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Research Article

Formulation and Investigation of an Integral Equation for Characteristic Functions of Positive Random Variables

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Abstract. Functional equations of characteristic functions constitute power research tools for establishing new results in several significant areas of probability theory. The present paper makes use of the characteristic functions of two Poisson random sums and the concept of equality in distribution for introducing an important selfdecomposable distribution.

Keywords. Characteristic function; Random sum; Functional equation

MSC. 97K50; 91B70; 60G50; 60E10

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1. Introduction

Let V be a positive random variable with characteristic function $\varphi_V(u)$ and W be a random variable following the power distribution with distribution function $F_W(w) = w^a$, where $0 \le w \le 1$ and a is a positive real number. If V and W are independent then the random variable $C = VW$ is said a-unimodal [2, 3, 7]. It is readily shown that the characteristic function of C has the form

$$\varphi_C(u) = a\int_0^1 \varphi_V(uw)w^{a-1}dw\,.$$

Moreover, let N be a discrete random variable following the Poisson distribution with parameter λ and $\{T_n, n = 1, 2, \ldots\}$ be a sequence of positive and independent random variables distributed as the random variable T with characteristic function $\varphi_T(u)$.

If N, $\{T_n, n = 1,2,\ldots\}$ are independent then the random variable

$$J = T_1 + T_2 + \ldots + T_N$$

is said a Poisson random sum [6]. It is also readily shown that the characteristic function of has the form

$$\varphi_J(u) = \exp\{\lambda[\varphi_T(u) - 1]\}.$$

It is generally recognized that the class of a-unimodal random variables constitute a useful analytical tool for investigating ordinary unimodality, stability, convexity, exponentiality, normality, concavity and other fundamental properties of a wide variety of probability distributions [5]. It is also generally recognized that Poisson random sums are extremely powerful analytical tools for studying infinitely divisible distributions, mixtures of distributions, service systems, power mixtures of distributions, stochastic processes and many other structural classes of probability theory [10].

From a theoretical point of view it can be said that the results of the present paper constitute the continuous analogue of the discrete results established by Artikis and Artikis [4]. The main contribution of this paper is incorporated in the following section.

2. Characterization of a Probability Distribution

The present section of the paper characterizes an important probability distribution by incorporating the fundamental concept of equality in distribution of two Poisson random sums and making use of characteristic functions. More precisely, this section concentrates on the solution of an integral equation for characteristic functions [1].

Theorem. *Let X be a positive random variable with differential infinite divisible characteristic function $\varphi_X(u)$ and let Y be a Poisson random sum with characteristic function*

$$\varphi_Y(u) = \exp\{\lambda[\varphi_X(u) - 1]\}, \quad \lambda > 0.$$

We suppose that S is a positive random variable with finite mean and characteristic function $\varphi_S(u)$ and L is a Poisson random sum of a-unimodal random variables with characteristic function

$$\varphi_L(u) = \exp\left\{\lambda\left[a\int_0^1 \varphi_X(uw)\varphi_S(uw)w^{a-1}dw - 1\right]\right\},$$

then the characteristic function of the random variable X has the form

$$\varphi_X(u) = \exp\left\{a\int_0^u \frac{\varphi_S(w) - 1}{w}dw\right\}$$

if, and only if

$$Y \stackrel{d}{=} L, \tag{2.1}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. If we use the characteristic function $\varphi_Y(u)$ and the characteristic $\varphi_L(u)$ in (2.1) we get the integral equation

$$\exp\{\lambda[\varphi_X(u) - 1]\} = \exp\left\{\lambda\left[a\int_0^1 \varphi_X(uw)\varphi_S(uw)w^{a-1}dw - 1\right]\right\},$$

or equivalently the integral equation

$$\exp\left\{\lambda[\varphi_X(u)-1]\right\}=\exp\left\{\lambda\left[\frac{a}{u^a}\int_0^u \varphi_X(w)\varphi_S(w)w^{a-1}dw-1\right]\right\}. \tag{2.2}$$

It is easily shown that the integral equation in (2.2) can be written in the form

$$\varphi_X(u)=\frac{a}{u^a}\int_0^u \varphi_X(w)\varphi_S(w)w^{a-1}dw. \tag{2.3}$$

Multiplying both sides of the integral equation in (2.3) by u^a, we get the integral equation

$$u^a\varphi_X(u)=a\int_0^u \varphi_X(w)\varphi_S(w)w^{a-1}dw \tag{2.4}$$

and then differentiating the integral equation in (2.4), we get the integral equation

$$au^{a-1}\varphi_X(u)+u^a\frac{d\varphi_X(u)}{du}=a\varphi_X(u)\varphi_S(u)u^{a-1}$$

which for $u\neq 0$ can be written in the form

$$\frac{u}{a}\frac{d\varphi_X(u)}{du}=\varphi_X(u)\varphi_S(u)-\varphi_X(u). \tag{2.5}$$

If we integrate in (2.5) with due regard to the boundary conditions

$$\varphi_X(0)=1$$

and

$$\varphi_L(0)=1,$$

we obtain the characteristic function

$$\varphi_X(u)=\exp\left\{a\int_0^u \frac{\varphi_S(w)-1}{w}dw\right\}.$$

It is of some particular practical importance to mention that the characteristic function $\varphi_X(u)$ has been established as a member of the extremely significant class of selfdecomposable characteristic functions [11]. More precisely, such an establishment substantially extents the practical and theoretical applicability of the random variable X in a very wide variety of scientific disciplines [8, 9, 12].

3. Conclusions

The presence of selfdecomposable distributions in various research areas of probability theory frequently facilitates the formulation, investigation and implementation of analytical activities in such areas. Incorporation of integral equations of characteristic functions, belonging to well known classes of probability distributions, strongly support the role of the property of selfdecomposability as strong research tool.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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