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On Hyponormality of Toeplitz Operators on the Weighted Bergman Space

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Abstract. In this paper we give necessary and sufficient conditions for the hyponormality of the Toeplitz operator T_φ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$ for $\varphi(z) = \bar{a}_{-3}\bar{z}^3 + \bar{a}_{-1}\bar{z} + a_1 z + a_3 z^3$ and $\varphi(z) = \bar{a}_{-3}\bar{z}^3 + \bar{a}_{-2}\bar{z}^2 + a_2 z^2 + a_3 z^3$.

1. **Introduction**

Let $\mathbb{D}$ denote the open unit disc in the complex plane. For $-1 < \alpha < \infty$, $L^2(\mathbb{D}, dA_\alpha)$ is the space of functions on $\mathbb{D}$ which are square integrable with respect to the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$, where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$. $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}\,dA_\alpha(z)$$

for $f, g \in L^2(\mathbb{D}, dA_\alpha)$. The weighted Bergman space A^2_α is the closed subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of analytic functions on $\mathbb{D}$. If $\alpha = 0$, A^2_0 is the Bergman space. For any non negative integer n and $z \in \mathbb{D}$, let $e_n(z) = \frac{z^n}{\gamma_n}$ where $\gamma_n^2 = \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}$, where $\Gamma(s)$ stands for the usual Gamma function. Then $\{e_n\}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D})$. Also, the reproducing kernel of $A^2_\alpha(\mathbb{D})$ is given by $k_z^{(\alpha)}(w) = \frac{1}{(1-\bar{z}w)^{2+\alpha}}$ for $z, w \in \mathbb{D}$. If $L^\infty(\mathbb{D})$ denotes the space of all essentially bounded, measurable functions then for $\varphi \in L^\infty(\mathbb{D})$ the multiplication operator M_φ on $A^2_\alpha(\mathbb{D})$ is defined by $M_\varphi(f) = \varphi f$. The orthogonal projection P_α of $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$ is given by

$$(P_\alpha f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^{2+\alpha}}\,dA_\alpha(w),$$

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for $f \in L^2(\mathbb{D}, dA_\alpha)$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ with symbol φ is defined on $A^2_\alpha(\mathbb{D})$ by $T_\varphi f = P_\alpha(\varphi \cdot f)$. Thus we have

$$T_\varphi f(z) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w),$$

for $f \in A^2_\alpha(\mathbb{D})$ and $z \in \mathbb{D}$. Similarly, the Hankel operator H_φ on $A^2_\alpha(\mathbb{D})$ is defined by $H_\varphi f = J(I - P_\alpha)(\varphi \cdot f)$, where J on $\mathbb{L}^2$ is defined as $J(e^{in\theta}) = e^{-i(n+1)\theta}$. As $\varphi \in L^\infty(\mathbb{D})$, the operators T_φ and H_φ are bounded. For details on these results, we refer the reader to [10].

A bounded linear operator A on a Hilbert space is said to be hyponormal if its self commutator $[A^*, A] := A^*A - AA^*$ is positive semi-definite. Hyponormality of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ was characterized by Cowen [2], and subsequently by Nakazi and Takahashi [7]. The solution is based on a dilation theorem of Sarason [9]. As no such dilation theorem is available on the Bergman space, the question concerning the characterization of hyponormal Toeplitz operators on the Bergman space is still open. For $\varphi = f + \bar{g}$ with f, g bounded analytic, H. Sadraoui proved the following result:

Theorem 1.1 ([8]). *Let f, g be bounded and analytic in $L^2(\mathbb{D}, dA)$. Then the followings are equivalent:*

(i) *$T_{f+\bar{g}}$ is hyponormal ;*

(ii) *$H^*_{\bar{g}} H_{\bar{g}} \leq H^*_{\bar{f}} H_{\bar{f}}$;*

(iii) *$H_{\bar{g}} = C H_{\bar{f}}$, where C is of norm less than or equal to one.*

The above result also holds on weighted Bergman space $A^2_\alpha(\mathbb{D})$ with $-1 < \alpha < \infty$. Also, for a certain trigonometric polynomial φ, I.S. Hwang proved the following result:

Theorem 1.2 ([3]). *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$. If $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$, then T_φ on $A^2(\mathbb{D})$ is hyponormal*

$$\iff \begin{cases} \dfrac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}| \end{cases}$$

For the weighted Bergman space $A^2_\alpha(\mathbb{D})$, a similar result is the following:

