} + \frac{(p+q)x}{(2-s-t)^2}\right]. \tag{2.22}$$

For any real μ,

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)a_2^2 + (a_3 - a_2^2) \\ &= \frac{U_1(x,\mathfrak{s},p,q)}{2}\left[\left(y(\mu) + \frac{1}{(3-s^2-t^2-st)}\right)c_2 + \left(y(\mu) - \frac{1}{(3-s^2-t^2-st)}\right)d_2\right], \end{aligned}$$

where

$$y(\mu) = \frac{U_1^2(x,\mathfrak{s},p,q)(1-\mu)}{(3-2s-2t+st)U_1^2(x,\mathfrak{s},p,q) - (2-s-t)^2 U_2(x,\mathfrak{s},p,q)}.$$

Hence, we have reached the desired assertion of the Theorem 1,

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{(p+q)x}{(3-s^2-t^2-st)}, & 0 \le |y(\mu)| \le \frac{1}{(3-s^2-t^2-st)}, \\ (p+q)x|y(\mu)| & |y(\mu)| \le \frac{1}{(3-s^2-t^2-st)}. \end{cases}$$

Next, we have some special case of Theorem 1 for different values of the parameters p, q, $\mathfrak{s}$ as corollaries.

Corollary 1. *Let $f \in \mathscr{A}$ be in the class $\mathrm{S}_\Sigma(\mathscr{H}, z, x)$. Then*

$$|a_2| \le \frac{2x\sqrt{2x}}{\sqrt{|4(3-2s-2t+st)x^2 - (2-s-t)^2(4x^2-1)|}},$$

$$|a_3| \le 2x\left[\frac{1}{(3-s^2-t^2-st)} + \frac{2x}{(2-s-t)^2}\right]$$

and for any real μ,

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{2x}{(3-s^2-t^2-st)} & \text{if } |\mu-1| \le \sigma_2 \\ \dfrac{8x^3|\mu-1|}{(3-2s-2t+st)2x^2-(2-s-t)^2[4x^2(p+q)-1]} & \text{if } |\mu-1| \ge \sigma_2, \end{cases}$$

where

$$\sigma_2 = \frac{(3-2s-2t+st)-(2-s-t)^2(4x^2-1)}{4x^2(3-s^2-t^2-st)}.$$

3. Conclusion

In the field of geometric function theory, the problem of finding Fekete-Szegö result has always been main interest and attention of active researchers. With the concept of (p,q)-Chebyshev polynomial of second kind, we have derived the initial coefficients and Fekete-Szegö theorem for the subclass od bi-univalent function with respect to the subordination defined in the open unit disk.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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