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Research Article

Solitary Wave Solutions for Time-Fractional Dispersive Long Wave Equations via Generalized Kudryashov-Auxaliry Method

A. A. Gaber*[1, 2]

[1] *Department of Mathematics, College of Science and Humanities at Howtat Sudair, Majmaah University, Majmaah 11952, Saudi Arabia;*

[2] *Department of Mathematics, Faculty of Education, Ain Shams University, Heliopolis, Roxy, Cairo, Egypt*

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Abstract. In this paper, we present *Generalized Kudryashov-Jacobian Method* (GKJM) to obtain new type of general exact solutions for nonlinear partial differential equations. GKJM is applied to time-fractional dispersive long wave equations. Seven types solutions of this equation are obtained and including trigonometric function and hyperbolic function. The obtained solutions represent kink wave, anti-kink wave, singular wave, and periodic wave.

Keywords. Generalized Kudryashov method; Dispersive long wave equations

Mathematics Subject Classification (2020). 02.30.Jr; 02.30.Hq; 04.20.Jb

1. Introduction

The applications of nonlinear differential equations have much attention by researchers because its describe various phenomena in many fields such as the fluid flow, electro chemistry, scattering theory, transport theory, elasticity, control theory, potential theory, signal processing, image processing, diffusion theory, kinetic theory, systems identification, biology and other areas [9, 12].

Travelling wave methods have an important role to obtain exact solutions that are described and explained these natural phenomena. Most famous of these effective methods

*Emails: a.gaber@mu.edu.sa, aagaber6@gmail.com

are (G'/G)-expansion method [2,3,18,23,25], Exp-function method [1,6,10,16,21], generalization of He's Exp-Function method [7,19], a new extended Auxiliary equation method [14], modified Kudryashov method [11,13,15] and generalized Kudryashov method [8].

Kudryashov [13] presented a wave method based on rotation function as form $\frac{a_i}{(1+\exp(\eta))^i}$, $i \in N$. Then, Gaber [8] developed a Kudryashov's method by replacing $\exp(\eta)$ with an arbitrary function that satisfies Riccati equation, and it has a large number of solutions. In this paper, we give another development of Kudryashov's method by replacing $\exp(\eta)$ with another arbitrary function which is a solution of auxiliary linear differential equation to give Jacobian elliptic function.

2. Description of GKAM for PDEs

In this part, we will present the detail description of the *Generalized Kudryashov Auxaliry Method* (GKAM).

We consider a given nonlinear partial differential equation for $u(t,x_1,x_2,\dots,x_n)$

$$\Theta(u,D_t^\beta u,D_{x_1}u,D_{x_1}^2u,D_{x_1}^3u,D_{x_1}D_{x_2}u,\dots)=0, \tag{2.1}$$

where Θ is is a polynomial in u. We seek its wave solution [20,22],

$$u=U(\eta),\quad \eta=h_ix_i+\frac{\omega t^\beta}{\Gamma(1+\alpha)},\quad i=1,2,\dots. \tag{2.2}$$

Consequently, eq. (2.1) is reduced to the *Ordinary Differential Equation* (ODE):

$$U(u,\omega u',h_iu',h_i^2u'',h^2u'',\dots)=0. \tag{2.3}$$

GKAM is based on the assumption that the travelling wave solutions can be expressed in the following form

$$u(\eta)=\sum_{i=0}^{m}\frac{a_i}{(1+\Psi(\eta))^i}, \tag{2.4}$$

where m is positive integers which are unknown to be further determined, a_i is constant. In addition, $\Psi(\eta)$ is a solution of the auxiliary linear differential equation

$$\Psi'^2(\eta)=R+Q\Psi^2(\eta)+P\Psi^4(\eta), \tag{2.5}$$

where R, Q and P are constants. The solutions $\Psi(\eta)$ are the Jacobian elliptic function which depend on the values of the constants R, Q and P (shown in Table 1). Equation (2.5) has more 40 different solutions [4].

When $m\to 1$ Jacobian elliptic function solutions are transformed to hypergeometric function as follow

{cn(η), dn(η)} → sech(η), {ds(η), cs(η)} → csch(η), sn(η) → tanh(η),

{sc(η), sd(η)} → sinh(η), {nc(η), nd(η)} → cosh(η), ns(η) → coth(η), {cd(η), dc(η)} → 1

When $m\to 0$ Jacobian elliptic function solutions are transformed to trigonometric function, yields

sc(η) → tan(η), {cn(η), cd(η)} → cos(η), {sn(η), sd(η)}→ sin(η),

{nc(η), dc(η)} → sec(η), {ns(η), ds(η)} → csc(η),cs(η) → cot(η), {dn(η), nd(η)} → 1.

