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Research Article

Solving Fuzzy Linear Programming Problem Using Defuzzification Method

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Abstract. *Linear Programming* (LP) has been one of the efficient, reliable and time tested techniques in Optimization. Conventional LP is not suitable for many real time problems which involve data with inherent vagueness or impreciseness. Fuzzy set theory is proved to be quite good in addressing the inherent vagueness or impreciseness and thus *Fuzzy Linear Programming* (FLP) is brought to light and developed over the years. A quite good number of techniques have been proposed for solving FLP problems to obtain optimal solution for real world problems involving fuzzy (vague or imprecise) environment. In this paper, "*Extended Geometric Mean Defuzzification*" is defined and based on it, a method is proposed for solving FLP problems. To showcase the advantages of the proposed method, different problems of FLP, available in the literature, are discussed. Numerical comparisons are also provided to validate the authentication of the proposed method.

Keywords. Fuzzy sets; Fuzzy numbers; Geometric mean defuzzification; Fuzzy linear programming

Mathematics Subject Classification (2020). 90C05; 90C70

1. Introduction

LP is most widely used and well known classical optimization technique, which ensures to strike a balance optimally among available resources and standing requirements. LP was initially designed and modelled for crisp environment. Hence conventionally, it works with parameters

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of crisp nature. Logically speaking, many real world problems, supposed to be solved by LP technique, involve parameters of non-crisp nature i.e. parameters with imprecise or vague data. Fuzzy set theory is a tested and proved tool in dealing with this type of vague or imprecise data. Thus, union of conventional LP with fuzzy set theory has stemmed as FLP which provides ample opportunities for researchers to deal with more realistic and complicated models of real time LP problems.

The idea and frame work of decision making in fuzzy environment was brought forward by Bellman and Zadeh [2]. Having understood the importance and significance of FLP, many researchers have shown keen interest and ventured to work on this. However, the authors worked on cases like where all the parameters and variables of the problems are not assumed to be fuzzy i.e. in some cases objective function is fuzzy but the constraints are crisp or in some cases objective function is crisp but constraints are fuzzy or sometimes right hand side of the constants are not fuzzy etc.

A fuzzy linear programming problem is said to be *Fully Fuzzy Linear Programming* (FFLP) problem if all the decision variables, constraints and coefficients of objective function in the problem are fuzzy [12]. These are further classified into two types as FFLP with inequality constraints and FFLP with equality constraints [12]. For solving FFLP problems with inequality constraints, different methods have been brought to the domain by the authors [1, 4, 9, 11]. The common point of all techniques [1, 4] is, firstly, FFLP is changed to crisp LP problem. Secondly, fuzzy optimal solution for the given FFLP was obtained from the corresponding crisp LP problem. However, the drawback of these methods is the obtained fuzzy optimal solution could not satisfy the constraints exactly. The authors [9, 11] brought forward methods to solve FFLP without converting into crisp LP problem.

Other authors [3, 6, 8, 10, 12, 13, 15] have proposed methods for solving FFLP problems with equality constraints. For solving fully fuzzy linear system of equations i.e. with equality constraints, a method was forwarded by Dehghan *et al*. [6] and this method was observed to get failed when dealing with negative fuzzy numbers. Lotfi *et al.* [13] proposed a method to obtain the approximate solution of FFLP problems with equality constraints and dealing only with "triangular symmetric fuzzy numbers" is observed to be its drawback. Making use of arithmetic operations and fuzzy equality, Amit Kumar *et al.* [12] introduced a method which overcame the deficiencies of [6, 13] and provides fuzzy optimal solution unlike other existing methods which give crisp solution. Najafi and Edalatpanah [15] identified that Amit Kumar *et al.* [12] method could not guarantee the non-negativity of the fuzzy solution and introduced a method [15] correcting those short comings. Ezzati *et al.* [8] introduced an algorithm which addressed the shortcomings of methods [6, 12]. The solutions, provided and claimed to be fuzzy by [8], are sometimes crisp numbers. Bhardwaj and Kumar [3] claimed that the algorithm, proposed by Ezzati *et al.* [8] to find the fuzzy optimal solution of FFLP problems with equality constraints, could not be used for finding the fuzzy optimal solution of FFLP problems with inequality constraints. Hosseinzadeh *et al.* [10] introduced a method and obtained better results than that of [6]. Das *et al.* [5] introduced a method for solving FLPP with both type of constraints using trapezoidal fuzzy numbers. Dong *et al.* [7] introduced a method using the concept of acceptance degree. Nasseri *et al.* [14] observed that method introduced by Ezzati *et al.* [8], failed to compare any arbitrary triangular fuzzy number and introduced a method [14]. Further, the triangular fuzzy solution given by [5] is observed not to follow monotonic condition.

