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Research Article

Strong Open Monophonic Number of a Graph

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Abstract. For a graph $G(V,E)$, the strong open monophonic problem is to find a set $S \subseteq V(G)$ such that each vertex in $V(G)$ lies on a unique fixed monophonic path between the vertices in S and the set S is called the strong open monophonic set. In this paper, we have discussed some results related to strong open monophonic sets and mainly we have the complexity property of strong open monophonic set problem for general graph. Also, some bounds for general graphs are derived.

Keywords. Strong geodetic number; Monophonic set; Monophonic number; Strong open monophonic number

Mathematics Subject Classification (2020). 05C05; 05C12; 05C38; 68R10

1. Introduction

The graphs $G(V,E)$ considered in the paper are simple connected graphs and we denote the order of graph by n and size of graph by m. The distance between two vertices, denoted by $d(u,v)$ where $u,v \in V(G)$, is the length of the shortest path linking the two vertices in G. A u-v geodesic is the u-v path with length $d(u,v)$. The eccentricity of a vertex u, $e(u)$, is the maximum distance between u and other vertices in G. The maximum and minimum eccentricity among the vertices of a graph G is called the diameter d and radius r of a graph G. Chord of a path is an edge linking two non-adjacent vertices in a graph G. For a simple connected graph G, a monophonic path is a u-v path without chords. A monophonic set S of a graph G is the set of vertices such that the pair of vertices in S forms a monophonic path that contains every

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vertex of G. The monophonic number $m(G)$ is the minimum cardinality of a monophonic set of a graph G. The monophonic distance $d_m(u,v)$ between two vertices u,v in G is the longest length of a u-v monophonic path. The monophonic radius r_m and monophonic diameter d_m is the minimum and maximum monophonic eccentricity of a graph G, respectively [1].

One of the widely studied concept in graph theory is geodetic number of a graph [6] which is introduced by Harary et al. and further studied in [2–4]. The NP-completeness of geodetic decision problem is studied in [10]. Many researchers are studying geodetic concept and many variants of the concept are introduced. A recent variation called strong geodetic set is introduced by Paul Manuel et al. [8]. If $S \subseteq V$, then for each pair of vertices $x,y \in S$, $x \neq y$, let $\widetilde{g}(x,y)$ be a selected fixed shortest x,y-path. Then we set $\widetilde{I}(S) = \{\widetilde{g}(x,y) : x,y \in S\}$,and let $V(\widetilde{I}(S)) = \bigcup_{P \in \widetilde{I}(S)} V(P)$. If $V(\widetilde{I}(S)) = V$, for some $\widetilde{I}(S)$, then the set S is called a strong geodetic set [9].

In this paper, we introduce the concept of strong open monophonic set followed by strong open monophonic number. Further the complexity of strong open monophonic number of a graph G is discussed. With respect to the notion of strong open monophonic number, we formulate some bounds for simple graphs and present the strong open monophonic number for some graphs. The monophonic number [12] is one of the recently studied topic which has many applications especially in networking and other security purposes. Different variants of monophonic number has been studied in [7, 11, 13].

2. Strong Open Monophonic Number of a Graph

Definition 1. Let $G = (V,E)$ be a graph. The strong monophonic set $S \subseteq V(G)$ is the set of vertices that covers every vertex of G by unique fixed u-v monophonic paths, $u,v \in S$. The cardinality of a minimum monophonic strong set is the strong monophonic number of G and denoted by $sm(G)$ [13].

Definition 2. If $S \subseteq V(G)$, then for each pair of vertices $x,y \in S$, $x \neq y$, let $\widetilde{g}_m(x,y)$ be a unique fixed monophonic x-y path. Then we set, $I_m(S) = \{\widetilde{g}_m(x,y) : x,y \in S\}$ and let $V(\widetilde{I}_m(S)) = \bigcup_{g_m \in \widetilde{I}_m(S)} V(g_m)$. If $V(\widetilde{I}_m(S)) = V(G)$ for some $\widetilde{I}_m(S)$, then the set S is called strong open monophonic set. The minimum cardinality of a strong open monophonic set is the strong open monophonic number and is denoted by $som(G)$. For graph shown in Figure 1, the set $S_1 = \{v_1, v_3, v_4, v_6\}$ and $S_2 = \{v_1, v_4, v_5\}$ are strong open monophonic sets and $som(G) = 3$.