Table 1. Jacobian elliptic function solution $\Psi(\eta)$ of auxiliary equation (2.5)

Cases	R	Q	P	Solutions of auxiliary equation
1	$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\eta)$
2	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}(\eta)\pm\mathrm{cs}(\eta), \frac{\mathrm{sn}(\eta)}{1\pm\mathrm{cn}(\eta)}$
3	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\mathrm{nc}(\eta)+\mathrm{sc}(\eta)$
4	1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\eta)$
5	1	$2m^2-1$	$-m(1-m^2)$	$\mathrm{sd}(\eta)$
6	1	$2-4m^2$	1	$\frac{\mathrm{sn}(\eta)\mathrm{dn}(\eta)}{\mathrm{cn}(\eta)}$
7	1	2	m^2	$\frac{\mathrm{sn}(\eta)\mathrm{cn}(\eta)}{\mathrm{dn}(\eta)}$

3. Application PDE

We apply the *GKAM* to find exact solutions for the following the $(2+1)$-dimensions of dispersive long wave equations which are called Wu-Zhang (WZ) equations [17, 24]

$$\begin{aligned}
&D_t^\beta u + u\,D_x u + vD_y u + D_x w = 0,\\
&D_t^\beta v + u\,D_x v + vD_y v + D_y w = 0,\\
&D_t^\beta w + D_x(uw) + D_y(vw) + \frac{1}{3}(D_x^3 u + D_y^2 D_x u + D_x^2 D_y v + D_y^3 v) = 0,
\end{aligned} \tag{3.1}$$

where $u(t,x,y)$, $v(t,x,y)$ and $w(t,x,y)$, this equations describe nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth.

We perform the transformation $\eta = hx + ky + \frac{\omega t^\beta}{\Gamma(1+\alpha)}$, eq. (3.1) can be reduced into an ODEs

$$\begin{aligned}
&gU' + hVU' + kVU' + hW' = 0,\\
&gV' + hUV' + kVV' + kW' = 0,\\
&gW' + h(VU' + UV) + k(VW' + WV') + \frac{1}{3}(h^3 + hk^2)U''' + \frac{1}{3}(kh^2 + k^3)V''' = 0,
\end{aligned} \tag{3.2}$$

where $U' = \frac{\partial U}{\partial \eta}$.

We can freely know that the solution do not depend on the balancing the highest order linear and nonlinear terms [5], we have:

$$U(\eta) = a_0 + \frac{a_1}{1+\Psi(\eta)}, \quad V(\eta) = b_0 + \frac{b_1}{1+\Psi(\eta)}, \quad W(\eta) = c_0 + \frac{c_1}{1+\Psi(\eta)} + \frac{c_2}{(1+\Psi(\eta))^2}. \tag{3.3}$$

Substituting from eq. (3.3) into eq. (3.2) and by comparing the coefficients of all powers of $\Psi(\eta)$, we get the following equations,

$$\begin{aligned}
&-\omega a_1 - ha_1a_0 - ka_1b_0 - hc_1 = 0, \quad -\omega b_1 - hb_1a_0 - kb_1b_0 - kc_1 = 0,\\
&-\omega a_1 - ha_1a_0 - ha_1^2 - ka_1b_0 - ka_1b_1 - hc_1 - 2hc_2 = 0,\\
&-\omega b_1 - hb_1a_0 - hb_1a_1 - kb_1b_0 - kb_1^2 - kc_1 - 2kc_2 = 0,\\
&-ha_1c_0 - \frac{1}{3}h^3a_1Q - kb_0c_1 - kb1c_0 - ha_0c_1 - 2h^3a_1P - 2k^3b_1P\\
&\qquad -\frac{1}{3}k^2ha_1Q - 2k^2ha_1P - \frac{1}{3}k^3b_1Q - \frac{1}{3}h^2kb_1Q - 2h^2kb_1P - \omega c_1 = 0,
\end{aligned}$$

$$\frac{4}{3}k^2ha_1Q-2gc_2-2ha_1c_0-2ha_0c_2+\frac{4}{3}h^3a_1Q+\frac{4}{3}k^3b_1Q-2kbc_1$$
$$-2kb_1c-2kbc_2-2kb_1c_1+\frac{4}{3}h^2kb_1Q-2ha_1c_1-2\omega c_1-2ha_0c_1=0,$$
$$-2ha_0c_2-2k^2ha_1R-2h^2kb_1R-\omega c_1-2\omega c_2-2h^3a_1R-2k^3b_1R$$
$$-\frac{1}{3}k^3b_1Q-ha_0c_1-ha_1c_0-2ha_1c_1-3ha_1c_2-3kb_1c_2-kbc_1$$
$$-kb_1c-2kb_1c1-\frac{1}{3}h^2kb_1Q-2kbc_2-\frac{1}{3}h^3a_1Q-\frac{1}{3}k^2ha_1Q=0. \tag{3.4}$$