The above mentioned different methods for solving FFLP problems have some drawbacks like failing to deal with negative fuzzy numbers or dealing with only triangular symmetrical fuzzy numbers or non-guarantee of the non negative fuzzy solution or failing to compare any two arbitrary triangular fuzzy numbers or method being difficult to apply on or not following monotonic conditions of fuzzy numbers.

These observed drawbacks of most of the existing methods are due to short comings occurred in their respective ranking functions, involved in defuzzification process. Further, after obtaining fuzzy optimal solutions, in order to compare solutions, ranking function has to be employed again. Obviously, it has to be better if the final solution is in defuzzified form. So that either the FLPP is converted to crisp LP or the fuzzy solution obtained is converted to crisp form. Veeraju *et al.* [16] introduced a ranking method based on geometric mean and height which gives defuzzified value directly to make ordering easier.

Therefore, a need and a necessity are there to introduce an integrated FLP method which may address the drawbacks of different FFLP methods. To deal with mixed fuzzy numbers also, "Extended geometric mean defuzzication" is defined as an extension to the ranking method used in Veeraju *et al.* [16]. Hence, in this paper, a novel FLP technique based on "Extended geometric mean defuzzication", is proposed which is simple to apply and efficient in giving better results.

This paper is organised as following. The required preliminaries of fuzzy numbers are mentioned in Section 2. Section 3 is utilised for illustration of the proposed method. Section 4 is kept for necessary discussions and useful comparisons of this work with other author works. With a briefing of key points in a nut shell, the paper is concluded with Section 5.

2. Preliminaries [16]

Let X be a universe of discourse, a fuzzy set $\widetilde{A}$ is defined by $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)), X \in x\}$ and $\mu_{\widetilde{A}}(x) : X \to [0,1]$, where $\mu_{\widetilde{A}}(x)$ is membership function. Support of a fuzzy set $\widetilde{A}$ is $S(\widetilde{A}) = \{x \in X/\mu_{\widetilde{A}} > 0\}$ and height of $\widetilde{A}$ is $H(\widetilde{A}) = \text{Supremum}(\mu_{\widetilde{A}}(x))$. A fuzzy set $\widetilde{A}$ is said to be convex if $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2))$ for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$. $\widetilde{A}$ is said to be normal if there exists a $x_i \in X$ satisfying $\mu_{\widetilde{A}}(x) = 1$. A fuzzy number is a fuzzy set which is both convex and normal. The most commonly used fuzzy numbers are Triangular Fuzzy Number (TFN) and Trapezoidal Fuzzy Number (TrFN) these are respectively defined as follows.

A TrFN is denoted by an ordered quadruple as $\widetilde{A} = (a, b, c, d)$ whose membership function $\mu_{\widetilde{A}}(x)$ is described as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{(d-x)}{(d-c)}, & c \leq x \leq d \\ 0, & \text{otherwise} \end{cases}$$

A TFN is denoted by an ordered triple as $\widetilde{A} = (a,b,c)$ whose membership function $\mu_{\widetilde{A}}(x)$ is described as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \le x \le b \\ \frac{(c-x)}{(c-b)}, & b \le x \le c \\ 0, & \text{otherwise} \end{cases}$$

The image of the fuzzy number $\widetilde{A} = (x, \mu_{\widetilde{A}}(x))$, denoted by "$-\widetilde{A}$", is obtained by multiplying each element of support of the $\widetilde{A}$ by '-1' that is $-\widetilde{A} = (-x, \mu_{\widetilde{A}}(x))$. If each '$x$' in $\widetilde{A}$ is negative then the fuzzy number is considered to be negative fuzzy number. The image (or opposite) of a fuzzy number $\widetilde{A} = (a,b,c,d)$ can be given by a fuzzy number $-\widetilde{A} = (-d,-c,-b,-a)$.