Theorem 1.3 ([4]). *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_1 z + a_2 z^2$ and $g(z) = a_{-1} z + a_{-2} z^2$. If $a_1 \bar{a}_2 = a_{-1} \bar{a}_{-2}$ and $\alpha \geq 0$, then T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal*

$$\iff \begin{cases} \dfrac{1}{\alpha+3}(|a_2|^2 - |a_{-2}|^2) \geq \frac{1}{2}(|a_{-1}|^2 - |a_1|^2) & \text{if } |a_{-2}| \leq |a_2| \\ 4(|a_{-2}|^2 - |a_2|^2) \leq |a_1|^2 - |a_{-1}|^2 & \text{if } |a_2| \leq |a_{-2}| \end{cases}$$

In this paper we continue this line of investigation and establish hyponormality conditions of T_φ on $A^2_\alpha(\mathbb{D})$ for the following cases:

1. Where $\varphi = \bar{g} + f$ and $f(z) = a_1 z + a_3 z^3$, $g(z) = a_{-1} z + a_{-3} z^3$ with $a_1 \bar{a}_3 = a_{-1}\bar{a}_{-3}$ and $\alpha \geq 0$.
2. Where $\varphi = \bar{g} + f$ and $f(z) = a_2 z + a_3 z^3$, $g(z) = a_{-2} z + a_{-3} z^3$ with $a_2 \bar{a}_3 = a_{-2}\bar{a}_{-3}$ and $\alpha \geq 0$.

2. **Preliminaries**

Since the hyponormality of operators is translation invariant, we may assume that $f(0) = g(0) = 0$. We recall the following properties of Toeplitz operators:

If $f, g \in L^\infty(\mathbb{D},)$, then

(i) $T_{f+g} = T_f + T_g$;

(ii) $T^*_f = T_{\bar{f}}$;

(iii) $T_{\bar{f}} T_g = T_{\bar{f}g}$ if f or g is analytic.

Also, if P_α denotes the projection of $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$, then we have the following useful lemmas:

Lemma 2.1 ([4])**.** *For any s, t nonnegative integers,*

$$P_\alpha(\bar{z}^t z^s) = \begin{cases} \dfrac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t \end{cases}$$

Lemma 2.2 ([1], [6])**.** *Fix $m \geq 1$. Then for $\alpha \geq -1$,*

(i) $$H_{\bar{z}^m}(z^k)(\xi) = \begin{cases} \bar{\xi}^m \xi^k & \text{if } 0 \leq k < m \\ \bar{\xi}^m \xi^k - \dfrac{\gamma_k^2}{\gamma_{k-m}^2} & \text{if } m \leq k \text{ ;} \end{cases}$$

(ii) *the functions $\{H_{\bar{z}^m}(z^k)\}_{k=0}^\infty$ are orthogonal in $L^2(\mathbb{D}, dA_\alpha)$;*

(iii) $H^*_{\bar{z}^m} H_{\bar{z}^m}(z^k)(\xi) = \omega^2_{mk} \xi^k$ $k = 0, 1, 2, \ldots$*, where*

$$\omega^2_{mk} = \begin{cases} \dfrac{\gamma^2_{k+m}}{\gamma^2_k} & \text{if } 0 \leq k < m \\ \dfrac{\gamma^2_{k+m}}{\gamma^2_k} - \dfrac{\gamma^2_k}{\gamma^2_{k-m}} & \text{if } m \leq k \text{ ;} \end{cases}$$

(iv) $\|H_{\bar{z}^m}(z^k)\|_\alpha = \omega_{mk}\gamma_k$.

Moreover, writing $k_i(z) := \sum_{n=0}^\infty c_{Nn+i} z^{Nn+i}$ we have:

Lemma 2.3. *For $0 < m < N$ and $i = 0, 1, \ldots, N-1$, we have*

(i) $$\|\bar{z}^m k_i(z)\|^2_\alpha = \sum_{n=0}^\infty \frac{(Nn+m+i)!\Gamma(\alpha+2)}{\Gamma(Nn+m+i+\alpha+2)} |c_{Nn+i}|^2$$

(ii) $\|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2$

$$= \begin{cases} \sum_{k=1}^{\infty} \frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}|c_{Nk+i}|^2 & \text{if } m > i \\ \sum_{k=0}^{\infty} \frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}|c_{Nk+i}|^2 & \text{if } m \le i \end{cases}$$

Proof. (i) $\|\bar{z}^m k_i(z)\|_\alpha^2$

$$= \left\langle \bar{z}^m \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}, \bar{z}^m \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i} \right\rangle_\alpha$$

$$= \left\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i+m}, \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i+m} \right\rangle_\alpha$$

$$= \sum_n \sum_k c_{Nn+i}\bar{c}_{Nk+i} \langle z^{Nn+i+m}, z^{Nk+i+m}\rangle_\alpha$$

$$= \sum_n \sum_k c_{Nn+i}\bar{c}_{Nk+i} \gamma_{Nn+i+m}\gamma_{Nk+i+m} \langle e_{Nn+i+m}(z), e_{Nk+i+m}(z)\rangle_\alpha$$

$$= \sum_n |c_{Nn+i}|^2 \gamma_{Nn+i+m}^2$$

$$= \sum_n \frac{(Nn+i+m)!\Gamma(\alpha+2)}{\Gamma(Nn+i+m+\alpha+2)}|c_{Nn+i}|^2$$