Solving the system of algebraic equations with bu using Maple, we obtain the following solutions:

$$a_0=-\frac{(2P+Q)(h^2+k^2)+(kb_0+\omega)\sqrt{3(Q+P+R)}}{h\sqrt{3(Q+P+R)}},$$
$$a_1=\frac{2}{3}h\sqrt{3(Q+P+R)},$$
$$b_1=\frac{2}{3}k\sqrt{3(Q+P+R)}, \quad (b_0 \text{ is arbitrary})$$
$$c_1=\frac{2}{3}(2P+Q)(h^2+k^2),$$
$$c_2=\frac{2}{3}(P+Q+R)(h^2+k^2),$$
$$c_0=-\frac{(2P^2+3PQ+RQ+6RP)(h^2+k^2)}{3(P+Q+R)}. \tag{3.5}$$

Substituting these results into eq. (3.2), we obtain the following solutions for WZ equations

$$u(t,x,y)=\frac{(2P+Q)(h^2+k^2)+(kb_0+\omega)\sqrt{3(Q+P+R)}}{h\sqrt{3(Q+P+R)}}+\frac{2}{3}\frac{h\sqrt{3(Q+P+R)}}{1+\phi(\eta)},$$
$$v(t,x,y)=b_0+\frac{2}{3}\frac{k\sqrt{3(Q+P+R)}}{1+\phi(\eta)},$$
$$w(t,x,y)=-\frac{(2P^2+3PQ+RQ+6RP)(h^2+k^2)}{3(P+Q+R)}+\frac{2}{3}\frac{(2P+Q)(h^2+k^2)}{1+\phi(\eta)}+\frac{2}{3}\frac{(P+Q+R)(h^2+k^2)}{(1+\phi(\eta))^2}, \tag{3.6}$$

where $\eta=hx+ky+\frac{\omega t^\beta}{\Gamma(1+\alpha)}$.

Substituting results eq. (3.6) into eq. (3.3) and with the aid of Table 1, we obtain the following exact solutions for WZ equations:

Case (1): When $R=1-m^2$, $Q=2-m^2$ and $P=1$

$$u(t,x,y)=\frac{(4-m^2)(h^2+k^2)+(kb_0+\omega)\sqrt{6(2-m^2)}}{h\sqrt{6(2-m^2)}}+\frac{2}{3}\frac{h\sqrt{6(2-m^2)}}{1+\mathrm{cs}(\eta)},$$
$$v(t,x,y)=b_0+\frac{2}{3}\frac{k\sqrt{6(2-m^2)}}{1+\mathrm{cs}(\eta)},$$
$$w(t,x,y)=-\frac{(14-9m^2+(1-m^2)(2-m^2))(h^2+k^2)}{6(2-m^2)}+\frac{2}{3}\frac{(4-m^2)(h^2+k^2)}{1+\mathrm{cs}(\eta)}+\frac{4}{3}\frac{(2-m^2)(h^2+k^2)}{(1+\mathrm{cs}(\eta))^2}, \tag{3.7}$$

where $\eta = hx + ky + \frac{\omega t^{\beta}}{\Gamma(1+\alpha)}$.

Case (2): When $R = \frac{1}{4}$, $Q = \frac{1-2m^2}{2}$ and $P = \frac{1}{4}$

$$
\begin{aligned}
u(t,x,y) &= \frac{(1-m^2)(h^2+k^2)+(kb_0+\omega)\sqrt{3(1-m^2)}}{h\sqrt{3(1-m^2)}} + \frac{2}{3}\frac{h\sqrt{3(1-m^2)}}{1+\mathrm{ns}(\eta)\pm\mathrm{cs}(\eta)},\\
v(t,x,y) &= b_0 + \frac{2}{3}\frac{k\sqrt{3(1-m^2)}}{1+\mathrm{ns}(\eta)\pm\mathrm{cs}(\eta)},\\
w(t,x,y) &= -\frac{1}{3}(h^2+k^2) + \frac{2}{3}\frac{(1-m^2)(h^2+k^2)}{1+\mathrm{ns}(\eta)\pm\mathrm{cs}(\eta)} + \frac{4}{3}\frac{(1-m^2)(h^2+k^2)}{(1+\mathrm{ns}(\eta)\pm\mathrm{cs}(\eta))^2},
\end{aligned} \tag{3.8}
$$

where $\eta = hx + ky + \frac{\omega t^{\beta}}{\Gamma(1+\alpha)}$.

Case (3): When $R = \frac{1-m^2}{4}$, $Q = \frac{1+m^2}{2}$ and $P = \frac{1-m^2}{4}$

$$
\begin{aligned}
u(t,x,y) &= -\frac{h^2+k^2+(kb_0+\omega)\sqrt{3}}{3h} + \frac{3h}{1+\mathrm{nc}(\eta)\pm\mathrm{sc}(\eta)},\\
v(t,x,y) &= b_0 + \frac{3k}{1+\mathrm{nc}(\eta)\pm\mathrm{sc}(\eta)},\\
w(t,x,y) &= -\frac{1}{3}(1-m^2)(h^2+k^2) + \frac{2}{3}\frac{(h^2+k^2)}{1+\mathrm{nc}(\eta)\pm\mathrm{sc}(\eta)} - \frac{2}{3}\frac{(h^2+k^2)}{(1+\mathrm{nc}(\eta)\pm\mathrm{sc}(\eta))^2},
\end{aligned} \tag{3.9}
$$

where $\eta = hx + ky + \frac{\omega t^{\beta}}{\Gamma(1+\alpha)}$.

