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Research Article

Linear Stability of Thermo Convection in a Nanofluid Through a Porous Medium

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Abstract. This paper aims to investigate the thermal stability of unsteady incompressible nanofluid enclosed within a porous medium using linear stability analysis. The governing equation of nanofluid is framed based on Buongiorno's model and Darcy law is incorporated to represent flow through a porous substrate. The resulting Eigen value problem is solved by applying normal mode analysis and method of small oscillation is employed to find closed form solutions. Stability characteristics of flow fields have been discussed numerically.

Keywords. Nanofluids; Natural convection; Linear stability; Method of small oscillations

Mathematics Subject Classification (2020). 74Axx; 81Q15; 03C45; 76R10; 82D80

1. Introduction

Nanofluids are fluids which consist of nano sized suspended particles in the base fluid and it is of great importance due to its enhanced heat transfer characteristics. Several researchers have investigated about nanofluids and its thermal convection properties over the last few decades and some of their earlier contributions are listed below. Das *et al*. [4] carried out an extensive research survey about the heat transfer characteristics of nanofluids. Buongiorno [3] reported about the heat transfer characteristics of nanofluids and proposed that the fluid properties may vary at the boundaries due to temperature gradients.

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Hwang and Jang [7] investigated the thermal conductivity of various nano particles and found that enhancement of heat transfer could be achieved by increasing the volume fraction of nano particles. Thermal instability of a nanofluid has been reported by Tzou [15].

Thermal instability of nanofluids was examined by Nield and Kuznetsov [11] by employing Brinkman model. They predicted that instability is induced in presence of thermal and concentration of nanoparticle and oscillatory convection sets in if nano particle is heavy at the bottom.

Bhadauria *et al*. [2], and Kumar *et al*. [8] studied the thermal instability of nanofluids using linear and non-linear stability analysis. Yadav *et al*. [17] investigated the transport phenomenon of binary nanofluid and interpreted that thermo diffusion, diffusion thermo and Lewis number stabilizes the system under certain conditions while modified diffusivity, Lewis number and nano particle Rayleigh number induces instability under some circumstances.

Nield and Kuznetsov [12] studied the influence of natural convection due to internal heat generation and found that the presence of nano particles induces instability into the system. Umavathi [16] analyzed the onset of convection of nanofluid under the influence of varying sinusoidal wall temperature.

Onset of convection of a nanofluid with realistic boundary condition has been analyzed by Nield and Kuznetsov [13]. Shivakumara and Dhananjaya [14] carried out a numerical analysis to predict the convective transport phenomenon of nanofluids by considering the viscosity of nanofluids to dependent upon temperature.

Agarwal and Bhadauria [1] reported about the onset of thermal convection of Newtonian nanofluid under the impact of thermal non-equilibrium using linear and weakly nonlinear analysis. Onset of thermal instability in a horizontal porous plane has been carried out using linear and weakly nonlinear analysis by Kumar *et al*. [9].

Reena and Amith [10] examined numerically using Chebyshev pseudospectral method,the effect of heat source and sink on thermal convection by taking into account Darcy Brinkman Model.

Dhiman and Nivedita [5] investigated the impact of temperature dependent viscosity on thermal convection of nanofluid which is heated from below considering both rigid and stress-free boundary conditions. Dipak and Darbhasayanam [6] analyzed using numerical method both free and forced convection of a nanofluid bounded by a vertical enclosure.

Although several researchers have reported about the thermal convection of nanofluid through a porous medium under different conditions, significance of stability characteristics of flow field has not been discussed so far. Hence the intent of this work is to examine the linear stability of thermal convection of viscous incompressible nanofluid saturated by a permeable medium.

2. Geometric Formulation of the Problem

Consider a horizontal infinite layer of nanofluid enclosed between the planes $z = 0$ and $z = H$ bounded by a porous medium. The upper and lower walls are considered to be impermeable and perfectly thermally conducting. The thermal gradient of the lower and upper walls are assumed as T_h and T_c where $T_c > T_h$ and T_c is taken as the reference temperature. The z-axis is aligned in the transverse direction under the influence of gravitational field. The permeable membrane is modeled using Darcy law and it is presumed to hold homogeneity and local thermal equilibrium everywhere.

Using Boussinesq approximation the governing equations of nanofluid are defined as follows.

$$\nabla\cdot\overline{q}=0, \tag{1}$$

$$\frac{\rho}{\varepsilon}\frac{\partial\overline{q}}{\partial t}=-\nabla p+[\phi\rho_p+(1-\phi)\{\rho(1-\beta(T-T_c))\}]-\frac{\mu}{K}\overline{q}, \tag{2}$$

$$(\rho c)_m\frac{\partial T}{\partial t}+(\rho c)_f\overline{q}\cdot\nabla T=k_m\nabla^2T+\varepsilon(\rho c)_p\left[D_B\nabla\phi\cdot\nabla T+\frac{D_T}{T_c}\nabla T\cdot\nabla T\right], \tag{3}$$

$$\frac{\partial\phi}{\partial t}+\frac{1}{\varepsilon}\overline{q}\cdot\nabla\phi=D_B\nabla^2\phi+\frac{D_T}{T_c}\nabla^2T. \tag{4}$$

Temperature and the volumetric fraction of nano particles are assumed to be constant and hence the corresponding conditions prescribed at the boundary are

$$\left.\begin{array}{l} w=0,\ T=T_h,\ \phi=\phi_0\ \text{ at } z=0 \\ w=0,\ T=T_c,\ \phi=\phi_1\ \text{ at } z=H \end{array}\right\} \tag{5}$$

The dimensionless variables are defined as follows:

$$(x^*,y^*,z^*)=(x,y,z)/H,\ \ (u^*,v^*,w^*)=(u,v,w)H/\alpha_m,\ \ p^*=pK/\mu\alpha_m,$$

$$\phi^*=\frac{\phi-\phi_0}{\phi_1-\phi_0},\ \ T^*=\frac{T-T_c}{T_h-T_c}$$

where $\alpha_m=\frac{k_m}{(\rho c)_f}$, $\sigma=\frac{(\rho c)_m}{(\rho c)_f}$.