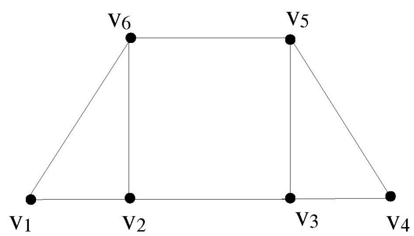

Figure 1. The strong open monophonic number of graph G, $som(G) = 3$

A graph is said to be a strong open monophonic graph if it contains a strong open monophonic set.

The graphs $K_n - e$, C_4 are not strong open monophonic graphs.
Throughout this paper, G is assumed to be a strong open monophonic graph.

3. Complexity of Strong Open Monophonic Number of a Graph

The proof of NP-completeness of the strong open monophonic problem for general graphs can be reduced from the problem of deciding, given three vertices whether there exists an induced path between the two vertices passing through the third vertex.

Theorem 1 ([5]). *Let x, y, z be three distinct vertices in a graph G. Deciding whether there is an induced path from x to y passing through z is NP-complete.*

Theorem 2. *The strong open monophonic number of a graph is NP-complete for general graphs.*

Proof. Given a graph G with distinct vertices x, y, z, construct G' as follows: Add two vertices u, v such that these vertices are adjacent to all vertices in $V(G)\backslash\{z\}$. Add a pendant vertex b adjacent to the vertex u. Similarly, add pendant vertices $\{a_1, a_2, \dots, a_{n-1}\}$ adjacent to the vertex v (refer Figure 2). Let $V' = \{b, a_1, a_2, \dots, a_{n-1}\}$. Since the vertices of V' forms a set of simplicial vertices, they belong to any strong open monophonic set of G'. The monophonic paths between the vertices in V' does not cover the vertex z. It is easy to see that every induced path in G is also an induced path in G'. Also, for any $v \in V(G)$, the monophonic paths between v and v' where $v' \in V'$ will not cover the vertex z. Consider $S = V' \cup \{x, y\}$ where $x, y \in V(G)$. From the construction of S, the vertices $b, a_1, a_2, \dots, a_{n-1}$ are extreme vertices and x, y are non extreme vertices. Also, x and y lies on fixed monophonic paths between b and $a_i; i = 1, 2, \dots, n-1$. It can be easily verified that S is a strong open monophonic set in G' if and only if the induced path between x and y contains z in G. □

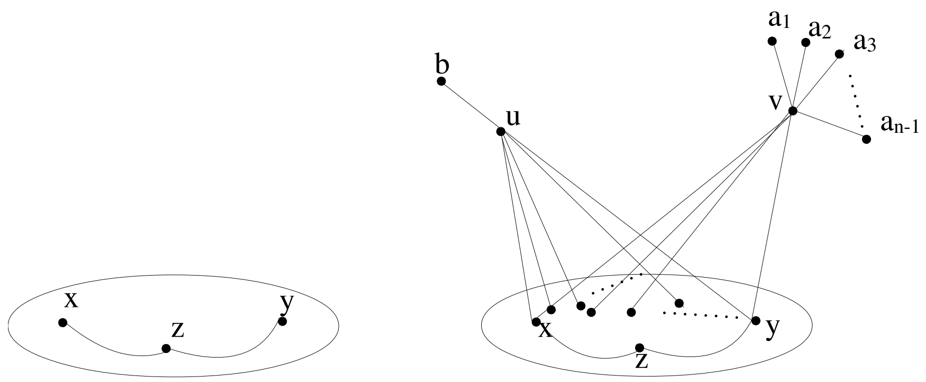

Figure 2. Illustration of NP-completeness of general graph

4. Results

Theorem 3. *For any graph G, $2 \le m(G) \le som(G) \le n$.*

Proof. Any monophonic set contains at least 2 vertices, $m(G) \ge 2$. Also, any strong open monophonic set is a monophonic set. Therefore, $m(G) \le som(G)$. Since, the strong open monophonic set contains the vertices of G, $som(G) \le n$. □

Corollary 1. *For a graph G of order n with k extreme vertices,* $\max(2,k) \le som(G) \le n$.