Case (4): When $R = 1$, $Q = 2-m^2$ and $P = 1-m^2$

$$
\begin{aligned}
u(t,x,y) &= -\frac{(4-3m^2)(h^2+k^2)+(kb_0+\omega)\sqrt{6(2-m^2)}}{h\sqrt{6(2-m^2)}} + \frac{2}{3}\frac{h\sqrt{6(2-m^2)}}{1+\mathrm{sc}(\eta)},\\
v(t,x,y) &= b_0 + \frac{2}{3}\frac{k\sqrt{6(2-m^2)}}{1+\mathrm{sc}(\eta)},\\
w(t,x,y) &= -\frac{(13-20m^2+5m^4)(h^2+k^2)}{6(2-m^2)} + \frac{2}{3}\frac{(4-3m^2)(h^2+k^2)}{1+\mathrm{sc}(\eta)} - \frac{4}{3}\frac{(2-m^2)(h^2+k^2)}{(1+\mathrm{sc}(\eta))^2},
\end{aligned} \tag{3.10}
$$

where $\eta = hx + ky + \frac{\omega t^{\beta}}{\Gamma(1+\alpha)}$.

Case (5): When $R = -m^2(1-m^2)$, $Q = 2m^2-1$ and $P = 1$

$$
\begin{aligned}
u(t,x,y) &= -\frac{(1+2m^2)(h^2+k^2)+(kb_0+\omega)\sqrt{3(2m^2+m(1-m^2)^2)}}{h\sqrt{3(2m^2+m(1-m^2)^2)}} + \frac{2}{3}\frac{\sqrt{3(2m^2+m(1-m^2)^2)}}{1+\mathrm{sd}(\eta)},\\
v(t,x,y) &= b_0 + \frac{2}{3}\frac{\sqrt{3(2m^2+m(1-m^2)^2)}}{1+\mathrm{sd}(\eta)},\\
w(t,x,y) &= -\frac{(6m^2-1+m(5-2m^4)(1-m^2)^2)(h^2+k^2)}{m(2m+(1-m^2)^2)} + \frac{2}{3}\frac{(1+2m^2)(h^2+k^2)}{1+\mathrm{sd}(\eta)}\\
&\quad - \frac{2}{3}\frac{m(2m+(1-m^2)^2)(h^2+k^2)}{(1+\mathrm{sd}(\eta))^2},
\end{aligned} \tag{3.11}
$$

where $\eta = hx + ky + \frac{\omega t^{\beta}}{\Gamma(1+\alpha)}$.

Case (6): When $R = 1$, $Q = 2-4m^2$ and $P = 1$

$$u(t,x,y) = -\frac{4(1-m^2)(h^2+k^2)+(kb_0+\omega)\sqrt{12(1-m^2)}}{h\sqrt{12(1-m^2)}} + \frac{2}{3}\frac{\mathrm{cn}(\eta)\sqrt{12(1-m^2)}}{\mathrm{cn}(\eta)+\mathrm{sn}(\eta)\mathrm{dn}(\eta)},$$

$$v(t,x,y) = b_0 + \frac{2}{3}\frac{\mathrm{cn}(\eta)\sqrt{12(1-m^2)}}{\mathrm{cn}(\eta)+\mathrm{sn}(\eta)\mathrm{dn}(\eta)},$$

$$w(t,x,y) = -\frac{4(1-m^2)(h^2+k^2)}{3(1-m^2)} + \frac{8}{3}\frac{\mathrm{cn}(\eta)(1-m^2)(h^2+k^2)}{\mathrm{cn}(\eta)+\mathrm{sn}(\eta)\mathrm{dn}(\eta)} - \frac{8}{3}\frac{\mathrm{cn}^2(\eta)(1-m^2)(h^2+k^2)}{(\mathrm{cn}(\eta)+\mathrm{sn}(\eta)\mathrm{dn}(\eta))^2}, \tag{3.12}$$

where $\eta = hx + ky + \frac{\omega t^\beta}{\Gamma(1+\alpha)}$.