Equations (1)-(4) are represented in terms of dimensionless quantities as follows:

$$\nabla\cdot q=0, \tag{6}$$

$$\gamma_a\frac{\partial q}{\partial t}=-\nabla p-q-Ra_m\hat{e}_z+Ra_TT\hat{e}_z-Ra_n\phi\hat{e}_z, \tag{7}$$

$$\frac{\partial T}{\partial t}+q\cdot\nabla T=\nabla^2T+\frac{N_B}{Le}\nabla\phi\cdot\nabla T+\frac{N_AN_B}{Le}\nabla T\cdot\nabla T, \tag{8}$$

$$\frac{1}{\sigma}\frac{\partial\phi}{\partial t}+\frac{1}{\varepsilon}q\cdot\nabla\phi=\frac{1}{Le}\nabla^2\phi+\frac{N_A}{Le}\nabla^2T. \tag{9}$$

The transformed boundary conditions are

$$\left.\begin{array}{l} w=0,\ T=1,\ \phi=0\ \text{ at } z=0 \\ w=0,\ T=0,\ \phi=1\ \text{ at } z=1 \end{array}\right\} \tag{10}$$

where

$$\gamma_a=\frac{\eta}{Va},$$

$$Va=\frac{\varepsilon^2Pr}{Da}\ \text{(Vadasz number)},$$

$$\eta=\frac{\varepsilon}{\gamma}\ \text{(normalized porosity)},$$

$$Da=\frac{k}{H^2}\ \text{(Darcy number)},$$

$$Pr = \frac{\nu}{\alpha_m} \text{ (Prandtl number)},$$
$$Le = \frac{\alpha_m}{D_B} \text{ (Lewis number)},$$
$$Ra_T = \frac{\rho g \beta K H (T_h - T_c)}{\mu \alpha_m} \text{ (Thermal Rayleigh Darcy number)},$$
$$Ra_n = \frac{[(\rho_p - \rho)(\phi_1 - \phi_0)] g K H}{\mu \alpha_m} \text{ (concentration Rayleigh number)},$$
$$Ra_m = \frac{[\phi_0 \rho_p + \rho(1 - \phi_0)] g K H}{\mu \alpha_m} \text{ (Basic density Rayleigh number)},$$
$$N_A = \frac{D_T (T_h - T_c)}{(\phi_1 - \phi_0) D_B T_c} \text{ (modified diffusivity ratio)},$$
$$N_B = \frac{\varepsilon (\rho c)_p}{(\rho c)_f} (\phi_1 - \phi_0) \text{ (modified particle density increment)}.$$

3. Linear Stability Analysis

In quiescent state, the basic state of the system is assumed to be of the form

$$q = 0,\ T = T_b(z),\ \phi = \phi_b(z). \tag{11}$$

Then equations (7)-(9), on solving with appropriate boundary conditions yields

$$T_b = 1 - z \text{ and } \phi_b = z. \tag{12}$$

To analyze the stability of the system, the basic state is subjected to small perturbation of the form

$$q = q', p = p(z) + p', T = T_b(z) + T',\ \phi = \phi_b(z) + \phi'. \tag{13}$$

On substituting equation (13) in equations (6)-(9), linearized equations are obtained by omitting the product of prime quantities:

$$\nabla \cdot q' = 0, \tag{14}$$

$$\gamma_a \frac{\partial q'}{\partial t} = -\nabla p' - q' + Ra_T T' \hat{e}_z - Ra_n \phi' \hat{e}_z, \tag{15}$$

$$\frac{\partial T'}{\partial t} - w' = \nabla^2 T' + \frac{N_B}{Le}\left(\frac{\partial T'}{\partial z} - \frac{\partial \phi'}{\partial z}\right) + \frac{N_A N_B}{Le}\left(\frac{\partial T'}{\partial z}\right), \tag{16}$$

$$\frac{1}{\sigma}\frac{\partial \phi'}{\partial t} + \frac{1}{\varepsilon} w' = \frac{1}{Le}\nabla^2 \phi' + \frac{N_A}{Le}\nabla^2 T', \tag{17}$$

with corresponding boundary conditions

$$\left.\begin{array}{l} w' = 0,\ T' = 1,\ \phi' = 0 \text{ at } z = 0 \\ w' = 0,\ T' = 0,\ \phi' = 1 \text{ at } z = 1 \end{array}\right\} \tag{18}$$

The parameter Ra_m is just a measure of basic static pressure gradient and hence it is omitted in the subsequent equations. Eliminating pressure gradient from equation (15) by operating curl twice on both sides we get the following equation which is represented as follows,

$$\nabla^2 w + \gamma_a \frac{\partial}{\partial t}(\nabla^2 w) = Ra_T \nabla_H^2 T' - Ra_n \nabla_H^2 \phi', \tag{19}$$

where $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

Applying normal mode analysis to analyze the disturbance and assuming the perturbed quantities of the form

$$(w', T', \phi') = [W(z), \theta(z), \Phi(z)]e^{i(lx+my)+st} \tag{20}$$

where, l and m represents wave numbers in x and y directions respectively, while s denotes the rate of growth rate.

Upon substituting equation (20) in equations (19), (16) and (17) we get

$$(1+\gamma_a s)(D^2-\alpha^2)W = -Ra_T\alpha^2\theta + Ra_n\alpha^2\phi, \tag{21}$$

$$\left[D^2 + D\left(\frac{N_B}{Le} - 2\frac{N_A N_B}{Le}\right) - (s+\alpha^2)\right]\theta = -W + \frac{N_B}{Le}D\phi, \tag{22}$$

$$\left(D^2 - \alpha^2 - \frac{Le\ s}{\sigma}\right)\phi = \frac{Le}{\varepsilon}W - N_A(D^2-\alpha^2)\theta, \tag{23}$$

where $D = \frac{\partial}{\partial z}$ and $\alpha = \sqrt{l^2+m^2}$.