(ii) $\|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2$

$$= \left\| \sum_{n=0}^{\infty} c_{Nn+i} P_\alpha(\bar{z}^m z^{Nn+i}) \right\|_\alpha^2$$

$$= \begin{cases} \left\| \sum_{n=0}^{\infty} c_{Nn+i} \frac{\Gamma(Nn+i+1)\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)\Gamma(Nn+i-m+1)} z^{Nn+i-m} \right\|_\alpha^2 & \text{if } m \le i \\ \left\| \sum_{n=1}^{\infty} c_{Nn+i} \frac{\Gamma(Nn+i+1)\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)\Gamma(Nn+i-m+1)} z^{Nn+i-m} \right\|_\alpha^2 & \text{if } m > i \end{cases}$$

$$= \begin{cases} \sum_{n=0}^{\infty} \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)^2} \gamma_{Nn+i-m}^2 |c_{Nn+i}|^2 & \text{if } m \le i \\ \sum_{n=1}^{\infty} \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)^2} \gamma_{Nn+i-m}^2 |c_{Nn+i}|^2 & \text{if } m > i \end{cases}$$

$$= \begin{cases} \sum_{n=0}^{\infty} \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2 & \text{if } m \le i \\ \sum_{n=1}^{\infty} \frac{(Nn+i)!^2\Gamma(Nn+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nn+i+\alpha+2)^2\Gamma(Nn+i-m+1)} |c_{Nn+i}|^2 & \text{if } m > i \end{cases} \qquad \square$$

Lemma 2.4. *For $m \geq 1$, $\langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha = 0$ for $i \neq j$.*

Proof. $\langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha = \sum_n \sum_k c_{Nn+i}\bar{c}_{Nk+j} \langle H_{\bar{z}^m} z^{Nn+i}, H_{\bar{z}^m} z^{Nk+j}\rangle_\alpha$.

Now, $Nn+i = Nk+j$ if and only if $N(n-k) = j-i$. As $0 \leq i,\ j \leq N-1$, so $0 \leq j-i \leq N-1$. Thus, $N(n-k) = j-i \Longrightarrow n-k = 0$ i.e. $n = k$. This gives $i = j$. Thus if $i \neq j$ then $Nn+i \neq Nk+j$ for all n, k and so by Lemma 2.2, $\langle H_{\bar{z}^m} z^{Nn+i}, H_{\bar{z}^m} z^{Nk+j}\rangle_\alpha = 0$. Thus, $\langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha = 0$ if $i \neq j$. □

Lemma 2.5. *Let $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ with $0 < m < N$. Let $\alpha > -1$ and $a_m \bar{a}_N = a_{-m}\bar{a}_{-N}$. Then for $i \neq j$ we have*

$$\langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z)\rangle_\alpha = \langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z)\rangle_\alpha .$$

Proof.

$$\begin{aligned}\langle H_{\bar{f}} k_i(z), H_{\bar{f}} k_j(z)\rangle_\alpha &= \langle \bar{a}_m H_{\bar{z}^m} k_i(z) + \bar{a}_N H_{\bar{z}^N} k_i(z), \bar{a}_m H_{\bar{z}^m} k_j(z) + \bar{a}_N H_{\bar{z}^N} k_j(z)\rangle_\alpha \\ &= a_m\bar{a}_N \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha + \bar{a}_m a_N \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z)\rangle_\alpha \\ &\qquad \text{(by Lemma 2.4)}\end{aligned}$$

and similarly,

$$\langle H_{\bar{g}} k_i(z), H_{\bar{g}} k_j(z)\rangle_\alpha = a_{-m}\bar{a}_{-N} \langle H_{\bar{z}^N} k_i(z), H_{\bar{z}^m} k_j(z)\rangle_\alpha + \bar{a}_{-m} a_{-N} \langle H_{\bar{z}^m} k_i(z), H_{\bar{z}^N} k_j(z)\rangle_\alpha .$$

Since $a_m\bar{a}_N = a_{-m}\bar{a}_{-N}$, hence we have the result. □

Lemma 2.6. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_m z^m + a_N z^N$, $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$. If $\alpha > -1$ and $a_m\bar{a}_N = a_{-m}\bar{a}_{-N}$ then T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal iff*

$$\sum_{t=0}^{\infty} \left\{ (|a_m|^2 - |a_{-m}|^2) N_\alpha(t) + (|a_N|^2 - |a_{-N}|^2) D_\alpha(t) \right\} \frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)} |c_t|^2 \geq 0$$

where $N_\alpha(t) = \dfrac{\prod_{j=1}^{m}(t+j)}{\prod_{j=1}^{m}(t+\alpha+j+1)} - \dfrac{\prod_{j=1}^{m}(t-j+1)}{\prod_{j=1}^{m}(t+\alpha-j+2)}$ *and* $D_\alpha(t) = \dfrac{\prod_{j=1}^{N}(t+j)}{\prod_{j=1}^{N}(t+\alpha+j+1)} - \dfrac{\prod_{j=1}^{N}(t-j+1)}{\prod_{j=1}^{N}(t+\alpha-j+2)}$.