Case (7): When $R = 1$, $Q = 2$ and $P = m^2$

$$u(t,x,y) = -\frac{2(1+m^2)(h^2+k^2)+(kb_0+\omega)\sqrt{3(3+m^2)}}{h\sqrt{3(3+m^2)}} + \frac{2}{3}\frac{\mathrm{dn}(\eta)\sqrt{3(3+m^2)}}{\mathrm{dn}(\eta)+\mathrm{sn}(\eta)\mathrm{cn}(\eta)},$$

$$v(t,x,y) = b_0 + \frac{2}{3}\frac{\mathrm{dn}(\eta)\sqrt{3(3+m^2)}}{\mathrm{dn}(\eta)+\mathrm{sn}(\eta)\mathrm{cn}(\eta)},$$

$$w(t,x,y) = -\frac{4(1+6m^2+m^4)(h^2+k^2)}{3(3+m^2)} + \frac{4}{3}\frac{\mathrm{dn}(\eta)(1+m^2)(h^2+k^2)}{\mathrm{dn}(\eta)+\mathrm{sn}(\eta)\mathrm{cn}(\eta)} - \frac{2}{3}\frac{\mathrm{dn}^2(\eta)(3+m^2)(h^2+k^2)}{(\mathrm{dn}(\eta)+\mathrm{sn}(\eta)\mathrm{cn}(\eta))^2}, \tag{3.13}$$

where $\eta = hx + ky + \frac{\omega t^\alpha}{\Gamma(1+\alpha)}$.

4. Graphics and Discussion

In this section, we show graphical simulations of the solutions eqs. (3.7)-(3.13). GKAM can be present a huge number of solutions, including trigonometric function, hyperbolic function and Jacobi elliptic function. The obtained solutions of govern PDE represent kink wave, anti-kink wave, singular wave, and periodic wave. Figure 1 shows anti-kink wave solution of eq. (3.7) when $\beta = 0.25$ and $m = 0$. Figure 2 shows singular periodic wave solution eq. (3.8) when $\beta = 0.5$ and $m = 0$. Figure 3 shows kink wave solution of eq. (3.9) when $\beta = 0.5$ and $m = 1$. Figure 4 shows anti-kink wave solution of eq. (3.10) when $\beta = 0.75$ and $m = 1$. Figure 5 shows kink wave solution of eq. (3.11) when $\beta = 0.5$ and $m = 1$. Figure 6 shows solitary wave solution of (3.12) when $\beta = 0.5$ and $m = 0$. Figure 7 shows periodic wave solution of eq. (3.13) when $\beta = 0.5$ and $m = 0$.

5. Conclusion

In this paper, new modification of Kudryashov method is represented to obtain more general exact solutions for time fractional WZ equations. A fractional similarity transformation is implemented to reduced time fractional WZ equations to ordinary differential equations. GKJM is applied to obtained a new type solutions for WZ equations. The exact solution of WZ equations are drawn by using of MAPLE 15. The exact solutions is shown a various waves as kink wave, anti-kink wave, singular wave, and periodic wave.

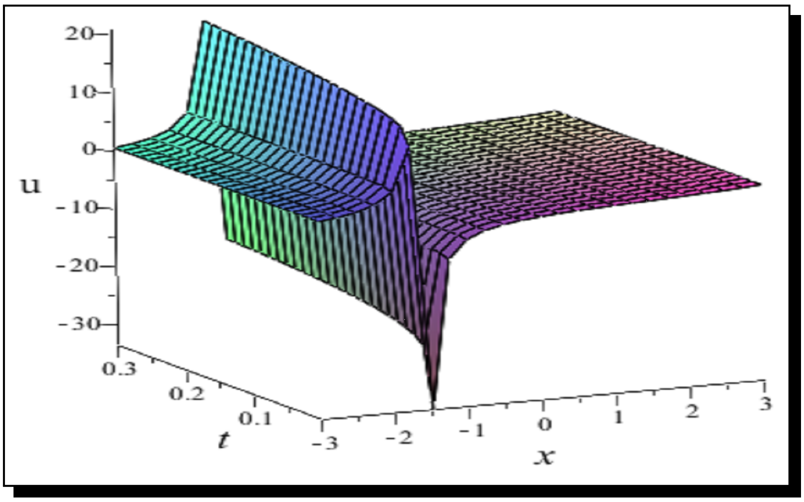

Figure 1. 3D graph of u-solution (3.7) when $b_0 = -1$, $h = 1$, $k = 0.1$, $\omega = 1$

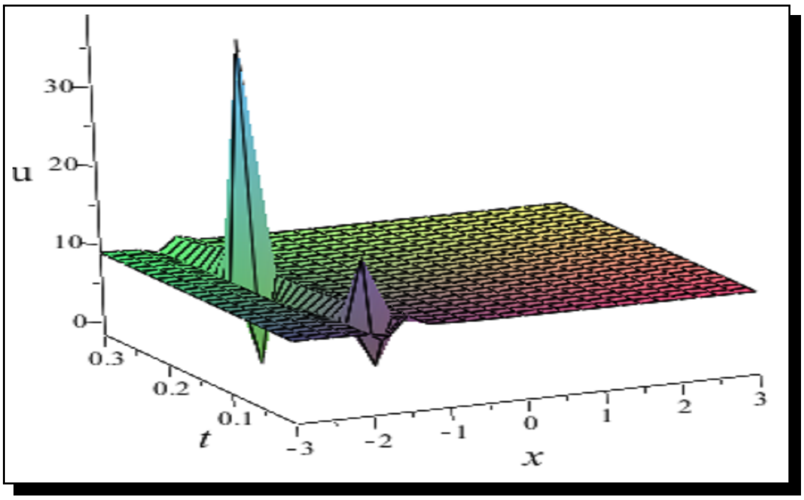

Figure 2. 3D graph of u-solution (3.8) when $b_0 = -1$, $h = -0.1$, $k = 0.1$, $\omega = 1$

