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Research Article

Quaternion Stieltjes Transform and Quaternion Laplace-Stieltjes Transform

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Abstract. In this study, we introduce quaternion Stieltjes transform. Various properties of the quaternion Stieltjes transform are derived. We also define quaternion Laplace-Stieltjes transform and operational properties of quaternion Laplace-Stieltjes transform are established. The theory is supported with applications in mathematical physics.

Keywords. Quaternion Stieltjes transform; Quaternion Laplace-Stieltjes transform; Convolution theorem; Applications

Mathematics Subject Classification (2020). 44A35; 44A05; 46S10

1. Introduction

Stieltjes transform was introduced by T. J. Stieltjes in 1895. The classical work of Stieltjes transform was carried forward in [13] which was followed by [4] and many others. The generalized Stieltjes transform and its inverse was studied in [11]. Stieltjes transform for Boehmians was introduced in [8]. The quaternion Stieltjes transform and it's inverse are derived in the study that are capable in transferring signals from real-valued domain to quaternion frequency domain.

The Laplace-Stieltjes transform, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. The inversion theorem for

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Laplace-Stieltjes transform was studied in [10]. In [1], the author has derived the tauberian theorem for Laplace-Stieltjes transform. Authors in [14, 15] studied the growth properties of different orders of Laplace-Stieltjes transforms.

In this study, the authors introduce quaternion Stieltjes transform. Authors have analysed various properties of the quaternion Stieltjes transform. Quaternion Laplace-Stieltjes transform is defined. Operational properties of quaternion Laplace-Stieltjes transform are derived. In the concluding section, appications in mathematical physics are demonstrated.

2. Preliminary Results

In quaternions, every element is a linear combination of a real scalar and three imaginary units **i**, **j** and **k** with real coefficients.

Let q be a quaternion defined in

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_1 + \mathbf{k}q_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\} \tag{2.1}$$

be the division ring of quaternions, where **i**, **j**, **k** satisfy Hamilton's multiplication rules in [5].

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.2}$$

The quaternion conjugate of q is defined by

$$\bar{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3; \;\; q_0, q_1, q_2, q_3 \in \mathbb{R}. \tag{2.3}$$

The norm of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{2.4}$$

For $f \in L^1(\mathbb{R};\mathbb{H})$, the function is expressed as

$$f(x) = f_0(x) + \mathbf{i}f_1(x) + \mathbf{j}f_2(x) + \mathbf{k}f_3(x). \tag{2.5}$$

Alternatively, in [9] the quaternions are defined as

$$\mathbb{H} = \{q = x_1 + jx_2 : x_1, x_2 \in \mathbb{C}\} \tag{2.6}$$

where j is the imaginary number satisfying following conditions:

$j^2 = -1$, $jr = rj$, $\forall\; r \in \mathbb{R}$, $ji = -ij$, where i is the imaginary number.

Similarly, the quaternion-valued function can be expressed as

$$f(x) = f_1(x) + jf_2(x), \tag{2.7}$$

where f_1 and f_2 are complex functions.

Definition 2.1. Let $f(t)$ be a quaternion-valued function defined for all real numbers $t \geq 0$. Then

$$L^l\{f\}(s) = F(s) = \int_0^\infty e^{-ts} f(t)dt, \tag{2.8}$$

for the quaternions $s \in \mathbb{H}$ such that the integral is convergent, is defined as the left quaternionic Laplace transform of f [3].

3. Main Results

3.1 Quaternion Stieltjes Transform

We use the left quaternionic Laplace transform of (2.8) with respect to s,

$$\begin{aligned} L^l\{F(s)\} &= \int_0^\infty e^{-sr}F(s)ds \\ &= \int_0^\infty e^{-sr}\int_0^\infty e^{-ts}f(t)dtds \\ &= \int_0^\infty \frac{f(t)}{t+r}dt. \end{aligned}$$

Definition 3.1. The *quaternion Stieltjes Transform* (QST) of a quaternion-valued function $f(t)$ on $0 \le t < \infty$ is defined as

$$\mathscr{S}_q\{f(t)\} = \tilde{f}(r) = \int_0^\infty \frac{f(t)}{t+r}dt, \tag{3.1}$$

where $r \in \mathbb{H}$ such that the integral is convergent.