The appropriate boundary conditions are

$$\left.\begin{aligned} &W=0,\ \theta = D\theta = 0,\ \phi = 0 \ \text{ at } z=0 \\ &W=0,\ \theta = D\theta = 0,\ \phi = 0 \ \text{ at } z=1 \end{aligned}\right\} \tag{24}$$

4. Eigen Values and Eigen Functions

In order to obtain the solution, we assume the form

$$\left.\begin{aligned} W &= W_0 + \alpha^2 W_1 + \alpha^4 W_2 + \dots \\ \theta &= \theta_0 + \alpha^2\theta_1 + \alpha^4\theta_2 + \dots \\ \phi &= \phi_0 + \alpha^2\phi_1 + \alpha^4\phi_2 + \dots \\ s &= s_0 + \alpha^2 s_1 + \alpha^4 s_2 + \dots \end{aligned}\right\} \tag{25}$$

Collecting the coefficient of α of different order, we get the following equations:

$$(1+\gamma_a s)D^2W_0 = 0, \tag{26}$$

$$\left[D^2 + D\left(\frac{N_B}{Le} - 2\frac{N_A N_B}{Le}\right) - s_0\right]\theta_0 = -W_0 + \frac{N_B}{Le}D\phi_0, \tag{27}$$

$$\left(D^2 - \frac{Le\ s_0}{\sigma}\right)\phi_0 = \frac{Le}{\varepsilon}W_0 - N_A D^2\theta_0, \tag{28}$$

$$(1+\gamma_a s)D^2W_1 = -Ra_T\theta_0 + Ra_n\phi_0 + \gamma_a s_1 D^2 W_0 + (1+\gamma_a s)W_0, \tag{29}$$

$$\left[D^2 + D\left(\frac{N_B}{Le} - 2\frac{N_A N_B}{Le}\right) - s_0\right]\theta_1 = -W_1 + -(s_1+1)\theta_0 + \frac{N_B}{Le}D\phi_1, \tag{30}$$

$$\left(D^2 - \frac{Le\ s_0}{\sigma}\right)\phi_1 = \left(1 + \frac{Le\ s_1}{\sigma}\right)\phi_0 + \frac{Le}{\varepsilon}W_1 - N_A D^2\theta_1 + N_A\theta_0. \tag{31}$$

On solving the equations we get the solution as follows:

$$W_0 = 0,$$

$$\theta_0 = A\cosh r_1 z + B\sinh r_1 z + Ce^{r_2 z}\cosh r_3 z + De^{r_2 z}\sinh r_3 z,$$

$$\phi_0 = A_1\cosh r_1 z + B_1\sinh r_1 z + C_3 z\sinh r_1 z + C_4 z\cosh r_1 z + C_5 e^{r_2 z}\cosh r_3 z + C_6 e^{r_2 z}\sinh r_3 z,$$

$$
\begin{aligned}
W_1 &= A_2 + B_2 z + C_7 \cosh r_1 z + C_8 \sinh r_1 z + C_9 z \sinh r_1 z + C_{10} z \cosh r_1 z \\
&\quad + C_{11} e^{r_2 z} \cosh r_3 z + C_{12} e^{r_2 z} \sinh r_3 z, \\
\theta_1 &= A_3' \cosh r_1 z + B_3' \sinh r_1 z + B_4' e^{r_2 z} \cosh r_3 z + B_5' e^{r_2 z} \sinh r_3 z + D_{19} \\
&\quad + D_{20} z + B_6 z \cosh r_1 z + B_7 z \sinh r_1 z + B_8 z^2 \cosh r_1 z + B_9 z^2 \sinh r_1 z, \\
\phi_1 &= A_5 \cosh r_1 z + A_6 \sinh r_1 z + A_7 + A_8 z + A_9 z \cosh r_1 z \\
&\quad + A_{10} z \sinh r_1 z + A_{11} z^2 \cosh r_1 z + A_{12} z^2 \sinh r_1 z + A_{13} e^{r_2 z} \cosh r_3 z \\
&\quad + A_{14} e^{r_2 z} \sinh r_3 z + A_{15} z^3 e^{r_1 z} + A_{16} z^2 e^{r_1 z} + A_{17} z e^{r_1 z} + A_{18} z^3 e^{-r_1 z} \\
&\quad + A_{19} z^2 e^{-r_1 z} + A_{20} z e^{-r_1 z}.
\end{aligned}
$$

On applying boundary condition we get eigen value and eigen function as

$$
\begin{aligned}
&C(r_1 e^{r_2} \cosh r_3 - r_1 \cosh r_1 - r_2 \sinh r_1) + D(r_2 e^{r_2} \sinh r_3 - r_3 \sinh r_1) = 0 \\
&C(r_2 e^{r_2} \cosh r_3 + r_3 e^{r_2} \sinh r_3 - r_1 \sinh r_1 - r_2 \cosh r_1) + D(r_3 e^{r_2} \cosh r_3 \\
&\quad + r_2 e^{r_2} \sinh r_3 - r_3 \cosh r_1) = 0
\end{aligned}
$$

The solution of above expression will not give explicit values of s_0. Hence the approximate value of s_0 is calculated by using MATHEMATICA 8.0.

The higher order approximation of growth rate is as follows:

$$
s_1 = -\frac{D_{47}}{D_{48}}.
$$

5. Numerical Discussion

To interpret about the onset of thermal convection on unsteady incompressible flow on nanofluid confined between horizontal porous planes, the effect of dimensionless parameters like Lewis number (Le), Thermal Rayleigh Darcy number (Ra_T), Concentration Rayleigh number (Ra_n), modified particle density increment (N_B), modified diffusivity ratio (N_A), porosity parameter (ε) and thermal conductivity (σ) on flow characteristics has been illustrated in Figures 1–12. The parameters under considerations take the values as follows: $Le = 50.0$, $Ra = 5.0$, $N_A = 5.0$, $N_B = 0.5$, $R_n = 1.0$, $\sigma = 10.0$, $\varepsilon = 0.9$, $\gamma_a = 0.01$, $\alpha = 0.1$.

The influence of Lewis number, thermal Rayleigh Darcy number, Concentration Rayleigh number,modified particle density increment and porosity on temporal growth rate is depicted in Figures 1–5 and it is observed that withincrease inLe, Ra_T, Ra_n, N_B and ε the growth rate seems to be increased, which implies that disturbance grows exponentially thereby inducing instable modes wheneverthe value ofdiffusivity ratio parameteris taken to be negative.

Figures 6–8 represents that as modified diffusivity ratio, Lewis number and thermal conductivity increases, the growth rate decreases. Hence under the influence of these parameters the system remains in stable state whenever N_B assumes positive values.

Figures 9, 10 and 12 signifies the influence of Lewis number on flow fieldsand it is found that as Le increases temperature profile increases whereas velocity field and concentration profile is seen to be decreased.