Theorem 4 ([11]). *If G is a connected graph with a cut vertex v, then every open monophonic set of G contains at least one vertex from each component of $G - v$.*

Since every open monophonic set is a strong open monophonic set, we have the resulting theorem.

Theorem 5. *If G is a connected graph with a cut vertex v, then every strong open monophonic set of G contains at least one vertex from each component of $G - v$.*

Result 1. *Let T be a tree with k leaves, then $som(T) = k$.*

Proposition 1. *Let G be a connected graph of order n, $som(G) = 2$ if and only if G is a path.*

Proof. Consider a path P, it contains two extreme vertices x, y which forms a monophonic path, thus covering all the vertices in P. Conversely, $som(G) = 2$. Let $\{u, v\}$ be the minimum strong open monophonic set. This implies that u-v unique fixed monophonic path covers all other vertices of G, since $som(G) = 2$. Thus, G is a path. □

Proposition 2. *Let G be a connected graph of order n. If G is a complete graph, then $som(G) = n$.*

Proof. Let G be a complete graph. Since each vertex of G is a simiplicial vertex, $som(G) = n$. □

The converse of the theorem is not true. In Figure 3, the strong open monophonic number of G with 5 vertices is 5, i.e., all the vertices in G is contained in the strong open monophonic set.

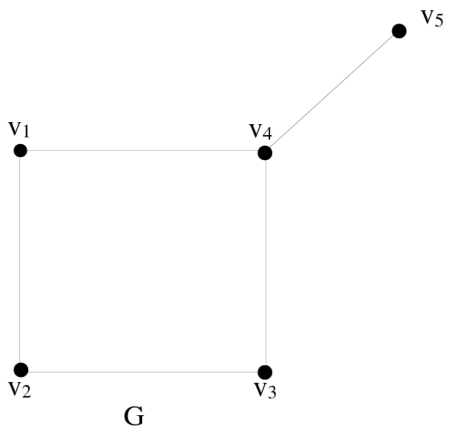

Figure 3. The strong open monophonic number of G, $som(G) = 5$

Theorem 6. *If G is a connected graph with no extreme vertices, then $som(G) \ge 3$.*

Proof. Since G has no extreme vertices, G is not path. By Proposition 1, $som(G) \ne 2$. Let $u, v \in V(G)$. Since u, v are not extreme vertices and u-v monophonic path does not cover u, v, there exists a unique fixed monophonic path covering them. Therefore, $som(G) \ge 3$. □

Theorem 7. *For any cycle C_n of order $n \ge 6$, then $som(C_n) = 3$.*

Proof. Let C_n be cycle with n vertices. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices. Then there exists $v_1 - v_{\lfloor \frac{n}{2} \rfloor}$ unique monophonic path which covers the vertices $\{v_n, v_{n-1}, \ldots, v_{\lfloor \frac{n}{2} \rfloor - 1}\}$ and $v_1 - v_{\lfloor \frac{n}{2} \rfloor + 2}$ unique monophonic path covers the vertices $\{v_2, v_3, \ldots, v_{\lfloor \frac{n}{2} \rfloor + 1}\}$. Similarly, other vertices are covered by $v_{\lfloor \frac{n}{2} \rfloor} - v_{\lfloor \frac{n}{2} \rfloor + 2}$ unique monophonic path. Thus $som(C_n) = 3$. □

Theorem 8. *For grid $G_{n,m}$, $som(G_{n,m}) \leq 4$, where $n, m \geq 3$.*

Proof. Without loss of generality, let us assume that $n \geq m$. Each vertex of the grid $G_{n,m}$ be (x_i, y_j) where $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $S = \{(x_1, y_1), (x_1, y_m), (x_n, y_1), (x_n, y_m)\}$.