For a quaternion-valued function $f(t)$, the quaternion Stieltjes transform can be represented as

$$\tilde{f}(r) = \tilde{f}_0(r) + \mathbf{i}\tilde{f}_1(r) + \mathbf{j}\tilde{f}_2(r) + \mathbf{k}\tilde{f}_3(r). \tag{3.2}$$

The inverse quaternion Stieltjes transform is defined as

$$f(t) = \mathscr{S}_q^{-1}\{\tilde{f}(r)\} = \frac{1}{4\pi^2 ij}\int_{\lambda-i\infty}^{\lambda+i\infty}\left(\int_{\sigma-j\infty}^{\sigma+j\infty}\tilde{f}(r)e^{rp}dr\right)e^{pt}dp, \tag{3.3}$$

where λ, σ are any suitably chosen large positive constants.

Property 3.2. *For the quaternion-valued function $f(t)$ and $c \in \mathbb{R}$,*

$$\mathscr{S}_q\{f(t+c)\} = \tilde{f}(r-c). \tag{3.4}$$

Proof. By using definition of QST,

$$\mathscr{S}_q\{f(t+c)\} = \int_0^\infty \frac{f(t+c)}{t+r}dt.$$

Substitute $t+c=\xi$, we get

$$\begin{aligned} \mathscr{S}_q\{f(\xi)\} &= \int_0^\infty \frac{f(\xi)}{\xi-c+r}d\xi \\ &= \int_0^\infty \frac{f_0(\xi)}{\xi+(r-c)}d\xi + \mathbf{i}\int_0^\infty \frac{f_1(\xi)}{\xi+(r-c)}d\xi + \mathbf{j}\int_0^\infty \frac{f_2(\xi)}{\xi+(r-c)}d\xi + \mathbf{k}\int_0^\infty \frac{f_3(\xi)}{\xi+(r-c)}d\xi \\ &= \tilde{f}_0(r-c) + \mathbf{i}\tilde{f}_1(r-c) + \mathbf{j}\tilde{f}_2(r-c) + \mathbf{k}\tilde{f}_3(r-c) \\ &= \tilde{f}(r-c). \end{aligned}$$

□

Property 3.3. *For the quaternion-valued function $f(t)$ and $c \in \mathbb{R}$,*

$$\mathscr{S}_q\{f(ct)\} = \tilde{f}(cr). \tag{3.5}$$

Proof. By using definition,

$$\mathscr{S}_q\{f(ct)\} = \int_0^\infty \frac{f(ct)}{t+r}dt.$$

Substitute $ct=\xi$, we get

$$\begin{aligned}\mathscr{S}_q\{f(\xi)\} &= \int_0^\infty \frac{f(\xi)}{\xi+cr}dt\\ &= \int_0^\infty \frac{f_0(\xi)}{\xi+cr}d\xi + \mathbf{i}\int_0^\infty \frac{f_1(\xi)}{\xi+cr}d\xi + \mathbf{j}\int_0^\infty \frac{f_2(\xi)}{\xi+cr}d\xi + \mathbf{k}\int_0^\infty \frac{f_3(\xi)}{\xi+cr}d\xi\\ &= \tilde{f}_0(cr)+\mathbf{i}\tilde{f}_1(cr)+\mathbf{j}\tilde{f}_2(cr)+\mathbf{k}\tilde{f}_3(cr)\\ &= \tilde{f}(cr).\end{aligned}$$