Geometric Mean Defuzzification (GMD)

Let $\widetilde{A}$ be an arbitrary non-negative fuzzy number with membership function $\mu_{\widetilde{A}}(x)$, support $S(\widetilde{A})$ and height $H(\widetilde{A})$ then the defuzzified value of $\widetilde{A}$ by geometric mean is defined to be

$$D_{\widetilde{A}} = \exp\left[\frac{\int_{S(\widetilde{A})} \mu_{\widetilde{A}}(x) \ln x dx}{\int_{S(\widetilde{A})} \mu_{\widetilde{A}}(x) dx}\right] \times H(\widetilde{A}), \quad \text{for all } x > 0.$$

If $\widetilde{A}$ be a negative fuzzy number then $-\widetilde{A}$ will be non-negative and then the defuzzified value is $D_{\widetilde{A}} = -D_{-\widetilde{A}}$.

3. Proposed Method

Extended Geometric Mean Defuzzification (EGMD)

If $\widetilde{A}$ be neither non-negative nor negative fuzzy number then it is considered to be mixed fuzzy number. The support $S(\widetilde{A})$ consists of both positive and negative values. In such case the GMD is not useful, thus EGMD is defined.

Let $S(\widetilde{A}) = N(\widetilde{A}) \cup P(\widetilde{A})$, where $N(\widetilde{A}) = \{x \in S(\widetilde{A}) : x < 0\}$ and $P(\widetilde{A}) = \{x \in S(\widetilde{A}) : x \ge 0\}$.

The defuzzified value of $\widetilde{A}$ by EGMD is

$$D_{\widetilde{A}} = \left(\exp\left[\frac{\int_{P(\widetilde{A})} \mu_{\widetilde{A}}(x) \ln x dx}{\int_{P(\widetilde{A})} \mu_{\widetilde{A}}(x) dx}\right] - \exp\left[\frac{\int_{-N(\widetilde{A})} \mu_{\widetilde{A}}(x) \ln x dx}{\int_{-N(\widetilde{A})} \mu_{\widetilde{A}}(x) dx}\right]\right) \times H(\widetilde{A}).$$

Example 3.1. Consider a mixed TFN $\widetilde{A} = (-1,1,2)$. The support of mixed $\widetilde{A}$ is $S(\widetilde{A}) = (-1,2)$ and it can be written as $S(\widetilde{A}) = N(\widetilde{A}) \cup P(\widetilde{A})$, where $N(\widetilde{A}) = (-1,0)$ and $P(\widetilde{A}) = [0,2)$. The defuzzified value of $\tilde{A}$ using above formula is $D_{\tilde{A}} = (0.6764 - 0.4346) \times 1 = 0.2418$.

It can be observed that if $\widetilde{A}$ is non-negative then $N(\widetilde{A}) = \phi$ and $S(\widetilde{A}) = P(\widetilde{A})$ follows that the defuzzification by GMD and EGMD are same.

Method for Solving Fuzzy Linear Programming Problem

The standard form of FLPP is as follows

$$\text{Maximize (or) Minimize } \widetilde{Z} = \sum_{k=1}^{n} \widetilde{c}_k \otimes \widetilde{x}_k$$

$$\text{Subject to } \sum_{k=1}^{n} \widetilde{a}_{ik} \otimes \widetilde{x}_k (\le = \ge) \widetilde{b}_i, \quad i = 1,2,3\ldots m,$$

where $\widetilde{c}_k$, $\widetilde{a}_{ik}$ and $\widetilde{b}_i$ are fuzzy numbers and $\widetilde{x}_k$ are non negative fuzzy numbers.

A method is proposed to solve FLP problem based on the sign in constraints of the problem. All the elements in cost vector $\widetilde{C} = [\widetilde{c}_k]_{1\times n}$ and coefficient matrix $\widetilde{A} = [\widetilde{a}_{ik}]_{m\times n}$ are defuzzified using EGMD. Thus left side of the constraints are defuzzfied and right side elements $\widetilde{B} = [\widetilde{b}_i]_{m\times 1}$ are left as fuzzy numbers. If the right side elements are also defuzzified, then there may be a chance of contraction of permissible region. In order to enhance the permissible region, the following ways are considered.

(i) *Constraint with equality sign*: After defuzzification, LHS of constraint is a non-fuzzy and it is considered to be equal to any one of the value in Support of the fuzzy number in RHS.

(ii) *Constraint with greater than sing*: After defuzzification, LHS of constraint is a non-fuzzy and it is considered to be greater than the infimum of Support of the fuzzy number in RHS.

(iii) *Constraint with less than sing*: After defuzzification, LHS of constraint is a non-fuzzy and it is considered to be less than the supremum of Support of the fuzzy number in RHS.

Using EGMD, the FLP problem is converted into the following crisp LP problem.