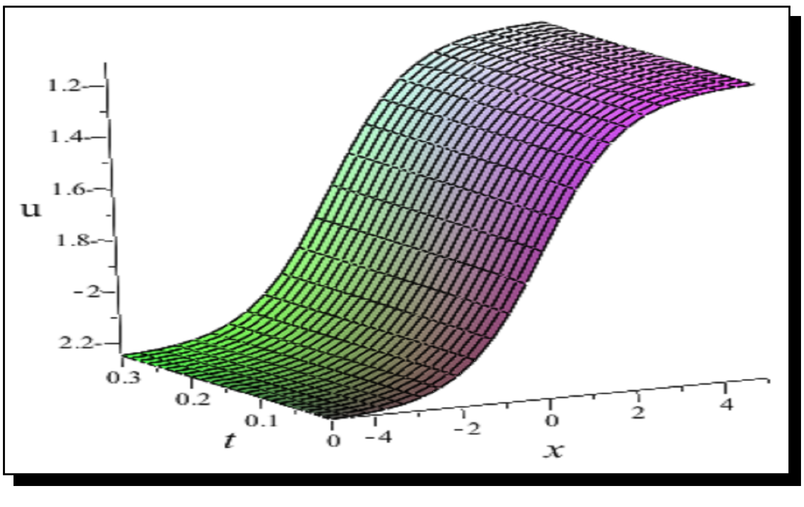

Figure 3. 3D graph of u-solution (3.9) when $b_0 = k = -1$, $h = 1$, $\omega = 0.1$

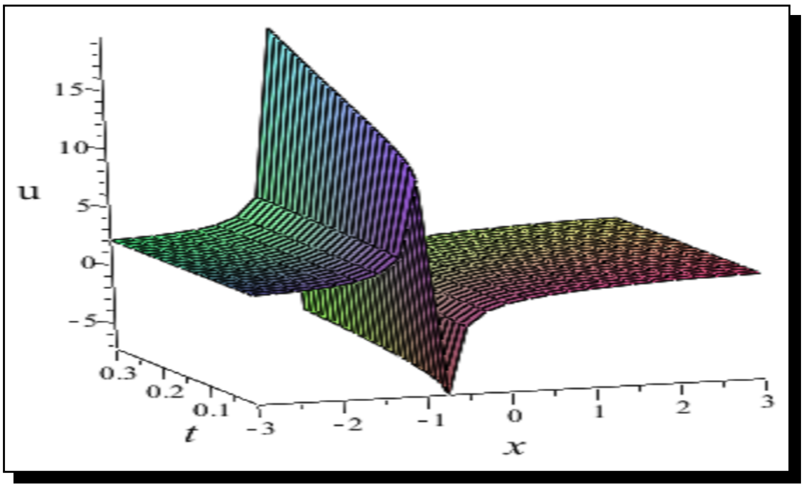

Figure 4. 3D graph of u-solution (3.10) when $b_0 = k = 1$, $h = -1$, $\omega = 0.1$

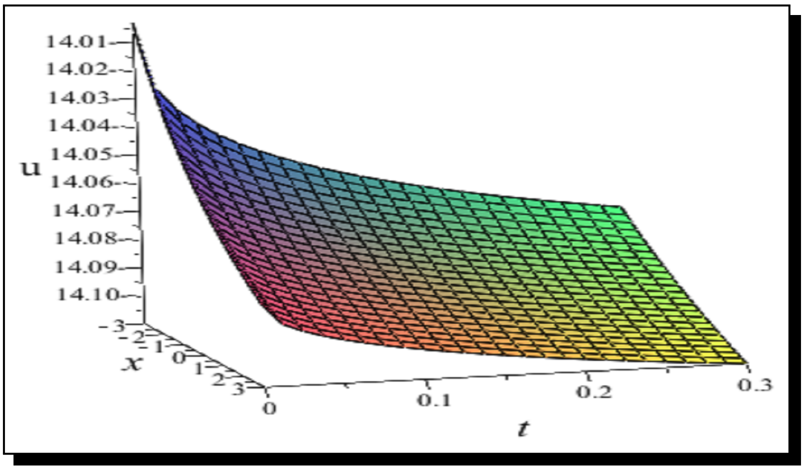

Figure 5. 3D graph of u-solution (3.11) when $b_0 = k = -1$, $h = 0.1$, $\omega = 1$