□

Property 3.4. *For the quaternion-valued function $f(t)$,*

$$\mathscr{S}_q\{tf(t)\} = \int_0^\infty f(t)dt - r\tilde{f}(r). \tag{3.6}$$

Proof. By using definition,

$$\begin{aligned}\mathscr{S}_q\{tf(t)\} &= \int_0^\infty \frac{tf(t)}{t+r}dt\\ &= \int_0^\infty \frac{tf_0(t)}{t+r}dt + \mathbf{i}\int_0^\infty \frac{tf_1(t)}{t+r}dt + \mathbf{j}\int_0^\infty \frac{tf_2(t)}{t+r}dt + \mathbf{k}\int_0^\infty \frac{tf_3(t)}{t+r}dt\\ &= \int_0^\infty \frac{(t+r-r)f_0(t)}{t+r}dt + \mathbf{i}\int_0^\infty \frac{(t+r-r)f_1(t)}{t+r}dt\\ &\quad + \mathbf{j}\int_0^\infty \frac{(t+r-r)f_2(t)}{t+r}dt + \mathbf{k}\int_0^\infty \frac{(t+r-r)f_3(t)}{t+r}dt\\ &= \int_0^\infty f_0(t)dt - \int_0^\infty \frac{rf_0(t)}{t+r}dt + \mathbf{i}\int_0^\infty f_1(t)dt - \mathbf{i}\int_0^\infty \frac{rf_1(t)}{t+r}dt\\ &\quad + \mathbf{j}\int_0^\infty f_2(t)dt - \mathbf{j}\int_0^\infty \frac{rf_2(t)}{t+r}dt + \mathbf{k}\int_0^\infty f_3(t)dt - \mathbf{k}\int_0^\infty \frac{rf_3(t)}{t+r}dt\\ &= \int_0^\infty f(t)dt - r\tilde{f}(r)\end{aligned}$$

provided the integral exists. □

Property 3.5. *Let $f(t)$ be quaternion-valued function,*

$$\mathscr{S}_q\left\{\frac{f(t)}{t+c}\right\} = \frac{1}{c-r}\left[\tilde{f}(r)-\tilde{f}(c)\right]. \tag{3.7}$$

Proof. By using definition,

$$\begin{aligned}\mathscr{S}_q\left\{\frac{f(t)}{t+c}\right\} &= \int_0^\infty \frac{f(t)}{(t+r)(t+c)}dt\\ &= \frac{1}{c-r}\int_0^\infty \left[\frac{1}{t+r}-\frac{1}{t+c}\right]f_0(t)dt + \mathbf{i}\frac{1}{c-r}\int_0^\infty \left[\frac{1}{t+r}-\frac{1}{t+c}\right]f_1(t)dt\\ &\quad + \mathbf{j}\frac{1}{c-r}\int_0^\infty \left[\frac{1}{t+r}-\frac{1}{t+c}\right]f_2(t)dt + \mathbf{k}\frac{1}{c-r}\int_0^\infty \left[\frac{1}{t+r}-\frac{1}{t+c}\right]f_3(t)dt\\ &= \frac{1}{c-r}\left[\tilde{f}_0(r)-\tilde{f}_0(c)\right] + \mathbf{i}\frac{1}{c-r}\left[\tilde{f}_1(r)-\tilde{f}_1(c)\right]\end{aligned}$$

$$+\mathbf{j}\frac{1}{c-r}\left[\tilde{f}_2(r)-\tilde{f}_2(c)\right]+\mathbf{k}\frac{1}{c-r}\left[\tilde{f}_3(r)-\tilde{f}_3(c)\right]$$
$$=\frac{1}{c-r}\left[\tilde{f}(r)-\tilde{f}(c)\right].$$
□

Theorem 3.6 (QST of Derivatives)**.** *Let $\mathscr{S}_q\{f(t)\}$ be the quaternion Stieltjes transform of the quaternion-valued function $f(t)$, then*