Maximize (or) Minimize $Z = \sum_{k=1}^{n} c_k \cdot x_k$

Subject to

(i) *Equality constraints:* $\text{Infimum}\{S(\widetilde{b}_i)\} \le \sum_{k=1}^{n} a_{ik} \cdot x_k \le \text{Supremum}\{S(\widetilde{b}_i)\},\ i = 1,2,3\ldots m$

(ii) *Constraints with grater than sign:* $\sum_{k=1}^{n} a_{ik} \cdot x_k \ge \text{Infimum}\{S(\widetilde{b}_i)\},\ i = 1,2,3\ldots m$

(iii) *Constraints with less than sign:* $\sum_{k=1}^{n} a_{ik} \cdot x_k \le \text{Supremum}\{S(\widetilde{b}_i)\},\ i = 1,2,3\ldots m$

where c_k and a_{ik} are defuzzified values of $\widetilde{c}_k$and $\widetilde{a}_{ik}$. x_k is crisp decision variable.

4. Numerical Comparisons

In this section, the proposed method is illustrated through the some numerical examples which are considered from the literature. The advantages of the proposed are explained by comparing the results with other existing methods.

Example 4.1. Consider the following FLP problem, studied by Nasseri *et al.* [14] and Ezzati *et al.* [8]

$$\text{Max } \widetilde{Z} = (10,15,17)\otimes\widetilde{x}_1 \oplus (10,16,20)\otimes\widetilde{x}_2 \oplus (10,14,17)\otimes\widetilde{x}_3 \oplus (10,12,14)\otimes\widetilde{x}_4$$

Subject to

$$(8,10,13)\otimes\widetilde{x}_1 \oplus (10,11,13)\otimes\widetilde{x}_2 \oplus (9,12,13)\otimes\widetilde{x}_3 \oplus (11,15,17)\otimes\widetilde{x}_4 = (271.75,411.75,573.75)$$

$$(11,14,16)\otimes\widetilde{x}_1 \oplus (14,18,19)\otimes\widetilde{x}_2 \oplus (14,17,20)\otimes\widetilde{x}_3 \oplus (13,14,18)\otimes\widetilde{x}_4 = (385.5,539.5,759.5)$$

where $\widetilde{x}_k$, $k = 1,2,3,4$ are non-negative fuzzy numbers.

Here, all the constraints in the problem are with equality sign. These fuzzy equality constraints are converted to corresponding crisp inequalities as mentioned in *Case* (i) of the proposed method. Hence, the corresponding crisp LP of the FLP by the proposed method is

$$\text{Max } Z = 13.9179x_1 + 15.1907x_2 + 13.5899x_3 + 11.9721x_4$$

Subject to

$$271.75 \le 10.2824x_1 + 11.3163x_2 + 11.3006x_3 + 14.2777x_4 \le 573.75$$

$$385.5 \le 13.675x_1 + 16.9649x_2 + 16.9557x_3 + 14.9618x_4 \le 759.5$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0$$

Using MATHEMATIC 9, the solution is obtained as $x_1 = 55.73$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and optimum value is $Z_{\text{Proposed method}} = 775.785$.

Nasseri *et al.* [14], obtained the fuzzy solution as $\widetilde{x}_1 = (25.5, 31.94, 26.35)$, $\widetilde{x}_2 = (0,0,17.78)$, $\widetilde{x}_3 = (7.5,0,0)$ and $\widetilde{x}_4 = (0,6.6,0)$ and optimum value as $\widetilde{Z}_{\text{Nasseri}} = (330, 558.3, 803.55)$. Here, the values of the variables are expected to be in TFN of the form (a,b,c) where in $a \le b \le c$. If $a = b = c$ then the TFN becomes crisp number. Clearly, $\widetilde{x}_1$, $\widetilde{x}_3$ and $\widetilde{x}_4$ are not in that form. Ezati *et al.* [8], obtained the fuzzy solution as $\widetilde{x}_1 = (17.27, 17.27, 17.27)$, $\widetilde{x}_2 = (2.16, 2.16, 2.16)$, $\widetilde{x}_3 = (4.64, 9.97, 16.36)$ and $\widetilde{x}_4 = (6.36, 6.36, 6.36)$ and optimum value as $\widetilde{Z}_{\text{Ezzati}} = (304.58, 509.79, 704.37)$. In this case, $\widetilde{x}_1$, $\widetilde{x}_2$ and $\widetilde{x}_4$ are crisp numbers though they were represented in TFN form. In order to compare the results obtained by different methods, it is required to rely on ranking methods. Using the EGMD ranking method, the defuzzified values of the fuzzy optimum values obtained by Nasseri *et al.* [14] and Ezzati *et al.* [8] respectively are $Z_{\text{Nasseri}} = 555.455$, $Z_{\text{Ezzati}} = 499.479$. Hence, the optimum value obtained by proposed method is grater than that of obtained by Nasseri *et al.* [14] and Ezzati *et al.* [8] methods.