Temperature rises steadily with increase in modified diffusivity ratio parameter as shown in Figure 11.

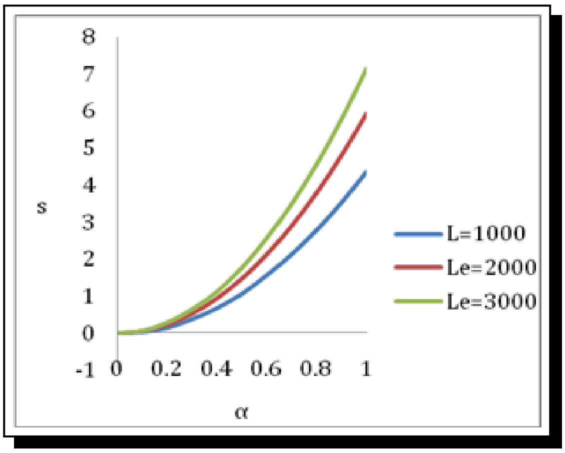

Figure 1. Variation of growth rate with respect to Lewis Number (Le)

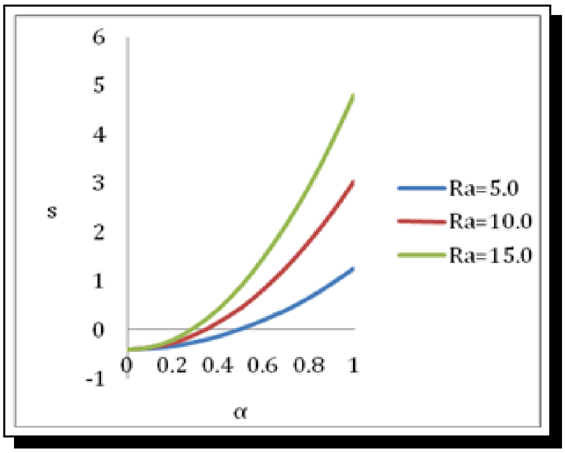

Figure 2. Variation of growth rate with respect to Thermal Rayleigh Darcy number (Ra_T)

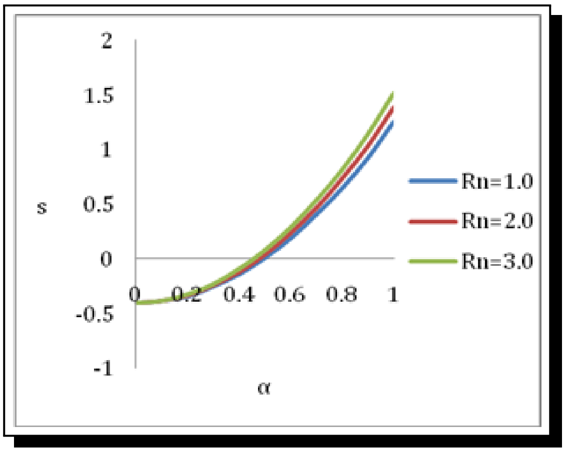

Figure 3. Variation of growth rate with respect to Concentration Rayleigh number (Ra_n)

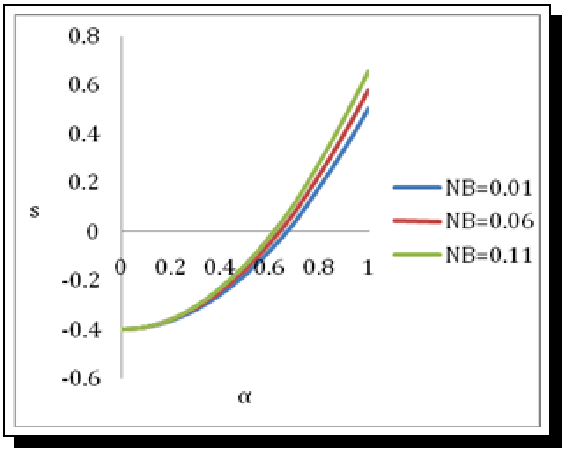

Figure 4. Variation of growth rate with respect to modified particle density increment (N_B)

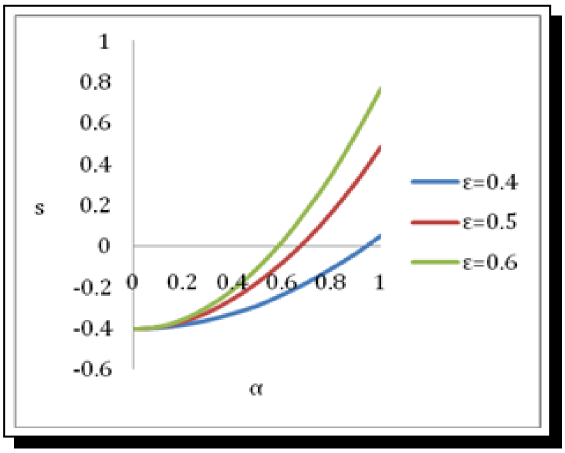

Figure 5. Variation of growth rate with respect to porosity parameter (ε)

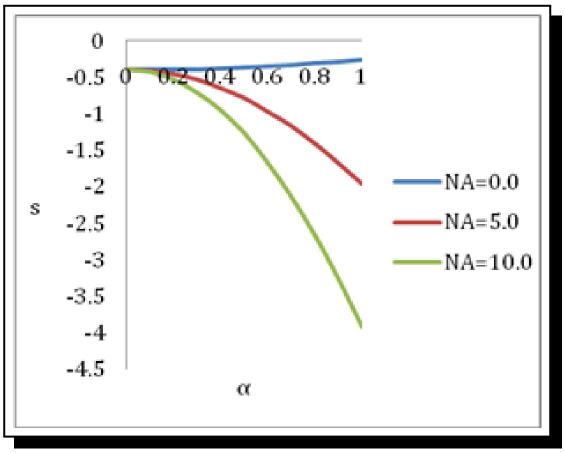

Figure 6. Variation of growth rate with respect to modified diffusivity ratio (N_A)