$$\mathscr{S}_q\{f'(t)\}=-\frac{1}{r}f(0)-\frac{d}{dr}\tilde{f}(r). \tag{3.8}$$

Proof. By using definition of quaternion Stieltjes transform,

$$\begin{aligned}\mathscr{S}_q\{f'(t)\}&=\int_0^\infty \frac{f'(t)}{t+r}dt\\
&=\left[\frac{f_0(t)}{t+r}\right]_0^\infty+\int_0^\infty \frac{f_0(t)}{(t+r)^2}dt+\mathbf{i}\left\{\left[\frac{f_1(t)}{t+r}\right]_0^\infty+\int_0^\infty \frac{f_1(t)}{(t+r)^2}dt\right\}\\
&\quad+\mathbf{j}\left\{\left[\frac{f_2(t)}{t+r}\right]_0^\infty+\int_0^\infty \frac{f_2(t)}{(t+r)^2}dt\right\}+\mathbf{k}\left\{\left[\frac{f_3(t)}{t+r}\right]_0^\infty+\int_0^\infty \frac{f_3(t)}{(t+r)^2}dt\right\}\\
&=-\frac{1}{r}f_0(0)-\frac{d}{dr}\tilde{f}_0(r)+\mathbf{i}\left\{-\frac{1}{r}f_1(0)-\frac{d}{dr}\tilde{f}_1(r)\right\}\\
&\quad+\mathbf{j}\left\{-\frac{1}{r}f_2(0)-\frac{d}{dr}\tilde{f}_2(r)\right\}+\mathbf{k}\left\{-\frac{1}{r}f_3(0)-\frac{d}{dr}\tilde{f}_3(r)\right\}\\
&=-\frac{1}{r}f(0)-\frac{d}{dr}\tilde{f}(r).\end{aligned}$$
□

In general, we can have the following result

$$\mathscr{S}_q\left\{f^{(n)}(t)\right\}=-\left[\frac{1}{r}f^{(n-1)}(0)+\frac{1}{r^2}f^{(n-2)}(0)+\cdots\frac{1}{r^n}f(0)\right]-\frac{d^n}{dr^n}\tilde{f}(r). \tag{3.9}$$

Definition 3.7. Generalized quaternion Stieltjes transform of a quaternion-valued function $f(t)$ on $0\le t<\infty$ is defined as

$$\mathscr{S}_q^n\{f(t)\}=\tilde{f}(r,n)=\int_0^\infty \frac{f(t)}{(t+r)^n}dt \tag{3.10}$$

where $r\in\mathbb{H}$ and $n\in\mathbb{R}^+$ such that the integral is convergent.

The generalized quaternion Stieltjes transform satisfies the following properties:

(i) $\mathscr{S}_q^n\{f(ct)\}=c^{n-1}\tilde{f}(cr,n),\ \ c>0.$ (3.11)

(ii) $\mathscr{S}_q^n\{tf(t)\}=\tilde{f}(r,n-1)-r\tilde{f}(r,n).$ (3.12)

4. Quaternion Laplace-Stieltjes Transform

Definition 4.1. Quaternion Laplace-Stieltjes transform of a well-defined quaternion-valued function $S(t)$ for $t\ge 0$ and $r\in\mathbb{H}$ is given by the following Stieltjes integral:

$$\mathscr{L}^*\{S(t)\}=S^*(r)=\int_0^\infty e^{-rt}dS(t)=r\int_0^\infty e^{-rt}S(t)dt \tag{4.1}$$

if the integral exists and converges for some r_0 such that $Re(r)>Re(r_0)$.

The inversion formula of quaternion Laplace-Stieltjes transform is given by

$$S(t) = \mathscr{L}^{-1}\left[S^*(r)\right] = \int_{\sigma-i\infty}^{\sigma-i\infty} \frac{e^{rt}}{r} S^*(r)dr. \tag{4.2}$$

Property 4.2 (Linearity). *Let $\mathscr{L}^*\{R(t)\}$ and $\mathscr{L}^*\{S(t)\}$ be quaternion Laplace-Stieltjes transform of functions $R(t)$ and $S(t)$ respectively, then*

$$\mathscr{L}^*\{aR(t)+bS(t)\} = a\mathscr{L}^*\{R(t)\} + b\mathscr{L}^*\{S(t)\}; \quad a,b \in \mathbb{R}. \tag{4.3}$$

Proof. By applying definition of quaternion Laplace-Stieltjes transform,

$$\begin{aligned}\mathscr{L}^*\{aR(t)+bS(t)\} &= \int_0^\infty e^{-rt} d\left(aR(t)+bS(t)\right)\\ &= a\int_0^\infty e^{-rt}dR(t) + b\int_0^\infty e^{-rt}dS(t)\\ &= a\mathscr{L}^*\{R(t)\} + b\mathscr{L}^*\{S(t)\}.\end{aligned}$$