Case 1: If m is even.

(a) When $m \cong 0 (\text{mod } 4)$

All the vertices in the even columns of $G_{n,m}$ except that of the mth column will lie on the (x_1, y_1)-(x_n, y_m) fixed monophonic path. The path is as follows, $P : (x_1, y_1), (x_1, y_2)$-$(x_n, y_2)$-$(x_n, y_4), (x_{n-1}, y_4)$-$(x_1, y_4), (x_1, y_5)(x_1, y_6)$-$(x_n, y_6)$-$(x_n, y_8), (x_{n-1}, y_8) \ldots (x_1, y_{m-2})$-$(x_n, y_{m-2}), (x_n, y_{m-1})$, (x_n, y_m).

Similarly, all the vertices in the odd columns will lie on the $(x_1, y_1), (x_1, y_m)$ fixed unique monophonic path. The vertices in the mth column, the first and last row will lie in the fixed unique (x_1, y_1)-(x_n, y_1) monophonic path. Also, the vertices in the first column, first row and last row will lie in the fixed unique (x_1, y_m)-(x_n, y_m) monophonic path.

(b) When $m \cong 2 (\text{mod } 4)$

All the vertices in the even columns of $G_{n,m}$ except that of the mth column will lie on the (x_1, y_1)-(x_1, y_m) fixed monophonic path. The path is as follows, $(x_1, y_1), (x_1, y_2), (x_2, y_2)$-$(x_n, y_2)$-$(x_n, y_4), (x_{n-1}, y_4), (x_1, y_5)$-$(x_1, y_7)$-$(x_n, y_7) \ldots (x_1, y_{m-2})$-$(x_1, y_m))$.

Similarly, all the vertices in the odd columns except mth column will lie on the $(x_1, y_1), (x_n, y_m)$ fixed unique monophonic path. The vertices in the mth column and the first and last row will lie in the fixed unique (x_1, y_1)-(x_n, y_1) monophonic path. Also, the vertices in the first column, first row and last row will lie in the fixed unique (x_1, y_m)-(x_n, y_m) monophonic path.

Case 2: If m is odd.

(a) When $m \cong 1 (\text{mod } 4)$

All the vertices in the even column will lie in the (x_1, y_1)-(x_1, y_m) fixed unique monophonic path. The path as follows: $(x_1, y_1), (x_1, y_2)$-(x_2, y_2)-$(x_n, y_2), (x_n, y_4)$, (x_{n-1}, y_4)-(x_1, y_4), $(x_1, y_5), (x_1, y_6) \ldots (x_n, y_{m-1})$-$(x_1, y_{m-1}), (x_1, y_m)$.

Similarly, all the vertices in the odd columns of $G_{m,n}$ will lie on (x_1, y_1)-(x_n, y_m) fixed unique monophonic path. The vertices in the mth column and the first and last row will lie in the fixed unique (x_1, y_1)-(x_n, y_1) monophonic path. Also, the vertices in the first column, the first and last row will lie in the unique (x_1, y_m)-(x_n, y_m) monophonic path.

(b) When $m \cong 3 (\text{mod } 4)$

All the vertices in the even columns of $G_{m,n}$ will lie on the (x_1, y_1)-(x_n, y_m) fixed unique monophonic path. The path is as follows: $(x_1, y_1), (x_1, y_2), (x_2, y_2)$-$(x_n, y_2)$-$(x_n, y_4), (x_{n-1}, y_4) \ldots$ (x_1, y_{m-1})-$(x_m, y_{m-1}), (x_n, y_m)$.