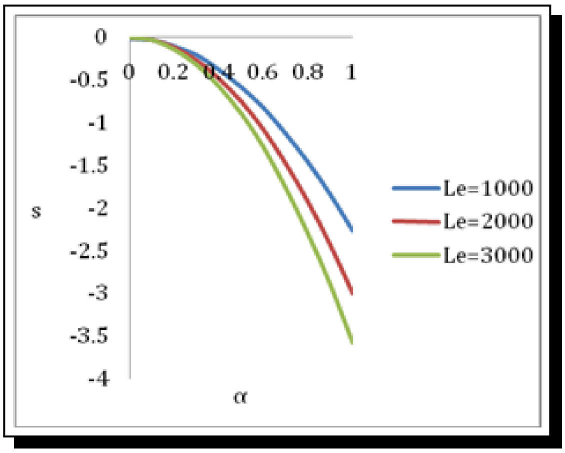

Figure 7. Variation of growth rate with respect to Lewis Number (Le)

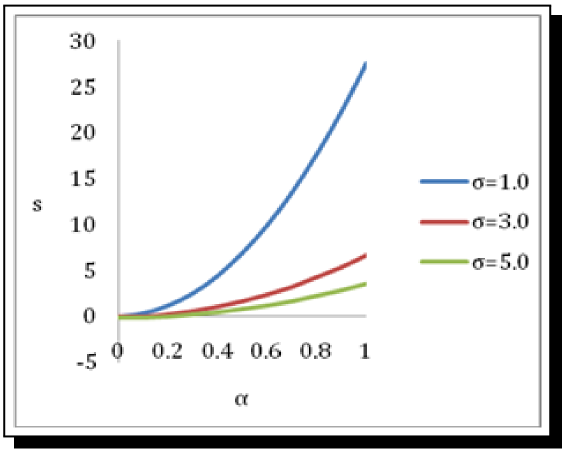

Figure 8. Variation of growth rate with respect to thermal conductivity (σ)

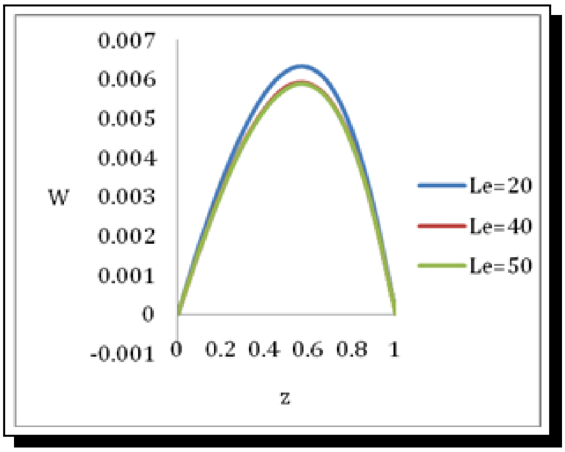

Figure 9. Variation of velocity profile with respect to Lewis number (Le)

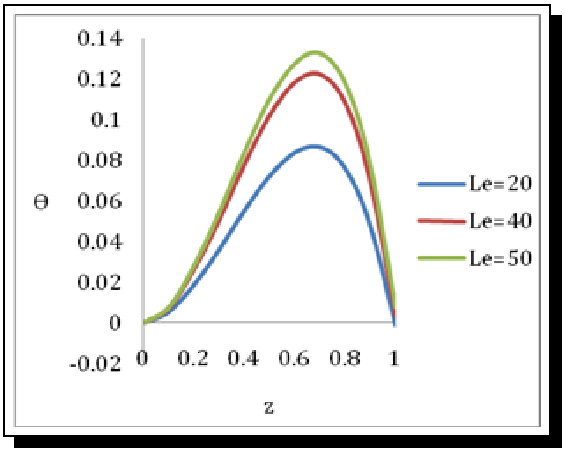

Figure 10. Variation of temperature profile with respect to Lewis number (Le)

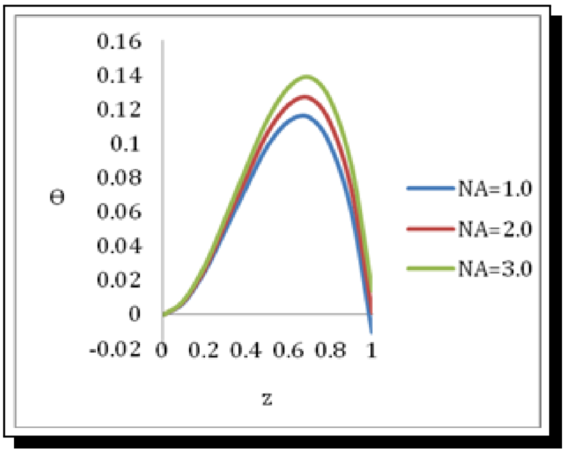

Figure 11. Variation of temperature profile with respect to modified diffusivity ratio (N_A)

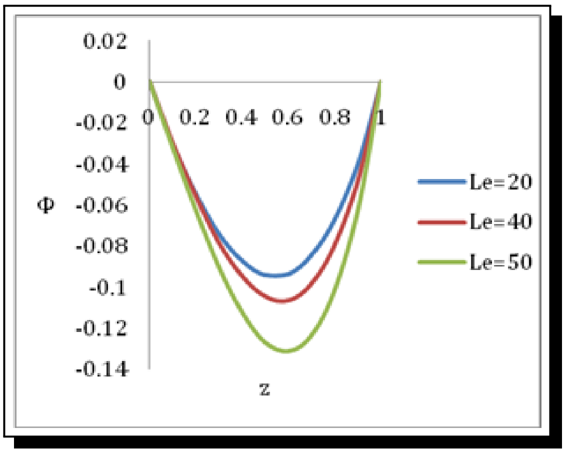

Figure 12. Variation of concentration profile with respect to Lewis number (Le)

6. Conclusion

A stability analysis is carried out to examine the onset of thermal convection of an incompressible nanofluid bounded by a saturated permeable porous medium using method of small oscillations. Influence of dimensionless parameters on stability of fluid flow has been represented graphically. Important finding is as follow:

- Temporal growth rate increases with increase in Lewis number, thermal Rayleigh Darcy number,concentration Rayleigh number, modified particle density increment and porosity. This indicates that instability sets in when N_B takes negative values.
- Increase in modified diffusivity ratio, Lewis number and thermal conductivity tend to stabilize the system when N_B takes positive values.