□

Property 4.3 (Change of Scale). *Let $\mathscr{L}^*\{S(t)\}$ be quaternion Laplace-Stieltjes transform of functions $S(t)$ and $\alpha \in \mathbb{R}$, then*

$$\mathscr{L}^*\{S(\alpha t)\} = \frac{1}{\alpha} S^*\left(\frac{r}{\alpha}\right). \tag{4.4}$$

Proof. By applying definition of quaternion Laplace-Stieltjes transform,

$$\mathscr{L}^*\{S(\alpha t)\} = \int_0^\infty e^{-rt} dS(\alpha t).$$

Substituting $\alpha t = u$, we get

$$\begin{aligned}&= \int_0^\infty e^{-r\frac{u}{\alpha}} \frac{dS(u)}{\alpha}\\ &= \frac{1}{\alpha} S^*\left(\frac{r}{\alpha}\right).\end{aligned}$$

□

Property 4.4. *Let $\mathscr{L}^*\{S(t)\}$ be quaternion Laplace-Stieltjes transform of functions $S(t)$,then*

$$\mathscr{L}^*\left\{S'(t)\right\} = r\left[S^*(r) - S(0)\right]. \tag{4.5}$$

Proof. By using definition of Quaternion Laplace-Stieltjes transform,

$$\begin{aligned}\mathscr{L}^*\left\{S'(t)\right\} &= \int_0^\infty e^{-rt} dS'(t)\\ &= r\int_0^\infty e^{-rt} S'(t)dt\end{aligned}$$

By using [3, Proposition 4.7], we get

$$\begin{aligned}&= r\left[r\int_0^\infty e^{-rt}S'(t)dt - S(0)\right]\\ &= r\left[S^*(r) - S(0)\right].\end{aligned}$$

□

Property 4.5. *Let $F(t) = \int_0^t S(y)dy$ be the quaternion-valued function, then*

$$\mathscr{L}^*\{F(t)\} = \mathscr{L}^*\left\{\int_0^t S(y)dy\right\} = \frac{1}{r} S^*(r). \tag{4.6}$$

Proof. By using definition of Quaternion Laplace-Stieltjes transform,

$$\begin{aligned}\mathscr{L}^*\{F(t)\} &= \int_0^\infty e^{-rt} dF(t)\\ &= r\int_0^\infty e^{-rt} F(t)dt\\ &= r\int_0^\infty e^{-rt}\left(\int_0^t S(y)dy\right)dt\end{aligned}$$

By using [3, Proposition 4.10], we get

$$\begin{aligned}&= r\left(\frac{1}{r}\int_0^\infty e^{-rt} S(t)dt\right)\\ &= \frac{1}{r}S^*(r).\end{aligned}$$

□

Theorem 4.6 (Convolution Theorem)**.** *Let $G^*(r)$ and $H^*(r)$ be the quaternion Laplace-Stieltjes transform of $G(t)$ and $H(t)$ respectively. Then*

$$\mathscr{L}^*\{G*H\} = \frac{G^*(r)H^*(r)}{r}. \tag{4.7}$$

Proof. The convolution of two functions is given by

$$G(t)*H(t) = \int_0^t G(t-u)H(u)du \tag{4.8}$$

By applying quaternion Laplace-Stieltjes transform, we get

$$\begin{aligned}\mathscr{L}^*\{G(t)*H(t)\} &= \mathscr{L}^*\left\{\int_0^t G(t-u)H(u)du\right\}\\ &= r\int_0^\infty\int_0^t e^{-rt}G(t-u)H(u)dudt\end{aligned}$$

By changing the order of integration, we get

$$= r\int_0^\infty H(u)du\int_u^\infty e^{-rt}G(t-u)dt$$

Substituting $t-u=\mu$, we get

$$\begin{aligned}&= r\int_0^\infty H(u)du\int_0^\infty e^{-r(u+\mu)}G(\mu)d\mu\\ &= r\int_0^\infty e^{-ru}H(u)du\int_0^\infty e^{-r\mu}G(\mu)d\mu\\ &= r\int_0^\infty e^{-ru}H(u)du\times\frac{r}{r}\int_0^\infty e^{-r\mu}G(\mu)d\mu\\ &= \frac{G^*(r)H^*(r)}{r}.\end{aligned}$$