Proof. For $i = 0, 1, \ldots, N-1$, let $K_i := \left\{ k_i \in A^2_\alpha : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \right\}$.

By Theorem 1.1, T_φ is hyponormal if and only if

$$\left\langle (H^*_{\bar{f}} H_{\bar{f}} - H^*_{\bar{g}} H_{\bar{g}}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \right\rangle_\alpha \geq 0$$

Iff $$\left\langle H_{\bar{f}}\left(\sum_{i=0}^{N-1} k_i(z)\right), H_{\bar{f}}\left(\sum_{i=0}^{N-1} k_i(z)\right)\right\rangle_\alpha \geq \left\langle H_{\bar{g}}\left(\sum_{i=0}^{N-1} k_i(z)\right), H_{\bar{g}}\left(\sum_{i=0}^{N-1} k_i(z)\right)\right\rangle_\alpha$$

Iff $\displaystyle\sum_{i=0}^{N-1} \|H_{\bar{f}}(k_i(z))\|_\alpha^2 \geq \sum_{i=0}^{N-1} \|H_{\bar{g}}(k_i(z))\|_\alpha^2$, using Lemma 2.5.

Iff

$$\sum_{i=0}^{N-1} \left[\|M_{\bar{f}}(k_i(z))\|_\alpha^2 - \|T_{\bar{f}}(k_i(z))\|_\alpha^2\right] \geq \sum_{i=0}^{N-1} \left[\|M_{\bar{g}}(k_i(z))\|_\alpha^2 - \|T_{\bar{g}}(k_i(z))\|_\alpha^2\right] \quad (2.1)$$

We have,

$$\begin{aligned}\|M_{\bar{f}}(k_i(z))\|_\alpha^2 &= \|\bar{f}(k_i(z))\|_\alpha^2 \\ &= \langle (\bar{a}_m \bar{z}^m + \bar{a}_N \bar{z}^N) k_i(z), (\bar{a}_m \bar{z}^m + \bar{a}_N \bar{z}^N) k_i(z) \rangle_\alpha \\ &= |a_m|^2 \|\bar{z}^m k_i(z)\|_\alpha^2 + |a_N|^2 \|\bar{z}^N k_i(z)\|_\alpha^2 \\ &\quad + \bar{a}_m a_N \langle \bar{z}^m k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_m \bar{a}_N \langle \bar{z}^N k_i(z), \bar{z}^m k_i(z) \rangle_\alpha\end{aligned}$$

Similarly,

$$\begin{aligned}\|M_{\bar{g}}(k_i(z))\|_\alpha^2 &= |a_{-m}|^2 \|\bar{z}^m k_i(z)\|_\alpha^2 + |a_{-N}|^2 \|\bar{z}^N k_i(z)\|_\alpha^2 \\ &\quad + \bar{a}_{-m} a_{-N} \langle \bar{z}^m k_i(z), \bar{z}^N k_i(z) \rangle_\alpha + a_{-m} \bar{a}_{-N} \langle \bar{z}^N k_i(z), \bar{z}^m k_i(z) \rangle_\alpha\end{aligned}$$

Also,

$$\begin{aligned}\|T_{\bar{f}}(k_i(z))\|_\alpha^2 &= \|(\bar{a}_m T_{\bar{z}^m} + \bar{a}_N T_{\bar{z}^N}) k_i(z)\|_\alpha^2 \\ &= |a_m|^2 \|T_{\bar{z}^m} k_i(z)\|_\alpha^2 + |a_N|^2 \|T_{\bar{z}^N} k_i(z)\|_\alpha^2 \\ &\quad + \bar{a}_m a_N \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha + a_m \bar{a}_N \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha\end{aligned}$$

and similarly,

$$\begin{aligned}\|T_{\bar{g}}(k_i(z))\|_\alpha^2 &= |a_{-m}|^2 \|T_{\bar{z}^m} k_i(z)\|_\alpha^2 + |a_{-N}|^2 \|T_{\bar{z}^N} k_i(z)\|_\alpha^2 \\ &\quad + \bar{a}_{-m} a_{-N} \langle T_{\bar{z}^m} k_i(z), T_{\bar{z}^N} k_i(z) \rangle_\alpha + a_{-m} \bar{a}_{-N} \langle T_{\bar{z}^N} k_i(z), T_{\bar{z}^m} k_i(z) \rangle_\alpha\end{aligned}$$