Appendix

$$r_1=\sqrt{\frac{Les_0}{\sigma}}\ ;\ r_2=-\frac{\left(\frac{N_B}{Le}-\frac{N_AN_B}{Le}\right)}{2}\ ;\ r_3=\sqrt{\frac{\left(\frac{N_B}{Le}-\frac{N_AN_B}{Le}\right)^2-4s_0}{2}}\ ;\ A=1\ ;\ C=1;$$

$$B=\frac{r_2-Dr_3}{r_1}\ ;\ D=\frac{r_1e^{r_2}\cosh r_3-r_1\cosh r_1-r_2\sinh r_1}{r_1e^{r_2}\sinh r_3-r_3\sinh r_1}\ ;\ C_1=C\left(r_2^2+r_3^2\right)+2Dr_2r_3\ ;$$

$$C_2=2Cr_2r_3+D(r_2^2+r_3^2)\ ;\ C_3=\frac{Ar_1^2}{2r_1}\ ;\ C_4=\frac{Br_1^2}{2r_1}\ ;\ C_5=\frac{C_1}{r_2^2+r_3^2-r_1^2}-\frac{2r_2r_3C_2}{(r_2^2+r_3^2-r_1^2)^2};$$

$$C_6=\frac{C_2}{r_2^2+r_3^2-r_1^2}-\frac{2r_2r_3C_1}{\left(r_2^2+r_3^2-r_1^2\right)^2}\ ;\ A_1=-C_5\ ;$$

$$B_1=-\frac{1}{\sinh r_1}\left[A_1\cosh r_1+C_3\ \sinh r_1+C_4\cosh r_1+C_5\ e^{r_2}\cosh r_3+C_6e^{r_2}\sinh r_3\right]\ ;$$

$$m_1=\frac{1}{(1+\gamma_a s_0)}\ ;\ C_7=\frac{(-R_aA+R_nA_1)m_1}{r_1^2}-\frac{2m_1}{r_1^3}\ ;\ C_8=\frac{(-R_aB+R_nB_1)m_1}{r_1^2}-\frac{2m_1}{r_1^3}\ ;$$

$$C_9=\frac{R_nC_3}{r_1^2}\ ;\ C_{10}=\frac{R_nC_4}{r_1^2}\ ;\ C_{11}=\frac{(-R_aC+R_nC_5)m_1}{r_2^2+r_3^2}-\frac{2r_2r_3m_1}{(r_2^2+r_3^2)^2}\ ;$$

$$C_{12}=\frac{(-R_aD+R_nC_6)m_1}{r_2^2+r_3^2}-\frac{2r_2r_3m_1}{(r_2^2+r_3^2)^2}\ ;$$

$$A_2=-(C_7+C_{11})\ ;$$

$$B_2=-\left[A_2+(C_7+C_{10})\cosh r_1+(C_8+C_9)\sinh r_1+C_{11}e^{r_2}\cosh r_3+C_{12}e^{r_2}\sinh r_3\right]\ ;$$

$$D_1=\frac{N_B}{\varepsilon}\ ;\ D_2=\frac{N_AN_B}{Le}\ ;\ D_3=\frac{N_B}{Le}\ ;\ D_4=\frac{N_B}{\sigma}\ ;\ D_5=r_1^2A_2+D_1B_2D_6=r_1^2B_2\ ;$$

$$D_7=-2r_1C_9+D_1r_1C_8+D_1C_{10}+D_2r_1B+D_3(r_1B_1+C_4)\ ;$$

$$D_8=D_4(r_1B_1+C_4)\ ;\ D_9=-2r_1C_{10}+D_1r_1C_7+D_1C_9+D_2r_1A+D_3(r_1A_1+C_3)\ ;$$

$$D_{10}=D_4(r_1A_1+C_3)\ ;\ D_{11}=D_1r_1C_9+D_3r_1C_3\ ;\ D_{12}=D_4r_1C_3\ ;\ D_{13}=D_1r_1C_{10}+D_3r_1C_4\ ;$$

$$D_{14}=D_4r_1C_4\ ;$$

$$D_{15}=\{-C_{11}(r_2^2+r_3^2-r_1^2)-2r_2r_3C_{12}+D_1r_2C_{11}+D_1r_3C_{12}+D_2Cr_2+D_2Dr_3+D_3(r_2C_5+C_6r_3)+C(r_2^2+r_3^2-r_1^2)+2Dr_2r_3\}\ ;$$

$$D_{16}=D_4(r_2C_5+C_6r_3)+C(r_2^2+r_3^2-r_1^2)+2Dr_2r_3\ ;$$

$$D_{17}=\{-2C_{11}r_2r_3-C_{12}(r_2^2+r_3^2-r_1^2)+D_1(r_2C_{12}+C_{11}r_3)+D_2(r_2D+Cr_3)+D_3(r_2C_6+C_5r_3)+2Cr_2r_3+D(r_2^2+r_3^2-r_1^2)\}\ ;$$

$$D_{18}=D_4(r_2C_6+C_5r_3)+D\left(r_2^2+r_3^2-r_1^2\right)+2Cr_2r_3\ ;$$

$$m_2=\left(\frac{N_B}{Le}-\frac{N_AN_B}{Le}\right)\ ;\ D_{19}=\frac{D_5}{r_1^2s_0}+\frac{D_6m_2}{r_1^2s_0^2}\ ;\ D_{20}=\frac{D_6}{r_1^2s_0}\ ;$$

$$m_{21}=r_1^2+m_2r_1-s_0\ ;\ m_3=r_1^2-m_2r_1-s_0\ ;$$

$$m_4=r_2^2+r_3^2-r_1^2\ ;\ m_5=r_2^2+r_3^2+m_2r_2-s_0\ ;\ m_6=\frac{2r_2r_3}{m_4}\ ;\ m_7=\frac{1}{m_4m_5}-\frac{(2r_2+m_2)m_6r_3}{m_4{m_5}^2}\ ;$$

$$m_8=\frac{m_6}{m_4m_5}+\frac{(2r_2+m_2)r_3}{m_4{m_5}^2}\ ;\ m_9=\frac{1}{m_4m_5}+\frac{(2r_2+m_2)m_6r_3}{m_4{m_5}^2}\ ;\ m_{11}=\frac{1}{4r_1m_{21}}+\frac{1}{4r_1m_3}\ ;$$

$$m_{12}=\frac{1}{2r_1m_{21}}\left(\frac{1}{2r_1}+\frac{2r_1+m_2}{m_{21}}\right)+\frac{1}{2r_1m_3}\left(\frac{1}{2r_1}+\frac{2r_1-m_2}{m_3}\right)\ ;$$