$Z_{\text{Ezzati}} = 499.479 < Z_{\text{Nasseri}} = 555.455 < Z_{\text{Proposed method}} = 775.785$. As the given FLP is maximization problem, the proposed method is out performed over the other.

Example 4.2. Consider the following FLP problem which is addressed by Nasseri *et al.* [14] and Ezzati *et al.* [8]

$$\begin{aligned}\text{Max } \widetilde{Z} = {} & (5,7,8)\otimes\widetilde{x}_{11} \oplus (3,5,6)\otimes\widetilde{x}_{12} \oplus (4,8,9)\otimes\widetilde{x}_{13} \oplus (3,5,7)\otimes\widetilde{x}_{21} \\ & \oplus (4,7,8)\otimes\widetilde{x}_{22} \oplus (8,9,10)\otimes\widetilde{x}_{23} \oplus (7,10,11)\otimes\widetilde{x}_{31} \oplus (6,8,10)\otimes\widetilde{x}_{32} \\ & \oplus (4,7,8)\otimes\widetilde{x}_{33} \oplus (4,6,8)\otimes\widetilde{x}_{41} \oplus (3,5,7)\otimes\widetilde{x}_{42} \oplus (7,9,11)\otimes\widetilde{x}_{43}\end{aligned}$$

Subject to

$$\sum_{i=1}^{4}\sum_{j=1}^{3}\widetilde{x}_{ij} = (25,30,40),\ \sum_{j=1}^{3}\widetilde{x}_{1j} \ge (2,3,5),\ \sum_{j=1}^{3}\widetilde{x}_{2j} \ge (4,5,6)$$

$$\sum_{j=1}^{3}\widetilde{x}_{3j} \ge (5,8,9),\ \sum_{j=1}^{3}\widetilde{x}_{4j} \ge (7,8,14),\ \widetilde{x}_{11} \le (4,6,7),\ \widetilde{x}_{12} \le (3,5,6),$$

$$\widetilde{x}_{13} \le (8,9,10),\ \widetilde{x}_{21} \le (5,7,8),\ \widetilde{x}_{22} \le (8,10,11),\ \widetilde{x}_{23} \le (3,4,5),\ \widetilde{x}_{31} \le (4,5,7)$$

$$\widetilde{x}_{32} \le (2,3,6),\ \widetilde{x}_{33} \le (4,7,9),\ \widetilde{x}_{41} \le (4,6,7),\ \widetilde{x}_{42} \le (4,5,9),\ \widetilde{x}_{43} \le (2,4,5),$$

where $\widetilde{x}_{ij}$, $i = 1,2,3,4$, $j = 1,2,3$ are non-negative fuzzy numbers.

Here, all the constraints in the problem are with inequality sign. These fuzzy inequality constraints are converted to crisp inequalities by the *Cases* (ii) and (iii) of the proposed method. Hence, the corresponding crisp LP of the FLP by the proposed method is

$$\text{Max } Z = 6.6367x_{11} + 4.6231x_{12} + 6.9103x_{13} + 4.9315x_{21} + 6.273x_{22} + 8.9907x_{23}$$
$$+ 9.2933x_{31} + 7.9579x_{32} + 6.273x_{33} + 5.9434x_{41} + 4.9315x_{42} + 8.9627x_{43}$$

Subject to

$$25 \le x_{11} + x_{12} + x_{13} + x_{21} + x_{2[-1pt]2} + x_{23} + x_{31} + x_{32} + x_{33} + x_{41} + x_{42} + x_{43} \le 40$$
$$x_{11} + x_{12} + x_{13} \ge 2,\ x_{21} + x_{22} + x_{23} \ge 4,\ x_{31} + x_{32} + x_{33} \ge 5,$$
$$x_{41} + x_{42} + x_{43} \ge 7,\ x_{11} \le 7,\ x_{12} \le 6,\ x_{13} \le 10,\ x_{21} \le 8,\ x_{22} \le 11,$$
$$x_{23} \le 5,\ x_{31} \le 7,\ x_{32} \le 6,\ x_{33} \le 9,\ x_{41} \le 7,\ x_{42} \le 9,\ x_{43} \le 5,$$
$$x_{11},\ x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{23},\ x_{31},\ x_{32},\ x_{33},\ x_{41},\ x_{42} \text{ and } x_{43} \ge 0.$$