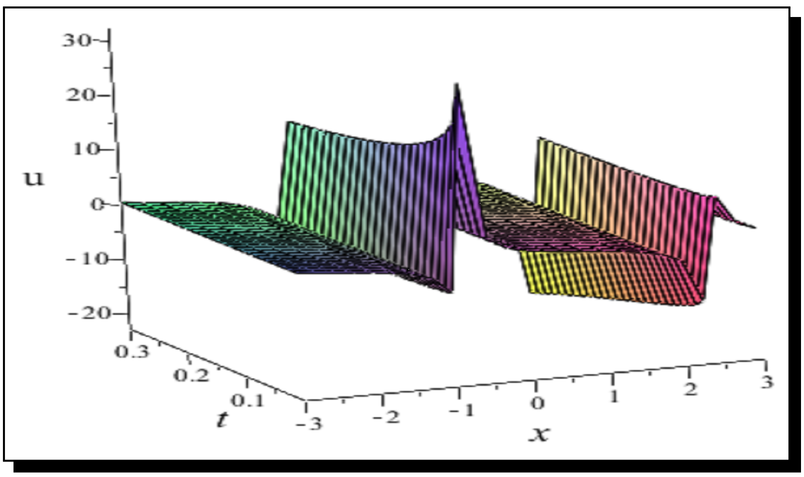

Figure 6. 3D graph of u-solution (3.12) when $b_0 = -1$, $h = k = \omega = 0.1$

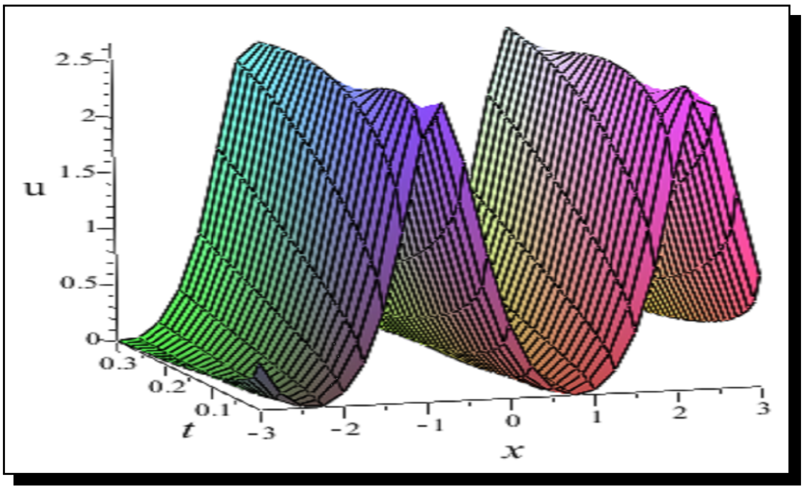

Figure 7. 3D graph of u-solution (3.13) when $b_0 = -1$, $h = k = \omega = 1$

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] K. Ayub, M. Y. Khan and Q. Mahmood-Ul-Hassan, Solitary and periodic wave solutions of Calogero-Bogoyavlenskii-Schiff equation via exp-function methods, *Computers & Mathematics with Applications* **74** (2017), 3231 – 3241, DOI: 10.1016/j.camwa.2017.08.021.

[2] Z. Bin, (G'/G)-Expansion method for solving fractional partial differential equations in the theory of mathematical physics, *Communications in Theoretical Physics* **58** (2012), 623 – 630, DOI: 10.1088/0253-6102/58/5/02.

[3] A. Biswas, A. Sonmezoglu, M. Ekici, M. Mirzazadeh, Q. Zhou, A. S. Alshomrani, S. P. Moshokoa and M Belic, Optical soliton perturbation with fractional temporal evolution by extended G'/G-expansion method, *Optik* **161** (2018), 301 – 320, DOI: 10.1016/j.ijleo.2018.02.051.

[4] A. Ebaid and E. H. Aly, Exact solutions for the transformed reduced Ostrovsky equation via the F-expansion method in terms of Weierstrass-elliptic and Jacobian-elliptic functions, *Wave Motion* **49** (2012), 296 – 308, DOI: 10.1016/j.wavemoti.2011.11.003.

[5] A. Ebaid, An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms, *Journal of Mathematical Analysis and Applications* **392** (2012), 1 – 5, DOI: 10.1016/j.jmaa.2011.04.025.

[6] M. Ekici, M. Mirzazadeh, A. Sonmezoglu, Q. Zhou, H. Triki, M. Z. Ullah, S. P. Moshokoa and A. Biswas, Optical solitons in birefringent fibers with Kerr nonlinearity by exp-function method, *Optik* **131** (2017), 964 – 976, DOI: 10.1016/j.ijleo.2016.12.015.

[7] A. A. El-Satar, Similarity reductions and new exact solutions for B-family equations, *American Journal of Mathematics and Statistics* **2**(3) (2012), 40 – 43, DOI: 10.5923/j.ajms.20120203.05.

[8] A. A. Gaber, A. F. Aljohani, A. Ebaid and J. T. Machado, The generalized version of Kudryashov method for nonlinear space-time fractional partial differential equations of Burgers type, *Nonlinear Dynamics* **95** (2019), 361 – 368, DOI: 10.1007/s11071-018-4568-4.