$$m_{13}=\frac{1}{2r_1m_{21}}\left(\frac{2r_1+m_2}{2r_1}-\frac{1}{m_{21}}\right)+\frac{1}{2r_1m_3}\left(\frac{2r_1-m_2}{2r_1}-\frac{1}{m_3}\right)\ ;$$

$$D_{21}=-\frac{D_7m_2}{2(r_1^2-s_0)^2}+\frac{D_9}{2r_1(r_1^2-s_0)}\ ;\ D_{22}=-\frac{D_8m_2}{2(r_1^2-s_0)^2}+\frac{D_{10}}{2r_1(r_1^2-s_0)}\ ;$$

$$D_{23}=-\frac{D_9m_2}{2(r_1^2-s_0)^2}+\frac{D_7}{2r_1(r_1^2-s_0)}\ ;\ D_{24}=-\frac{D_{10}m_2}{2(r_1^2-s_0)^2}+\frac{D_8}{2r_1(r_1^2-s_0)}\ ;\ D_{25}=\frac{(D_{11}+D_{13})}{8r_1m_{21}}\ ;$$

$$D_{26}=\frac{(D_{12}+D_{14})}{8r_1m_{21}}\ ;\ D_{27}=\frac{(D_{11}+D_{13})m_{11}}{4r_1m_{21}}\ ;\ D_{28}=\frac{(D_{12}+D_{14})m_{11}}{4r_1m_{21}}\ ;\ D_{29}=\frac{(D_{11}+D_{13})m_{12}}{4r_1m_{21}}\ ;$$

$$D_{30}=\frac{(D_{12}+D_{14})m_{12}}{4r_1m_{21}}\ ;\ D_{31}=\frac{(D_{13}-D_{11})}{8r_1m_3}\ ;$$

$$D_{32}=\frac{(D_{14}-D_{12})}{8r_1m_3}\ ;\ D_{33}=\frac{(D_{13}-D_{11})m_{13}}{4r_1m_3}\ ;\ D_{34}=\frac{(D_{14}-D_{12})m_{13}}{4r_1m_3}\ ;\ D_{35}=\frac{(D_{13}-D_{11})m_{14}}{4r_1m_3}\ ;$$

$$D_{36}=\frac{(D_{14}-D_{12})m_{14}}{4r_1m_3}\ ;\ D_{37}=D_{15}m_7-D_{17}m_{10}\ ;\ D_{38}=D_{16}m_7-D_{18}m_{10}\ ;$$

$$D_{39}=D_{17}m_9-D_{15}m_8\ ;\ D_{40}=D_{18}m_9-D_{16}m_8\ ;$$

$$\begin{aligned}D_{41}=\Bigg\{&D_{19}(\cosh r_1-1)+D_{20}\left(\frac{\sinh r_1}{r_1}-1\right)+D_{21}\left(\frac{\sinh r_1}{r_1}-\cosh r_1\right)-D_{23}\sinh r_1-D_{25}e^{r_1}\\&+D_{27}\left(-e^{r_1}+\frac{\sinh r_1}{r_1}\right)+D_{29}(\cosh r_1+\sinh r_1-e^{r_1})-D_{31}e^{-r_1}+D_{33}\left(-e^{-r_1}+\frac{\sinh r_1}{r_1}\right)\\&+D_{35}\left(\cosh r_1-\sinh r_1-e^{-r_1}\right)+D_{37}\left(\cosh r_1+\frac{r_2\sinh r_1}{r_1}-e^{r_2}\cosh r_3\right)\\&+D_{39}\left(\frac{r_3\sinh r_1}{r_1}-e^{r_2}\sinh r_3\right)\Bigg\}\ ;\end{aligned}$$

$$D_{42} = \left\{D_{22}\left(\frac{\sinh r_1}{r_1} - \cosh r_1\right) - D_{24}\sinh r_1 - D_{26}e^{r_1} + D_{28}\left(-e^{r_1} + \frac{\sinh r_1}{r_1}\right)\right.$$
$$+ D_{30}\left(\cosh r_1 + \sinh r_1 - e^{r_1}\right) - D_{32}e^{-r_1} + D_{34}\left(-e^{-r_1} + \frac{\sinh r_1}{r_1}\right)$$
$$+ D_{36}\left(\cosh r_1 - \sinh r_1 - e^{-r_1}\right) + D_{38}\left(\cosh r_1 + \frac{r_2\sinh r_1}{r_1} - e^{r_2}\cosh r_3\right)$$
$$\left. + D_{40}\left(\frac{r_3\sinh r_1}{r_1} - e^{r_2}\sinh r_3\right)\right\} ;$$

$$D_{43} = \{r_1D_{19}\sinh r_1 + D_{20}(\cosh r_1 - 1) - D_{21}(r_1\sinh r_1) - D_{23}(r_1\cosh r_1 + \sinh r_1)$$
$$- D_{25}(r_1e^{r_1} + 2e^{r_1}) + D_{27}(\cosh r_1 - r_1e^{r_1} - e^{r_1}) + D_{29}(r_1\cosh r_1 - r_1e^{r_1} + r_1\sinh r_1)$$
$$- D_{31}(-r_1e^{-r_1} + 2e^{-r_1}) + D_{33}(\cosh r_1 + r_1e^{-r_1} - e^{-r_1}) + D_{35}(-r_1\cosh r_1 + r_1e^{-r_1} + r_1\sinh r_1)$$
$$+ D_{37}(r_1\sinh r_1 - r_3e^{r_2}\sinh r_3 - r_2e^{r_2}\cosh r_3 + r_2\cosh r_1)$$
$$+ D_{33}(r_3\cosh r_1 - r_3e^{r_2}\cosh r_3 - r_2e^{r_2}\sinh r_3)\} ;$$