Using these in (2.1), we have, T_φ is hyponormal if and only if

$$\begin{aligned}&\sum_{i=0}^{N-1} [(|a_m|^2 - |a_{-m}|^2) \|\bar{z}^m k_i(z)\|_\alpha^2 + (|a_N|^2 - |a_{-N}|^2) \|\bar{z}^N k_i(z)\|_\alpha^2] \\ &\quad \geq \sum_{i=0}^{N-1} [(|a_m|^2 - |a_{-m}|^2) \|T_{\bar{z}^m} k_i(z)\|_\alpha^2 + (|a_N|^2 - |a_{-N}|^2) \|T_{\bar{z}^N} k_i(z)\|_\alpha^2]\end{aligned}$$

That is, iff

$$\begin{aligned}&(|a_m|^2 - |a_{-m}|^2) \sum_{i=0}^{N-1} [\|\bar{z}^m k_i(z)\|_\alpha^2 - \|P_\alpha(\bar{z}^m k_i(z))\|_\alpha^2] \\ &\quad + (|a_N|^2 - |a_{-N}|^2) \sum_{i=0}^{N-1} [\|\bar{z}^N k_i(z)\|_\alpha^2 - \|P_\alpha(\bar{z}^N k_i(z))\|_\alpha^2] \geq 0\end{aligned}$$

That is, iff

$$
\begin{aligned}
&\Bigg[\sum_{i=0}^{N-1}\sum_{k=0}^{\infty}\frac{(Nk+i+m)!\Gamma(\alpha+2)}{\Gamma(Nk+i+m+\alpha+2)}|c_{Nk+i}|^2 \\
&\quad-\sum_{i=0}^{m-1}\sum_{k=1}^{\infty}\frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}|c_{Nk+i}|^2 \\
&\quad+\sum_{i=m}^{N-1}\sum_{k=0}^{\infty}\frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}|c_{Nk+i}|^2\Bigg](|a_m|^2-|a_{-m}|^2) \\
&\quad+\sum_{i=0}^{N-1}\Bigg[\sum_{k=0}^{\infty}\frac{(Nk+i+N)!\Gamma(\alpha+2)}{\Gamma(Nk+i+N+\alpha+2)}|c_{Nk+i}|^2 \\
&\quad-\sum_{k=1}^{\infty}\frac{(Nk+i)!^2\Gamma(Nk+i-N+\alpha+2)\Gamma(\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-N+1)}|c_{Nk+i}|^2\Bigg](|a_N|^2-|a_{-N}|^2)\ge 0
\end{aligned}
$$

That is, iff

$$
\begin{aligned}
&\Bigg[\sum_{i=0}^{m-1}\Bigg\{\frac{(m+i)!}{\Gamma(m+i+\alpha+2)}|c_i|^2+\sum_{k=1}^{\infty}\Bigg\{\frac{(Nk+i+m)!}{\Gamma(Nk+i+m+\alpha+2)} \\
&\quad-\frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}\Bigg\}|c_{Nk+i}|^2\Bigg\} \\
&\quad+\sum_{i=m}^{N-1}\sum_{k=0}^{\infty}\Bigg\{\frac{(Nk+i+m)!}{\Gamma(Nk+i+m+\alpha+2)} \\
&\quad-\frac{(Nk+i)!^2\Gamma(Nk+i-m+\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(Nk+i-m+1)}\Bigg\}|c_{Nk+i}|^2\Bigg](|a_m|^2-|a_{-m}|^2) \\
&\quad+\sum_{i=0}^{N-1}\Bigg[\frac{(N+i)!}{\Gamma(N+i+\alpha+2)}|c_i|^2+\sum_{k=1}^{\infty}\Bigg\{\frac{(N(k+1)+i)!\Gamma(\alpha+2)}{\Gamma(N(k+1)+i+\alpha+2)} \\
&\quad-\frac{(Nk+i)!^2\Gamma(N(k-1)+i+\alpha+2)}{\Gamma(Nk+i+\alpha+2)^2\Gamma(N(k-1)+i+1)}\Bigg\}|c_{Nk+i}|^2\Bigg](|a_N|^2-|a_{-N}|^2)\ge 0
\end{aligned}
$$