Using MATHEMATICA 9, the solution of the problem is obtained as

$$x_{11} = 5,\ x_{12} = 0,\ x_{13} = 10,\ x_{21} = 0,\ x_{22} = 0,\ x_{23} = 5,$$
$$x_{31} = 7,\ x_{32} = 6,\ x_{33} = 0,\ x_{41} = 2,\ x_{42} = 0,\ x_{43} = 5,$$

and optimum value is $Z_{\text{Proposed method}} = 316.741$.

Nasseri *et al.* [14], obtained the fuzzy solution as $\tilde{x}_{11} = (4,0,0)$, $\tilde{x}_{12} = (0,0,0)$, $\tilde{x}_{13} = (0,3,5)$, $\tilde{x}_{21} = (0,0,0)$, $\tilde{x}_{22} = (5,1,1)$, $\tilde{x}_{23} = (3,4,5)$, $\tilde{x}_{31} = (4,5,7)$, $\tilde{x}_{32} = (2,3,3)$, $\tilde{x}_{33} = (0,0,0)$, $\tilde{x}_{41} = (4,0,0)$, $\tilde{x}_{42} = (1,0,0)$, $\tilde{x}_{43} = (2,14,19)$ and optimum value as $\tilde{Z}_{\text{Nasseri}} = (137,267,419)$. Ezzati *et al.* [8], obtained the fuzzy solution as $\tilde{x}_{11} = (0,0,1)$, $\tilde{x}_{12} = (0,0,1)$,$\tilde{x}_{13} = (8,9,9)$, $\tilde{x}_{21} = (0,0,1)$, $\tilde{x}_{22} = (1,1,1)$, $\tilde{x}_{23} = (3,4,4)$, $\tilde{x}_{31} = (4,5,5)$, $\tilde{x}_{32} = (2,3,3)$, $\tilde{x}_{33} = (0,0,1)$, $\tilde{x}_{41} = (4,4,5)$, $\tilde{x}_{42} = (1,1,5)$, $\tilde{x}_{43} = (2,3,4)$ and optimum value as $\tilde{Z}_{\text{Ezzati}} = (133,245,362)$. Using the EGMD, the defuzzified values are obtained as

$$Z_{\text{Nasseri}} = 268.075, \quad Z_{\text{Ezzati}} = 242.086.$$

Thus, clearly the following is obtained

$$Z_{\text{Nasseri}} = 268.075 < Z_{\text{Ezzati}} = 242.086 < Z_{\text{Proposed method}} = 316.741.$$

In this example also the proposed method performed in a better way.

Example 4.3. Consider the following FLP problem, which is a production problem studied by Das *et al.* [5]

$$\text{Max } \tilde{Z} = (11,13,15,17) \otimes \tilde{x}_1 \oplus (9,12,14,17) \otimes \tilde{x}_2 \oplus (13,15,17,19) \otimes \tilde{x}_3$$

Subject to

$$(9,11,13,15) \otimes \tilde{x}_1 \oplus (11,12,14,15) \otimes \tilde{x}_2 \oplus (9,11,13,15) \otimes \tilde{x}_3 \le (469,475,505,511)$$
$$(11,12,16,17) \otimes \tilde{x}_1 \oplus (11,12,14,15) \otimes \tilde{x}_3 \le (452,460,480,488)$$
$$(9,11,13,15) \otimes \tilde{x}_1 \oplus (11,14,16,19) \otimes \tilde{x}_3 \le (460,465,495,500)$$

where $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ are non negative fuzzy numbers.

Here, all the constraints in the problem are with inequality sign. These fuzzy inequality constraints are converted to crisp inequalities using *Case* (iii) of the proposed method. Hence, the corresponding crisp LP of the FLP by the proposed method is

$$\text{Max } Z = 13.94x_1 + 12.8894x_2 + 15.9476x_3$$

Subject to

$$11.9299x_1 + 12.9678x_2 + 11.9299x_3 \le 511$$
$$13.922x_1 + 12.9678x_3 \le 488$$
$$11.9299x_1 + 14.9045x_2 \le 500$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$$

By MATHEMATICA 9, the solution is $x_1 = 0$, $x_2 = 4.7855$, $x_3 = 37.6317$ and the optimum value is $Z_{\text{Proposed method}} = 661.8176$.