$$D_{44} = \{-D_{22}r_1\sinh r_1 - D_{24}(r_1\cosh r_1 + \sinh r_1) - D_{26}(r_1e^{r_1} + 2e^{r_1}) + D_{28}(\cosh r_1 - r_1e^{r_1} - e^{r_1})$$
$$+ D_{30}(r_1\cosh r_1 - r_1e^{r_1} + r_1\sinh r_1) - D_{32}(-r_1e^{-r_1} + 2e^{-r_1}) + D_{34}(\cosh r_1 + r_1e^{-r_1} - e^{-r_1})$$
$$+ D_{36}(-r_1\cosh r_1 + r_1e^{-r_1} + r_1\sinh r_1) + D_{38}(r_1\sinh r_1 - r_3e^{r_2}\sinh r_3 - r_2e^{r_2}\cosh r_3 + r_2\cosh r_1)$$
$$+ D_{40}(r_3\cosh r_1 - r_3e^{r_2}\cosh r_3 - r_2e^{r_2}\sinh r_3)\} ;$$

$$D_{45} = \left(r_3e^{r_2}\cosh r_3 + r_2e^{r_2}\sinh r_3 - r_3\cosh r_1\right) ; \quad D_{46} = \left(e^{r_2}\sinh r_3 - \frac{r_3}{r_1}\sinh r_1\right) ;$$

$$D_{47} = D_{41}D_{45} - D_{43}D_{46} ; \quad D_{48} = D_{42}D_{45} - D_{44}D_{46} ;$$

$$A_3 = -(B_4 + D_{19} + (D_{29} + s_1D_{30}) + (D_{35} + s_1D_{36}) + (D_{37} + s_1D_{38})) ;$$

$$B_3 = -\frac{1}{r_1}(r_2B_4 + r_3B_5 + D_{20} + (D_{21} + s_1D_{22}) + (D_{27} + s_1D_{28}) + (D_{29} + s_1D_{30})r_1 + (D_{33} + s_1D_{34})$$
$$- (D_{35} + s_1D_{36})r_1 + (D_{37} + s_1D_{38})r_2 + (D_{39} + s_1D_{40})r_3) ;$$

$$B_4 = 1 ; \quad B_5 = \frac{D_{41} + s_1D_{42} - \left(e^{r_2}\cosh r_3 - \cosh r_1 - \frac{r_2}{r_1}\sinh r_1\right)}{e^{r_2}\sinh r_3 - \frac{r_3}{r_1}\sinh r_1} ; \quad B_6 = (D_{21} + s_1D_{22}) ;$$

$$B_7 = (D_{23} + s_1D_{24}) ; \quad B_8 = (D_{25} + s_1D_{26}) ; \quad B_9 = (D_{27} + s_1D_{28}); B_{10} = (D_{29} + s_1D_{30}) ;$$

$$B_{11} = (D_{31} + s_1D_{32}) ; \quad B_{12} = (D_{33} + s_1D_{34}) ; \quad B_{13} = (D_{35} + s_1D_{36}) ; \quad B_{14} = (D_{37} + s_1D_{38}) ;$$

$$B_{15} = (D_{39} + s_1D_{40}) ;$$

$$B_4' = B_4 + B_{14} ; \quad B_5' = B_5 + B_{15} ; \quad B_{16} = \frac{Le}{\varepsilon} ; \quad B_{17} = \left(1 + \frac{Le\ s_1}{\sigma}\right) ;$$

$$B_{18} = B_{16}C_7 + B_{17}A_1 + N_AA - N_A(r_1^2A_3 + 2r_1B_7) ;$$

$$B_{19} = B_{16}C_8 + B_{17}B_1 + N_AB - N_A(r_1^2B_3 + 2r_1B_6) ; \quad B_{20} = B_{16}C_9 + B_{17}C_3 - N_Ar_1^2B_6 ;$$

$$B_{21} = B_{16}C_{10} + B_{17}C_4 - N_Ar_1^2B_7 ; \quad B_{22} = B_{16}C_{11} + B_{17}C_5 + N_AC - N_A((r_2^2 + r_3^2)B_4' + 2r_2r_3B_5') ;$$

$$B_{23} = B_{16}C_{12} + B_{17}C_6 + N_AD - N_A((r_2^2 + r_3^2)B_5' + 2r_2r_3B_4') ; \quad B_{24} = -N_AB_8 ;$$

$$B_{25} = -N_A(4r_1B_8 + {r_1}^2B_9) ; \quad B_{26} = -N_A(2B_8 + 2r_1B_9 + {r_1}^2B_{10}) ; \quad B_{27} = -N_Ar_1^2B_{11} ;$$

$$B_{28} = -N_A(4r_1B_{11} + {r_1}^2B_{12})\ ;\ B_{29} = -N_A(2B_{11} - 2r_1B_{12} + {r_1}^2B_{13})\ ;\ A_7 = -\frac{B_{16}A_2}{r_1^2}\ ;$$

$$A_8 = -\frac{B_{16}B_2}{r_1^2}\ ;\ A_9 = \frac{B_{19}}{2r_1} - \frac{B_{20}}{4r_1^2}\ ;\ A_{10} = \frac{B_{18}}{2r_1} - \frac{B_{21}}{4r_1^2}\ ;\ A_{11} = \frac{B_{21}}{4r_1};A_{12} = \frac{B_{20}}{4r_1}\ ;$$

$$A_{13} = \frac{B_{22}}{m_4} - \frac{2B_{23}r_2r_3}{m_4^2}\ ;\ A_{14} = \frac{B_{23}}{m_4} - \frac{2B_{22}r_2r_3}{m_4^2}\ ;$$

$$A_{15} = \frac{B_{24}}{6r_1}\ ;\ A_{16} = \frac{B_{25}}{4r_1} - \frac{B_{24}}{4r_1^2}\ ;\ A_{17} = \frac{B_{26}}{2r_1} - \frac{B_{25}}{4r_1^2}\ ;\ A_{18} = -\frac{B_{27}}{6r_1}\ ;$$

$$A_{19} = -\frac{B_{28}}{4r_1} - \frac{B_{27}}{4r_1^2}\ ;\ A_{20} = -\frac{B_{29}}{2r_1} - \frac{B_{28}}{4r_1^2}\ ;\ A_5 = -(A_7 + A_{13})\ ;$$

$$A_6 = -\frac{1}{\sinh r_1}[(A_5 + A_9 + A_{11})\cosh r_1 + A_7 + A_8 + (A_{10} + A_{12})\sinh r_1 + A_{13}e^{r_2}\cosh r_3 + A_{14}e^{r_2}\sinh r_3 + (A_{15} + A_{16} + A_{17})e^{r_1} + (A_{18} + A_{19} + A_{20})e^{-r_1}].$$

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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