Similarly, all the vertices in the odd columns of $G_{m,n}$ will lie on (x_1, y_1)-(x_1, y_m) fixed unique monophonic path. The vertices in the mth column and the first and last row will lie in the fixed

unique (x_1,y_1)-(x_n,y_1) monophonic path. Also, the vertices in the first column, first row and last row will lie in the unique (x_1,y_m)-(x_n,y_m) monophonic path.
Thus the $som(G_{n,m}) \le 4$. □

Theorem 9. *Let P_n be the path of order n and C_m be the cycle of order m. For $n \ge 3$, $m \ge 5$,*

$$som(C_m \square P_n)\begin{cases} \le 4 & \text{if } m \cong 1(\text{mod } 4),\\ = 3 & \text{otherwise}. \end{cases}$$

Proof. Without loss of generality, Let each vertex of the $C_m \square P_n$ be (x_i,y_j), where $1 \le i \le m$, $1 \le j \le n$.

(a) When $m \cong 0(\text{mod } 4)$

Let $S = \{(x_1,y_1),(x_2,y_n),(x_{m-1},y_n)\}$. All the vertices in the odd rows except the last odd row lie on (x_1,y_1)-(x_{m-1},y_n) unique fixed monophonic path. The path P_1 is as follows: (x_1,y_1)-$(x_1,y_n),(x_2,y_n),(x_3,y_n)$-$(x_3,y_1)$-$(x_5,y_1)$-$(x_5,y_n)\ldots(x_{m-3},y_n)(x_{m-2},y_n),(x_{m-1},y_n)$. Similarly, all vertices in the even rows except the first even row is covered by (x_1,y_1)-(x_2,y_n) unique fixed monophonic path. All the remaining vertices lie on (x_2,y_n)-$(x_{m-1}y_n)$ unique fixed monophonic path which is as follows, $P_2 : (x_2,y_n)$-$(x_2,y_1),(x_1,y_1),(x_m,y_1),(x_{m-1},y_1)$-$(x_{m-1},y_n)$. Also, each of the vertex in the set S is covered by the other two pair of vertices in S. Thus S is the strong open monophonic set and $som(P_n \square C_m) = 3$.

(b) When $m \cong 2(\text{mod } 4)$

Let $S = \{(x_1,y_1),(x_2,y_n),(x_{m-1},y_n)\}$. All the vertices in the odd rows lie on unique fixed (x_1,y_1)-(x_{m-1},y_n) monophonic path. The path P_3 is as follows: (x_1,y_1)-(x_1,y_n)-(x_3,y_n)-(x_3,y_1)-(x_5,y_1)-$(x_5,y_n)\ldots(x_{m-3},y_n)$-$(x_{m-3},y_1)$-$(x_{m-1},y_1)$-$(x_{m-1},y_n)$. Similarly, all vertices in the even rows except the first even row is covered by unique fixed (x_1,y_1)-(x_{m-1},y_1) monophonic path. All the remaining vertices lie on unique fixed $(x_2,y_n)(x_{m-1},y_n)$ monophonic path which is as follows, $P_4 : (x_2,y_n)$-$(x_2,y_1)(x_1,y_1)(x_m,y_1)$-$(x_m,y_n),(x_{m-1},y_n)$. Also, each of the vertex in the set S is covered by the other two pair of vertices in S. Thus S is the strong open monophonic set and $som(P_n \square C_m) = 3$.