Das *et al.* [5], obtained the fuzzy solution as $\tilde{x}_1 = (0,0,0,0)$, $\tilde{x}_2 = (2.57, 4.23, 4.23, 5.89)$, $\tilde{x}_3 = (32.28, 34.28, 34.28, 36.28)$, and optimum value is $\tilde{Z}_{\text{Das}} = (483.98, 564.96, 680.98, 761.98)$. Kumar and Kaur [11], obtained the fuzzy solution as $\tilde{x}_1 = (0,0,0,0)$, $\tilde{x}_2 = (4.32, 4.32, 4.32, 4.32)$, $\tilde{x}_2 = (36.15, 36.15, 36.15, 36.15)$ and optimum value as $\tilde{Z}_{\text{Kumar}} = (508.84, 594.1, 675.08, 760.34)$. Ganesan and Veeramani [9] obtained fuzzy solution as $\tilde{x}_1 = (0,0,0,0)$, $\tilde{x}_2 = (1.43, 2.46, 6.18, 7.21)$, $\tilde{x}_3 = (34.76, 35.38, 36.92, 37.54)$ and optimum value is $\tilde{Z}_{\text{Ganesan}} = (447.33, 557.6, 711.6, 821.87)$. Using EGMD, the defuzzified values of the optimum values obtained by different methods are calculated and obtained as

$$Z_{\text{Das}} = 619.914 < Z_{\text{Ganesan}} = 629.143 < Z_{\text{Kumar}} = 632.287 < Z_{\text{Proposed method}} = 661.8176.$$

Thus, from above relation, it is clear that for the given maximization FLP, the proposed method gave maximum value for objective function.

Example 4.4. Consider the FLP problem which is a fuzzy knapsack problem studied by Dong [7]

$$\text{Max } \tilde{Z} = (7,8,12,13) \otimes x_1 \oplus (13,14,16,17) \otimes x_2 \oplus (18,19,21,22) \otimes x_3$$
$$\oplus (10,11,13,14) \otimes x_4 \oplus (16,17,19,20) \otimes x_5 \oplus (23,24,26,27) \otimes x_6$$

Subject to

$$(7.8,7.9,8.1,9) \otimes x_1 \oplus (11.6,11.9,12.1,13.6) \otimes x_2 \oplus (12.4,12.8,13.2,13.8) \otimes x_3$$
$$\oplus (63,63.8,64.2,64.6) \otimes x_4 \oplus (21.3,21.8,22.2,23.3) \otimes x_5 \oplus (40,40.8,41.2,41.6) \otimes x_6$$
$$\le (76,78,82,84),$$
$$0 \le x_i \le 1,\ (i = 1,2,3,4,5,6)$$

Here, the decision variables are not fuzzy and all the constraints in the problem are with inequality sign. These in equality constraints are converted to inequalities as mentioned in Case (iii) of the proposed method. Hence, the corresponding crisp LP of the FLP by the proposed method is

$$\text{Max } Z = 9.8899x_1 + 14.9721x_2 + 19.9791x_3 + 11.9651x_4 + 17.9768x_5 + 24.9833x_6$$

Subject to

$$8.2432x_1 + 12.3742x_2 + 13.0559x_3 + 68.8791x_4 + 22.1793x_5 + 40.8786x_6 \le 84$$
$$0 \le x_i \le 1,\ (i = 1,2,3,4,5,6)$$

By MATHEMATICA 9, the solution is obtained as $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$, $x_5 = 1$, $x_6 = 0.6886$ and optimum value is $Z_{\text{Proposed method}} = 80.0204$.

Dong [7] obtained the crisp solution as $x_1 = 0.6573$, $x_2 = 0.8353$, $x_3 = 0.9027$, $x_4 = 0.0502$, $x_5 = 0.799$, $x_6 = 0.7171$ and fuzzy optimum value is $\widetilde{Z}_{\text{Dong}} = (61.4875, 65.4491, 74.6868, 78.6484)$. Dong [5] derived decision variable values in crisp form, however, the objective function value in TrFN form. The defuzzified value of optimum value is calculated using EGMD and is given as $Z_{\text{Dong}} = 69.9548$. Hence, $Z_{\text{Dong}} = 69.9548 < Z_{\text{Proposed method}} = 80.0204$.

In all the above examples, non-negative fuzzy numbers are used. In the next example, the mixed fuzzy number is involved.