□

Theorem 4.7 (Quaternion Abelian Theorem)**.** *If for some non-negative τ and quaternion-valued function $S(t)$,*

$$\lim_{t\to\infty}\frac{S(t)}{t^\tau} = \frac{C}{\Gamma(\tau+1)} \tag{4.9}$$

then

$$\lim_{r\to 0^+} r^\tau S^*(r) = C. \tag{4.10}$$

Proof. Let the quaternion-valued well defined function $S(t)$ be represented as

$$S(t) = S_1(t) + jS_2(t).$$

By using Theorem A.1 [6, p. 242] and for some constants τ, C_1;

$$\lim_{t\to\infty} \frac{S_1(t)}{t^\tau} = \frac{C_1}{\Gamma(\tau+1)} \tag{4.11}$$

then we have

$$\lim_{r\to 0^+} r^\tau S_1^*(r) = C_1. \tag{4.12}$$

Similarly, for some constants τ, C_2;

$$\lim_{t\to\infty} \frac{S_2(t)}{t^\tau} = \frac{C_2}{\Gamma(\tau+1)} \tag{4.13}$$

then we have

$$\lim_{r\to 0^+} r^\tau S_2^*(r) = C_2. \tag{4.14}$$

By taking linear combination of (4.11) and (4.13), we have

$$\lim_{t\to\infty} \frac{S_1(t)}{t^\tau} + j \lim_{t\to\infty} \frac{S_2(t)}{t^\tau} = \frac{C_1}{\Gamma(\tau+1)} + j\frac{C_2}{\Gamma(\tau+1)}$$

$$\lim_{t\to\infty} \left\{ \frac{S_1(t)+jS_2(t)}{t^\tau} \right\} = \frac{C_1+jC_1}{\Gamma(\tau+1)}$$

$$\lim_{t\to\infty} \frac{S(t)}{t^\tau} = \frac{C}{\Gamma(\tau+1)}.$$

Therefore by taking linear combination of (4.12) and (4.14), we have

$$\lim_{r\to 0^+} r^\tau S_1^*(r) + j \lim_{r\to 0^+} r^\tau S_2^*(r) = C_1 + jC_2$$

$$\lim_{r\to 0^+} r^\tau \{S_1^*(r) + jS_2^*(r)\} = C_1 + jC_2$$

$$\lim_{r\to 0^+} r^\tau S^*(r) = C.$$

Hence the proof. □

Now, we will extend the Ikehara's Tauberian theorem as given in [12, p. 233] for quaternions.

Theorem 4.8 (Quaternion Tauberian Theorem)**.** *If $S(t)$ is non-decreasing quaternion-valued function and defined in $0 \le t \le \infty$, and the quaternion Laplace-Stieltjes transform given by*

$$\mathscr{L}^*\{S(t)\} = S^*(r) = \int_0^\infty e^{-rt} dS(t), \quad r = r_1 + jr_2 \in \mathbb{H} \tag{4.15}$$

converges for $Re(r) > 1$. If for some constant $C \in \mathbb{H}$ and for some function $h(y)$,

$$\lim_{Re(r)\to 1^+} S^*(r) - \frac{C}{r-1} = h(y) \tag{4.16}$$

then we have

$$\lim_{t\to\infty} e^{-t} S(t) = C. \tag{4.17}$$

Proof. By using [12], If for some constant C_1 and for some function h,

$$\lim_{Re(r_1)\to 1^+} \left[S^*(r_1) - \frac{C_1}{r_1 - 1} \right] = h(y_1) \tag{4.18}$$

then we have

$$\lim_{t\to\infty} e^{-t}S(t) = C_1. \tag{4.19}$$

Similarly, If for some constant C_2 and for some function h,

$$\lim_{Re(r_2)\to 1^+}\left[S^*(r_2) - \frac{C_2}{r_2-1}\right] = h(y_2) \tag{4.20}$$