[9] A. A. Gaber and M. H. Shehata, New approach of MHD boundary layer flow towards a porous stretching sheet via symmetry analysis and the generalized Exp-function method, *International Journal of Analysis and Applications* **18**(5) (2020), 738 – 747, DOI: 10.28924/2291-8639-18-2020-738.

[10] J.-H. He and M. A. Abdou, New periodic solutions for non-linear evolution equations using Exp-function method, *Chaos, Solitons & Fractals* **34** (2007), 1421 – 1429, DOI: 10.1016/j.chaos.2006.05.072.

[11] K. Hosseini, A. Bekir and R. Ansari, New exact solutions of the conformable time-fractional Cahn-Allen and Cahn-Hilliard equations using the modified Kudryashov method, *Optik* **132** (2017), 203 – 209, DOI: 10.1016/j.ijleo.2016.12.032.

[12] A.-A. Hyder and M. A. Barakat, General improved Kudryashov method for exact solutions of nonlinear evolution equations in mathematical physics, *Physica Scripta* **4**(95) (2020), 045212, DOI: 10.1088/1402-4896/ab6526.

[13] M. M. Kabir, A. Khajeh, E. A. Aghdam and A. Y. Koma, Modified Kudryashov method for finding exact solitary wave solutions of higher-order nonlinear equations, *Mathematical Methods in the Applied Sciences* **34** (2011), 213 – 219, DOI: 10.1002/mma.1349.

[14] M. M. A. Khater, A. R. Seadawy and D. Lu, Dispersive optical soliton solutions for higher order nonlinear Sasa-Satsuma equation in mono mode fibers via new auxiliary equation method, *Superlattices and Microstructures* **113** (2018), 346 – 358, DOI: 10.1016/j.spmi.2017.11.011.

[15] D. Kumar, A. R. Seadawy and A. K. Joardar, Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology, *Chinese Journal of Physics* **56** (2018), 75 – 85, DOI: 10.1016/j.cjph.2017.11.020.

[16] J.-G. Liu, L. Zhou and Y. He, Multiple soliton solutions for the new $(2+1)$-dimensional Korteweg-de Vries equation by multiple exp-function method, *Applied Mathematics Letters* **80** (2018), 71 – 78, DOI: 10.1016/j.aml.2018.01.010.

[17] Z. Y. Ma, Homotopy perturbation method for the Wu-Zhang equation in fluid dynamics, *Journal of Physics: Conference Series* **96** (2008), 012182, DOI: 10.1088/1742-6596/96/1/012182.

[18] J. Manafian, M. F. Aghdaei, M. Khalilian and R. S. Jeddi, Application of the generalized G'/G-expansion method for nonlinear PDEs to obtaining soliton wave solution, *Optik* **135** (2017), 395 – 406, DOI: 10.1016/j.ijleo.2017.01.078.

[19] M. H. M. Moussa and A. A. Gaber, Symmetry analysis and solitary wave solutions of nonlinear ion-acoustic waves equation, *International Journal of Analysis and Applications* **18** (2020), 448 – 460, DOI: 10.28924/2291-8639-18-2020-448.

[20] I. Naeem and M. D. Khan, Symmetry classification of time-fractional diffusion equation, *Communications in Nonlinear Science and Numerical Simulation* **42** (2017), 560 – 570, DOI: 10.1016/j.cnsns.2016.05.022.

[21] Rahmatullah, R. Ellahi, S. T. Mohyud-Din and U. Khan, Exact traveling wave solutions of fractional order Boussinesq-like equations by applying Exp-function method, *Results in Physics* **8** (2018), 114 – 120, DOI: 10.1016/j.rinp.2017.11.023.

[22] S. Sahoo and S. S. Ray, Invariant analysis with conservation laws for the time fractional Drinfeld-Sokolov-Satsuma-Hirota equations, *Chaos, Solitons and Fractals* **104** (2017), 725 – 733, DOI: 10.1016/j.chaos.2017.09.031.

[23] A. M. Shahoot, K. A. E. Alurrfi, M. O. M. Elmrid, A. M. Almsiri and A. M. H. Arwiniya, The G'/G-expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines, *Journal of Taibah University for Science* **13** (2018), 63 – 70, DOI: 10.1080/16583655.2018.1528663.

[24] N. Xiong, Y.-Q. Li, J. O. Chen and Y. Chen, One-dimensional optimal system and similarity reductions of Wu-Zhang equation, *Communications in Theoretical Physics* **66** (2016), 1 – 11, URL: https://ctp.itp.ac.cn/EN/Y2016/V66/I01/1.

[25] P. Ye and G. Cai, An extended (G'/G)-expansion method and travelling wave solutions to nonlinear Klein-Gordon equation, *International Journal of Nonlinear Science* **11** (2011), 225 – 229, URL: http://www.internonlinearscience.org/upload/papers/20110915021935124.pdf.