Example 4.5. Consider the following FLP problem, discussed by Kumar *et al.* [12].

$$\text{Max } \widetilde{Z} = (1,6,9) \otimes \widetilde{x}_1 \oplus (2,3,8) \otimes \widetilde{x}_2$$

Subject to

$$(2,3,4) \otimes \widetilde{x}_1 \oplus (1,2,3) \otimes \widetilde{x}_2 = (6,16,30)$$

$$(-1,1,2) \otimes \widetilde{x}_1 \oplus (1,3,4) \otimes \widetilde{x}_2 = (1,17,30)$$

where $\widetilde{x}_1$ and $\widetilde{x}_2$ are non-negative fuzzy numbers.

Here, all the constraints in the problem are with equality sign. These equality constraints are converted to inequalities as mentioned in Case (i) of the proposed method. Also, EGMD is used to defuzzify $(-1,1,2)$ as it is a mixed fuzzy number. Hence, the corresponding crisp LP of the FLP by the proposed method is

$$\text{Max } Z = 5.03x_1 + 4.14x_2$$

Subject to

$$6 \le 2.97x_1 + 1.96x_2 \le 30$$

$$1 \le 2.42x_1 + 2.59x_2 \le 30, \ x_1 \ge 0, \ x_2 \ge 0$$

Using MATHEMATIC 9, the solution is obtained as $x_1 = 6.4087$, $x_2 = 5.5949$ and the optimum value is $Z_{\text{Proposed method}} = 55.3989$.

Kumar *et al.* [12], obtained the fuzzy solution as $\widetilde{x}_1 = (1,2,3)$, $\widetilde{x}_2 = (4,5,6)$ and optimum value is $\widetilde{Z}_{\text{Kumar}} = (9,25,27)$. Using, EGMD the defuzzified optimum value is $Z_{\text{Kumar}} = 34.2931 < Z_{\text{Proposed method}} = 55.3989$.

Hence, the proposed method gives the better solution than the Kumar *et al.* [12] method.

Example 4.6. Consider the following FLP problem from Das *et al.* [5]

$$\text{Max } \widetilde{Z} = (2,3,4,5) \otimes \widetilde{x}_1 \oplus (2,4,6,8) \otimes \widetilde{x}_2$$

Subject to

$$(2,4,6,8) \otimes \widetilde{x}_1 \oplus (2,5,7,8) \otimes \widetilde{x}_2 = (-20,2,25,48)$$

$$(2,3,5,6) \otimes \widetilde{x}_1 \oplus (6,7,8,9) \otimes \widetilde{x}_2 = (-23,-4,18,36)$$

where $\widetilde{x}_1$ is non-negative fuzzy number and $\widetilde{x}_2$ is unrestricted fuzzy number.

Here, all the constraints in the problem are with equality sign and fuzzy numbers on the RHS are of mixed type. These equality constraints are converted to inequalities as mentioned in *Case* (i) of the proposed method. Hence, the corresponding crisp LP of the FLP by the proposed method is

$$\text{Max } Z = 3.4386x_1 + 4.8225x_2$$

Subject to

$$-20 \le 4.8225x_1 + 5.2375x_2 \le 48$$

$$-23 \le 3.8913x_1 + 7.4721x_2 \le 36,\ x_1 \ge 0$$

x_2 is unrestricted

Using MATHEMATIC 9, the solution is obtained as $x_1 = 10.8673$, $x_2 = -0.8415$ and optimum value is $Z_{\text{Proposed method}} = 33.31$.

Das *et al.* [5], obtained the fuzzy solution as $x_1 = (-3,-2,1.4,5)$, $x_2 = (1,2,4.84,6)$ and optimum value is $\tilde{Z}_{\text{Das}} = (-1,4,33.24,68)$. Using EGMD, the defuzzified value is $Z_{\text{Das}} = 18.5593$. Hence, $Z_{\text{Das}} < Z_{\text{Proposed method}}$ and it shows that the proposed method gives better solution.

5. Conclusions

In this paper, a new model has been forwarded to solve the FLP problem with both equality constraints and inequality constraints. Based on GMD and EGMD, the model was proposed to solve the fuzzy linear programming. This proposed method addressed problems of FLP with both equality and inequality constraints involving triangular and trapezoidal fuzzy numbers of different types (non-negative fuzzy numbers, negative fuzzy number and mixed fuzzy numbers). The proposed scheme exhibited better results from the aspects of computing efficiency, being simple to apply on and performance. Though, the developed method was explained using a production problem and other real-life decision problems in many areas, this method further can be applied to such areas as investment involving risk, engineering and supply chain management and transportation problem.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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