That is, iff

$$
\begin{aligned}
&(|a_m|^2-|a_{-m}|^2)\Bigg[\sum_{t=0}^{m-1}\frac{(t+m)!}{\Gamma(t+m+\alpha+2)}|c_t|^2 \\
&\quad+\sum_{t=m}^{\infty}\Bigg\{\frac{(t+m)!}{\Gamma(t+m+\alpha+2)}-\frac{t!^2\Gamma(t-m+\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-m+1)}\Bigg\}|c_t|^2\Bigg] \\
&\quad+(|a_N|^2-|a_{-N}|^2)\Bigg[\sum_{t=0}^{N-1}\frac{(t+N)!}{\Gamma(t+N+\alpha+2)}|c_t|^2 \\
&\quad+\sum_{t=N}^{\infty}\Bigg\{\frac{(t+N)!}{\Gamma(t+N+\alpha+2)}-\frac{t!^2\Gamma(t-N+\alpha+2)}{\Gamma(t+\alpha+2)^2\Gamma(t-N+1)}\Bigg\}|c_t|^2\Bigg]\ge 0
\end{aligned}
$$

That is, iff

$$(|a_m|^2-|a_{-m}|^2)\Bigg[\sum_{t=0}^{m-1}\frac{\prod_{j=1}^{m}(t+j)}{\prod_{j=1}^{m}(t+\alpha+j+1)}$$
$$+\sum_{t=m}^{\infty}\Bigg\{\frac{\prod_{j=1}^{m}(t+j)}{\prod_{j=1}^{m}(t+\alpha-j+2)}-\frac{\prod_{j=1}^{m}(t-j+1)}{\prod_{j=1}^{m}(t+\alpha+j+1)}\Bigg\}\Bigg]\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2$$
$$+(|a_N|^2-|a_{-N}|^2)\Bigg[\sum_{t=0}^{N-1}\frac{\prod_{j=1}^{N}(t+j)}{\prod_{j=1}^{N}(t+\alpha+j+1)}$$
$$+\sum_{t=N}^{\infty}\Bigg\{\frac{\prod_{j=1}^{N}(t+j)}{\prod_{j=1}^{N}(t+\alpha-j+2)}-\frac{\prod_{j=1}^{N}(t-j+1)}{\prod_{j=1}^{N}(t+\alpha+j+1)}\Bigg\}\Bigg]\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2\geq 0$$

That is, iff

$$\sum_{t=0}^{\infty}\Big\{(|a_m|^2-|a_{-m}|^2)N_\alpha(t)+(|a_N|^2-|a_{-N}|^2)D_\alpha(t)\Big\}\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2\geq 0$$

where $N_\alpha(t)=\frac{\prod\limits_{j=1}^{m}(t+j)}{\prod\limits_{j=1}^{m}(t+\alpha+j+1)}-\frac{\prod\limits_{j=1}^{m}(t-j+1)}{\prod\limits_{j=1}^{m}(t+\alpha-j+2)}$ and $D_\alpha(t)=\frac{\prod\limits_{j=1}^{N}(t+j)}{\prod\limits_{j=1}^{N}(t+\alpha+j+1)}-\frac{\prod\limits_{j=1}^{N}(t-j+1)}{\prod\limits_{j=1}^{N}(t+\alpha-j+2)}$. □

3. Hyponormality conditions

Theorem 3.1. *Let* $\varphi(z)=\overline{g(z)}+f(z)$, *where* $f(z)=a_1z+a_3z^3$, $g(z)=a_{-1}z+a_{-3}z^3$. *If* $\alpha\geq 0$ *and* $a_1\bar{a}_3=a_{-1}\bar{a}_{-3}$ *then* T_φ *on* $A^2_\alpha(\mathbb{D})$ *is hyponormal*

$$\Longleftrightarrow\begin{cases}9(|a_{-3}|^2-|a_3|^2)\leq(|a_1|^2-|a_{-1}|^2) & \text{if } |a_3|\leq|a_{-3}|\\ 6(|a_3|^2-|a_{-3}|^2)\geq(\alpha+3)(\alpha+4)(|a_{-1}|^2-|a_1|^2) & \text{if } |a_3|\geq|a_{-3}|\end{cases}$$

Proof. By Lemma 2.6, T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal if and only if

$$\sum_{t=0}^{\infty}[(|a_1|^2-|a_{-1}|^2)N_\alpha(t)+(|a_3|^2-|a_{-3}|^2)D_\alpha(t)]\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2\geq 0$$

where

$$N_\alpha(t)=\left\{\frac{t+1}{t+\alpha+2}-\frac{t}{t+\alpha+1}\right\}=\frac{\alpha+1}{(t+\alpha+2)(t+\alpha+1)}$$

and

$$D_\alpha(t)=\frac{(t+1)(t+2)(t+3)}{(t+\alpha+2)(t+\alpha+3)(t+\alpha+4)}-\frac{t(t-1)(t-2)}{(t+\alpha+1)(t+\alpha)(t+\alpha-1)}$$
$$=\frac{3(\alpha+1)(-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2n+3n^2+15\alpha n^2+12n^3+6\alpha n^3+3n^4)}{(t+\alpha+2)(t+\alpha+3)(t+\alpha+4)(t+\alpha+1)(t+\alpha)(t+\alpha-1)}$$