(c) When $m \cong 3(\text{mod } 4)$

Let $S = \{(x_1,y_1),(x_3,y_n),(x_m,y_n)\}$. All the vertices in the even rows lie on unique fixed (x_1,y_1)-(x_m,y_n) monophonic path. The path is as follows: $(x_1,y_1),(x_2,y_1)$-(x_2,y_n)-(x_4,y_n)-(x_4,y_1)-(x_6,y_1)-$(x_6,y_n)\ldots(x_{m-1},y_n),(x_m,y_n)$. Similarly, all the vertices in the odd rows except first row lie on the unique fixed (x_1,y_1)-(x_3,y_n) monophonic path which is as follows: $P : (x_1,y_1),(x_m,y_1)$-$(x_m,y_n),(x_{m-1},y_n),(x_{m-2},y_n)$-$(x_{m-2},y_1)\ldots(x_3,y_1)(x_3,y_n)$. All the remaining vertices lie on unique fixed (x_3,y_n)-(x_m,y_n) monophonic path which is as follows, $P : (x_3,y_n)$-(x_3,y_1)-(x_1,y_1)-$(x_1,y_n)(x_m,y_n)$. Also, each of the vertex in the set S is covered by the other two pair of vertices in S. Thus S is the strong open monophonic set and $som(P_n \square C_m) = 3$.

(d) When $m \cong 1(\text{mod } 4)$

Let $S = \{(x_1,y_1),(x_3,y_n),(x_{m-1},y_1),(x_2,y_n)\}$. The unique fixed (x_1,y_1)-(x_{m-1},y_1) monophonic path covers odd rows except the last row. The path is as follows: (x_1,y_1)-(x_1,y_n)-(x_3,y_n)-(x_3,y_1)-(x_5,y_1)-$(x_5,y_n)\ldots(x_{m-2},y_n)$-$(x_{m-2},y_1),(x_{m-1},y_1)$. The unique fixed (x_1,y_1)-(x_3,y_n) monophonic path covers $(x_m,y_1),(x_{m-1},y_1)$-(x_{m-1},y_n)-(x_{m-3},y_n)-$(x_{m-3},y_1)\ldots(x_4,y_1)$-$(x_4,y_n)$ vertices. The unique fixed (x_1,y_1)-(x_2,y_n) monophonic path covers (x_m,y_1)-$(x_m,y_n),(x_{m-1},y_n)$-(x_3,y_n) vertices.

The vertices (x_2, y_n)-$(x_2, y_1),(x_1, y_1),(x_m, y_1)$ is covered by the unique fixed (x_2, y_n)-(x_{m-1}, y_1) monophonic path. Thus S is the strong open monophonic set and the $som(P_n \Box C_m) \leq 4$. □

Theorem 10. *The strong open monophonic number of a complete Apollonian network is* 3^{r-1}.

Proof. Let $A(r)$ be a complete r-dimensional Apollonian network of order n. Let S be the set of extreme vertices of $A(r)$ and its cardinality is 3^{r-1}. By definition, the strong open monophonic set contains extreme vertices. Thus the extreme vertices of $A(r)$ is contained in the strong open monophonic set of $A(r)$, and also, any non-extreme vertex is adjacent to three extreme vertices. It is straightforward to see that every non-extreme vertex will lie on some monophonic path between the extreme vertices. Therefore, the strong open monophonic number of a complete Apollonian network, $som(A(r)) = 3^{r-1}$. □

Theorem 11. *The strong open monophonic number of a r-dimensional silicate graph, $SL(r)$ is k, where k is the number of extreme vertices of $SL(r)$.*

Proof. Let $SL(r)$ be the r-dimensional silicate graph and let k and u be the number of extreme vertices and the non-extreme vertex of $SL(r)$, respectively. It is straightforward that there exists extreme vertices x, y such that u lies in the $x - y$ unique monophonic path. Therefore, $som(SL(r)) = k$. □

Theorem 12. *Let G and H be connected graphs with $sm(G) = p$ and $sm(G) = q$, $2 \leq p \leq q$. If G and H have minimum strong monophonic set S and T such that $G[S]$ and $G[T]$ are totally disconnected. Then $som(G \Box H) \leq pq$.*