then we have

$$\lim_{t\to\infty} e^{-t}S(t) = C_2. \tag{4.21}$$

By taking linear combination, we get

$$\lim_{Re(r_1)\to 1^+}\left[S^*(r_1) - \frac{C_1}{r_1-1}\right] + \lim_{Re(r_2)\to 1^+}\left[S^*(r_2) - \frac{C_2}{r_2-1}\right] j = h(y_1) + j(y_2)$$

$$\lim_{Re(r)\to 1^+}\left[S^*(r_1+jr_2) - \frac{C_1+jC_2}{(r_1+jr_2)-1}\right] = h(y_1+jy_2)$$

$$\lim_{Re(r)\to 1^+} S^*(r) - \frac{C}{r-1} = h(y).$$ □

Example 1. Let us consider the equilibrium distribution as given in [6]

$$S_e(t) = \frac{1}{\mu}\int_0^t [1-S(y)]dy, \tag{4.22}$$

where $S(t)$ is the probability distribution function and μ is the mean of $S(t)$.

By using the definition of quaternion Laplace-Stieltjes transform,

$$S_e^*(t) = \int_0^\infty e^{-rt} dS_e(t)$$

By using Property 4.5, we get

$$= \frac{1}{\mu r}\int_0^\infty e^{-rt} d[1-S(t)]$$

$$= \frac{1}{\mu r}[1-S^*(r)].$$

5. Applications

(a) Stieltjes introduced the theory of moments and formulated the so-called Stieltjes problem of moments in the general form. The Stieltjes transform occurs naturally in connection with the Stieltjes moment problem and hence is related to certain continued fractions as stated in [7].

If the quaternion-valued funtion $e(t)$ has an exponential rate of decay as $t\to\infty$, then all of the moments exist and are given by

$$M_k = \int_0^\infty t^k e(t)dt; \quad k = 0,1,2,\cdots. \tag{5.1}$$

Analogous to [2, Sec 9.10], we can get

$$\tilde{e}(r) = \sum_{k=0}^{n-1} (-1)^k M_k r^{-(k+1)} + E_n(r), \tag{5.2}$$

where

$$E_n(r) \le r^{-(k+1)} \sup_{0<t<\infty} \left| \int_0^t \tau^n e(\tau) d\tau \right|.$$

(b) Consider the Volterra integral equation of first kind with a convolution type kernel

$$F(x) = \lambda \int_0^x K(r-t) G(t) dt. \tag{5.3}$$

Applying quaternion Laplace-Stieltjes transform and using the convolution theorem, we get

$$F^*(r) = \frac{\lambda K^*(r) G^*(r)}{r},$$

$$G^*(r) = \frac{rF^*(r)}{\lambda K^*(r)}.$$

By using inversion formula of quaternion Laplace-Stieltjes transform,

$$G(x) = \mathscr{L}^{-1}\left[\frac{rF^*(r)}{\lambda K^*(r)}\right] \tag{5.4}$$

which is the required solution.

(c) Consider the Volterra integral equation of second kind with a convolution type kernel

$$g(x) = f(x) + \lambda \int_0^x k(r-t) g(t) dt. \tag{5.5}$$

Applying quaternion Laplace-Stieltjes transform and using the convolution theorem, we get

$$G^*(r) = F^*(r) + \frac{\lambda K^*(r) G^*(r)}{r},$$

$$G^*(r) = \frac{rF^*(r)}{r - \lambda K^*(r)}.$$

By using inversion formula of quaternion Laplace-Stieltjes transform,

$$G(x) = \mathscr{L}^{-1}\left[\frac{rF^*(r)}{r - \lambda K^*(r)}\right] \tag{5.6}$$

which is the required solution.

6. Conclusion

In this paper, we have introduced quaternion Stieltjes transform. The inverse of Quaternion Stieltjes transform is also stated. Properties of quaternion Stiletes transform are established. We have established quaternion Laplace-Stieltjes transform with all the basic operational properties of it. Convolution theorem for quaternion Laplace-Stieltjes transform is proved. Applications to moment praoblem and Volterra integral equations are illustrated.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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