For $n\in\mathbb{N}$, define ξ_α as $\xi_\alpha(n):=\frac{N_\alpha(n)}{D_\alpha(n)}$ so that

$$\xi_\alpha(n)=\frac{(n+\alpha-1)(n+\alpha)(n+\alpha+3)(n+\alpha+4)}{3(-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2n+3n^2+15\alpha n^2+12n^3+6\alpha n^3+3n^4)}$$

From this we get,

$$\lim_{n\to\infty} \xi_\alpha(n) = \frac{1}{9}$$

Also, if $\xi'_\alpha(n)$ denotes the derivative of $\xi_\alpha(n)$, then we have

$$\xi'_\alpha(n) = -\frac{N}{3(-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2 n+3n^2+15\alpha n^2+12n^3+6\alpha n^3+3n^4)^2}$$

where,

$$\begin{aligned} N &= 10\alpha^2(n^5-1)+143\alpha^3(n^2-1)+49\alpha^4(n^3-1)+52\alpha n(n^4-1) \\ &\quad +278\alpha^2 n(n^2-1)+207\alpha n^2(n^2-1)+72n^3(\alpha^3-1) \\ &\quad +6n^2(n^4+4n^3-2n^2-12n+9)+192\alpha+7\alpha^5+3\alpha^6+66\alpha^3 n \\ &\quad +192\alpha^4 n+66\alpha^5 n+6\alpha^6 n+267\alpha^2 n^2+415\alpha^3 n^2+240\alpha^4 n^2 \\ &\quad +30\alpha^5 n^2+140\alpha n^3+314\alpha^2 n^3+360\alpha^3 n^3+11\alpha^4 n^3+36\alpha n^4 \\ &\quad +255\alpha^2 n^4+60\alpha^3 n^4+26\alpha n^5+20\alpha^2 n^5+6\alpha n^6, \end{aligned}$$

which is always positive for $n \geq 1$ and for $\alpha \geq 0$.

So $\xi_\alpha(n)$ is a strictly decreasing function for all $\alpha \geq 0$.

Case 1: Let $|a_{-3}| \leq |a_3|$.
As $\xi_\alpha(0) = \frac{(\alpha+3)(\alpha+4)}{6}$ and $\xi_\alpha(1) = \frac{(\alpha+1)(\alpha+4)(\alpha+5)}{24(\alpha+2)}$ so we have $\xi_\alpha(0) \geq \xi_\alpha(1)$. Hence T_φ is hyponormal if and only if $\frac{|a_3|^2-|a_{-3}|^2}{|a_1|^2-|a_{-1}|^2} \geq \frac{(\alpha+3)(\alpha+4)}{6}$ or equivalently, $6(|a_3|^2-|a_{-3}|^2) \geq (\alpha+3)(\alpha+4)(|a_1|^2-|a_{-1}|^2)$

Case 2: Let $|a_3| \leq |a_{-3}|$.

Since $\xi_\alpha(n) \geq \frac{1}{9}$ for all n, so T_φ is hyponormal if and only if $\frac{|a_3|^2-|a_{-3}|^2}{|a_1|^2-|a_{-1}|^2} \leq \frac{1}{9}$. That is, if and only if $9(|a_{-3}|^2-|a_3|^2) \leq |a_1|^2-|a_{-1}|^2$ □

Theorem 3.2. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_2z^2 + a_3z^3$, $g(z) = a_{-2}z^2 + a_{-3}z^3$. If $\alpha \geq 0$ and $a_2\bar{a}_3 = a_{-2}\bar{a}_{-3}$ then T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal*

$$\Longleftrightarrow \begin{cases} 9(|a_{-3}|^2-|a_3|^2) \leq 4(|a_2|^2-|a_{-2}|^2) & \text{if } |a_3| \leq |a_{-3}| \\ 3(|a_3|^2-|a_{-3}|^2) \geq (\alpha+4)(|a_{-2}|^2-|a_2|^2) & \text{if } |a_3| \geq |a_{-3}| \end{cases}$$

Proof. T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal if and only if

$$\begin{aligned} &(|a_2|^2-|a_{-2}|^2)\sum_{t=0}^{\infty}\left\{\frac{(t+2)(t+1)}{(t+\alpha+3)(t+\alpha+2)} - \frac{t(t-1)}{(t+\alpha+1)(t+\alpha)}\right\}\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2 \\ &\quad +(|a_3|^2-|a_{-3}|^2)\sum_{t=0}^{\infty}\left\{\frac{(t+3)(t+2)(t+1)}{(t+\alpha+4)(t+\alpha+3)(t+\alpha+2)}\right. \\ &\quad \left. -\frac{t(t-1)(t-2)}{(t+\alpha+1)(t+\alpha)(t+\alpha-1)}\right\}\frac{\Gamma(t+1)}{\Gamma(t+\alpha+2)}|c_t|^2 \geq 0 \end{aligned}$$