Proof. Let $S = \{g_1, g_2, \dots, g_p\}$ and $T = \{h_1, h_2, \dots, h_q\}$ be the minimum strong monophonic set of G and H respectively and $G[S]$ and $G[T]$ are totally disconnected. Let $U = \{(g_m, h_n); 1 \leq m \leq p, 1 \leq n \leq q\}$. To prove that $U = S \times T$ is the strong open monophonic set of $(G \Box H)$.
Let $(x, y) \in V(G \Box H)$. Clearly, (x, y) is not an extreme vertex. Then there exists indices i and i' and $i \neq i'$ such that x lies on the (g_i, g_i') unique fixed monophonic path P of G and there exists indices j and j' and $j \neq j'$ such that y lies on the (h_j, h_j') unique fixed monophonic path Q of H. Since $G[S]$ and $G[T]$ are independent sets, the monophonic distance of P and Q are greater than or equal to 2. Let $U = \{(g_i, h_i),(g_i, h_i'),(g_i', h_i),(g_i', h_i')\}$.

Case 1: When x is not an extreme vertex in G and y is not an extreme vertex in H.

Case 1(a): Suppose $x \neq g_i, g_{i'}$, $y \neq h_i, h_{i'}$.

Case 1(b): Suppose $x = g_i (or) g_{i'}$ and $y \neq h_i, h_{i'}$ or $x \neq g_i, g_{i'}$ and $y = h_i (or) h_{i'}$.

Case 1(c): Suppose $x = g_i (or) g_{i'}$ and $y = h_i (or) h_{i'}$.

In all these three cases, (x, y) lies on the grid formed by the fixed monophonic paths (g_i, h_i)-$(g_i, h_{i'}),(g_{i'}, h_i)$-$(g_{i'}, h_{i'}),(g_i, h_{i'})$-$(g_{i'}, h_{i'}),(g_i, h_i)$-$(g_{i'}, h_{i'})$. By Theorem 8, (x, y) is covered by unique fixed monophonic paths between the vertices of the set $U = \{(g_i, h_i),(g_i, h_{i'}),(g_{i'}, h_i),(g_{i'}, h_{i'})\}$. Therefore, U is a strong open monophonic set of $G \Box H$.

Case 2: When x is an extreme vertex in G and y is not an extreme vertex in H.
Since y is a non-extreme vertex, y lies on $h_r - h_s$ fixed monophonic path, $1 \leq r, s \leq q$. Therefore, (x, y) lies on the fixed monophonic (x, h_r)-(x, h_s).

Case 3: When both x and y are extreme vertices.
x is one of the end of a monophonic path $A : x - g_j$, where $A \in \widetilde{P_m}(S)$. Otherwise, consider A as some monophonic path $x - g_j$. Similarly, y is one of the end vertex of a monophonic path $B : y - h_j$ where $B \in \widetilde{P_m}(T)$. Otherwise, consider B as some monophonic path $y - h_j$. Therefore, (x, y) lies on the fixed monophonic A and B with corner vertices $(x, y), (x, h_j), (g_i, y), (g_i, h_j)$. Thus, $som(G \Box H) \le pq$. □

The bound is sharp for grid $P_n \Box P_m$, $n, m \ge 3$.

Theorem 13. *For graphs G and H with $o(G) \ge 5$, the $som(G \circ H) = o(G) som(H)$.*

Proof. Let G and H be two connected strong open monophonic graphs with order m and n, respectively. The corona product of graphs G and H has m copies of H. Let S be the set that contains strong open monophonic set of each copies of H_i, $1 \le i \le m$ in $G \circ H$. Let Y be the minimum strong open monophonic set of H and Y^g be the corresponding strong monophonic set of H_g. Let $x \in G \circ H$. Suppose $x \in H_g$, $g \in G$. Since Y_g is a strong open monophonic set of H_g, each vertex of H_g lies on some unique monophonic path between the vertices of Y_g. Suppose $x \in V(G)$. Since G is connected, there exists $y \in V(G)$, $xy \in E(G)$ such that x lies on the unique monophonic path between the vertices of H_x and H_y. Therefore, S is a strong open monophonic set of $G \circ H$. Thus $som(G \circ H) \le o(G) som(H)$.
Suppose there exists a strong open monophonic set S' such that $|S'| < o(G) som(H)$. There exists $g' \in G$ such that $|S' \cap H_{g'}| < som(H)$. Then, $A = S' \cap H_{g'}$ is not a strong open monophonic set for $H_{g'}$. Hence, there exists $u \in H_{g'}$ such that u is not covered by any fixed monophonic between the vertices of A. It is straight forward to see that u does not lie on any monophonic path between the vertices of A and $S' \backslash H_g$. Also, u does not lie on any monophonic paths between the vertices of $S' \backslash H_g$. Therefore, S' is not a strong open monophonic set, which is a contradiction. Hence $som(G \circ H) = o(G) som(H)$. □

Result 2. *For any wheel $W_n = K_1 + C_{n-1}$, where $n \ge 5$,*

$$som(W_n) = \begin{cases} 5 & \text{if } n = 5, \\ 6 & \text{if } n \ge 6. \end{cases}$$

5. Bounds for Strong Open Monophonic Number of a Graph

Theorem 14. *Let G be a non-trivial connected graph of order n and monophonic diameter, $d_m \ge 2$, then*

$$som(G) \ge \left\lfloor \frac{d_m - 1 + \sqrt{(d_m - 1)^2 + 8(n - \mathrm{Ext}(G))(d_m - 1)}}{2(d_m - 1)} \right\rfloor.$$

Proof. Let S be the minimum strong open monophonic set of G and let $Ext(G)$ be the extreme vertices of G. Each monophonic path will have length at most d_m and it covers at most $d_m - 1$ vertices of $V(G)$. All the vertices of G is covered with $\binom{som(G)}{2}$ monophonic paths. Therefore, $n - Ext(G) \le \binom{som(G)}{2}(d_m - 1)$ which implies $som^2(G)(d_m - 1) - som(G)(d_m - 1) - 2(n - Ext(G)) \ge 0$.

Also, $som(G) \geq 0$. Therefore

$$som(G) \geq \left\lfloor \frac{d_m - 1 + \sqrt{(d_m-1)^2 + 8(n - Ext(G))(d_m-1)}}{2(d_m-1)} \right\rfloor.$$

□

The bound attains equality when $G \cong P_n$, $n \geq 2$.

Theorem 15. *For positive integers r,d and $l \geq 3$ such that $r \leq d \leq 2r$, there exists a connected $G(V,E)$ with radius, $r(G) = r$, diameter, $d(G) = d$ and strong open monophonic number, $som(G) = l$.*

Proof. When $r = 1$, consider $G = K_{1,l}$, then $d = 2$ and by Theorem 5, $som(G) = l$. When $r \geq 2$, let us construct the graph with the given properties.

Let $C_{2r} : x_0, x_1, x_2, \dots, x_{2r-1}$ be a cycle of order $2r$ and let $P_{d-r+1} : y_0, y_1, \dots, y_{d-r}$ be a path of order $d - r + 1$. Let G be the graph obtained by identifying the vertex y_0 in P_{d-r+1} with the vertex x_0 in C_{2r} and join $l - 2$ new vertices, $z_1, z_2, \dots, z_{l-2}$ to the vertex y_{d-r-1} and also join the edge $x_{r-1}x_{r+1}$ as shown in Figure 4. Then radius of G, $r(G) = r$, diameter of G, $d(G) = d$ and $som(G) = l$. □

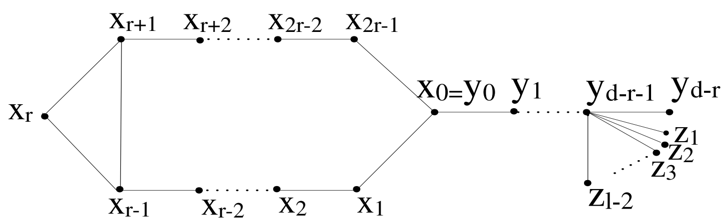

Figure 4. A graph G with radius r, diameter d and $som(G) = l$

6. Conclusion

This paper introduces the concept *strong open monophonic number* and some related results. The complexity is one of the core part and the concept has many applications.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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