For $n \in \mathbb{N}$, define ξ_α by

$$\xi_\alpha(n) := \frac{\left\{ \dfrac{(t+2)(t+1)}{(t+\alpha+3)(t+\alpha+2)} - \dfrac{t(t-1)}{(t+\alpha+1)(t+\alpha)} \right\}}{\left\{ \dfrac{(t+3)(t+2)(t+1)}{(t+\alpha+4)(t+\alpha+3)(t+\alpha+2)} - \dfrac{t(t-1)(t-2)}{(t+\alpha+1)(t+\alpha)(t+\alpha-1)} \right\}}$$

$$= \frac{2(n+\alpha-1)(n+\alpha+4)(2n^2+2\alpha n+4n+\alpha)}{3(-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2 n+3n^2+15\alpha n^2+3\alpha^2 n^2+12n^3+6\alpha n^3+3n^4)}$$

So $\lim_{t\to\infty} \xi_\alpha(n) = \frac{4}{9}$. Also, $\xi'_\alpha(n) = -\frac{2N}{3D}$ where,

$D = (-2\alpha+2\alpha^2-18n-3\alpha n+3\alpha^2 n+3n^2+15\alpha n^2+3\alpha^2 n^2+12n^3+6\alpha n^3+3n^4)^2$

and

$$\begin{aligned} N = {} & 29\alpha^3(n^4-1) + 14\alpha^4(n-1) + \alpha^5(n-1) + 8\alpha n(n^2-1) + 34\alpha^2 n(n^3-1) \\ & + 6n^2\{n(n+1)(n^2+n-4)+4\} + 40\alpha + 4\alpha^2 + 12\alpha^3 n + 10\alpha^4 n + 5\alpha^5 n \\ & + 87\alpha n^2 + 264\alpha^2 n^2 + 216\alpha^3 n^2 + 66\alpha^4 n^2 + 6\alpha^5 n^2 + 212\alpha n^3 + 368\alpha^2 n^3 \\ & + 168\alpha^3 n^3 + 24\alpha^4 n^3 + 189\alpha n^4 + 134\alpha^2 n^4 + 7\alpha^3 n^4 + 12n^5 + 66\alpha n^5 \\ & + 24\alpha^2 n^5 + 6\alpha n^6 \end{aligned}$$

which is always positive for $n \geq 1$ and for $\alpha \geq 0$.
Hence $\xi_\alpha(n)$ is a strictly decreasing sequence for all $\alpha \geq 0$.

Case 1: Let $|a_{-3}| \leq |a_3|$.
We have $\xi_\alpha(0) = \frac{\alpha+4}{3}$ and $\xi_\alpha(1) = \frac{\alpha+5}{4}$ and so $\xi_\alpha(0) \geq \xi_\alpha(1)$. Hence T_φ is hyponormal if and only if $\frac{|a_3|^2-|a_{-3}|^2}{|a_2|^2-|a_{-2}|^2} \geq \frac{\alpha+4}{3}$. That is,

$$3(|a_3|^2 - |a_{-3}|^2) \geq (\alpha+4)(|a_2|^2 - |a_{-2}|^2).$$

Case 2: Let $|a_3| \leq |a_{-3}|$.
Since $\xi_\alpha(n) \geq \frac{4}{9}$ for all n, so T_φ is hyponormal if and only if $\frac{|a_{-3}|^2-|a_3|^2}{|a_2|^2-|a_{-2}|^2} \leq \frac{4}{9}$. That is,

$$9(|a_{-3}|^2 - |a_3|^2) \leq 4(|a_2|^2 - |a_{-2}|^2). \qquad \square$$

4. **Conclusion**

Looking at these particular results, we propose the following general result:

Theorem 4.1 (Conjecture). *Let* $\varphi(z) = \overline{g(z)} + f(z)$*, where* $f(z) = a_m z^m + a_N z^N$*,* $g(z) = a_{-m} z^m + a_{-N} z^N$ *(*$0 < m < N$*). If* $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$ *and* α *is sufficiently large, then* T_φ *on* $A^2_\alpha(\mathbb{D})$ *is hyponormal*

$$\iff \begin{cases} \displaystyle\prod_{j=0}^{N-1} (\alpha+2+j)(|a_{-m}|^2 - |a_m|^2) \leq \prod_{j=m+1}^{N} (j)(|a_N|^2 - |a_{-N}|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}| \end{cases}$$

In Theorems 3.1 and 3.2 it has been shown that this is true for $m = 1$, $N = 3$ and $m = 2$, $N = 3$ respectively. In Theorem 1.3 it was shown that the result also holds for $m = 1$, $N = 2$.

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