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Fixed Point Theorems for Weakly Contractive Maps in Banach Spaces

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Abstract. In this paper, we prove fixed point theorems for certain classes of mappings called weakly contractive. These mappings were introduced by Alber and Guerre-Delabriere [1] and studied by the author and others via iterative processes [1, 2].

1. Introduction

Direct and iterative methods for finding fixed points of an operator defined in an appropriate Banach space have been studied by many authors. These studies have given rise to development of results and techniques which are now widely available in the literature (see, for example [4, 5, 6, 7, 8]).

Let E be a real normed linear space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single valued and if E^* is uniformly convex (equivalently if E is uniformly smooth) then J is uniformly continuous on bounded subsets of E (see, e.g., [9]). We shall denote the single-valued duality mapping by j.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called d-weakly contractive if there exists a continuous and nondecreasing function $\phi : [0,\infty] := \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is positive on $\mathbb{R}^+\backslash\{0\}$, $\phi(0) = 0$, $\lim\limits_{t\to\infty} \phi(t) = \infty$ and for

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$x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|^2). \tag{1.1}$$

It is called weakly contractive see, e.g., [1, 10, 11]) if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and ϕ as above such that

$$\|Tx - Ty\| \leq \|x-y\| - \phi(\|x-y\|). \tag{1.2}$$

These classes of operators were first introduced and studied by Alber and Guerre-Delabriere [1, 10, 11], and convergence results were obtained in both Hilbert and Banach spaces, assuming the existence of a fixed point in the domain, $D(T)$ of T. Further generalizations of the results of Alber and Guerre-Delabriere were made by the author, C.E. Chidume and H. Zegeye [1, 3], also assuming the existence of a fixed point in the interior of the domain of T. No attempt was made to find out if any of these operators has a fixed point, or to ascertain the minimum condition to ensure the existence of a fixed point.

The purpose of this paper is to prove that if T is a self map, then each of these operators actually has a unique fixed point. Thus,, given the conditions of Theorem 5.7 of Alber and Guerre-Delabriere [1], and others, we show that the operator T has a unique fixed point.

2. **Preliminaries**

Let E be a real Banach space with $\dim E \geq 2$. The modulus of smoothness of E is defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t, t > 0\right\}.$$

The Banach space E is called uniformly smooth if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$. In [12], Reich proved that if E is a real uniformly smooth Banach space, then there exists a nondecreasing continuous function $\beta : [0,\infty) \to [0,\infty)$ satisfying the following conditions: (i) $\beta ct \leq c\beta(t)$ for all $c \geq 1$, (ii) $\lim_{t\to 0^+} \beta(t) = 0$. and such that the following inequality holds

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|), \tag{2.1}$$

for all $x, y \in E$.

Typical examples of such spaces are the Lebesgue L_p, the sequence l_p, and the Sobolev W_p^m spaces, $1 < p < \infty$. We shall also use the following well known inequality.

Lemma 2.1. *Let E be a real Banach space and let J be the normalized duality map on E. Then for any given $x, y \in E$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \textit{for all } \; j(x+y) \in J(x+y).$$

We shall also need the following lemma which was stated and proved in Alber and Guerre-Delabriere [1], see also Chidume etal [2]. Since the proof does not depend on the existence of a fixed point and also since it plays a key role in our theorems and proof, we shall repeat the proof here.

Lemma 2.2. *Let E be an arbitrary real Banach space and let $T : D(T) \subseteq E \to E$ be a d-weakly contractive map, then $A = I - T$ is bounded.*

Proof. A is accretive since $\langle Ax - Ay, j(x-y)\rangle \geq \phi(\|x-y\|^2)$. Let x_0 be any vector in the interior of $D(T)$. Then by Lemma 5.5 of [1] (see also [2]) there exists a constant $r_0 > 0$ and a closed ball $\bar{B}(x_0, r_0) \subseteq D(A)$ such that for all $x \in D(A)$ we have

$$\langle Ax - Ax_0, j(x-x_0)\rangle \geq r_0\|Ax\| - c_0(\|x-x_0\|) + r_0), \tag{2.2}$$

where $c_0 = \sup\limits_{\eta \in \bar{B}(x_0,r_0)} \|A(\eta)\| < \infty$. On the other hand, for some $j(x-y) \in J(x-y)$ we have that

$$\begin{aligned} \langle Ax - Ax_0, j(x-x_0)\rangle &= \langle x - x_0, j(x-x_0)\rangle - \langle Tx - Tx_0, j(x-x_0)\rangle \\ &\leq \|x-x_0\|^2 + \langle Tx - Tx_0, j(x-x_0)\rangle \\ &\leq 2\|x-x_0\|^2. \end{aligned} \tag{2.3}$$

From 2.2 and 2.3 we get that

$$\|Ax\| \leq r_0^{-1}(2\|x-x_0\|^2 + c_0(\|x-x_0\| + r_0)).$$

Thus, A is bounded.

3. **Main results**

Theorem 3.1. *Let E be a real Banach space, K a closed, convex and bounded subset of E. Let $T : K \to K$ be a weakly contractive map, from K into itself. Then T has a unique fixed point in K.*

Proof. For the inequality 1.2 to be meaningful, $\|x-y\| \geq \phi(\|x-y\|)$. Since ϕ is continuous, $\phi(0) = 0$ and ϕ is positive on $\mathbb{R}^+\backslash\{0\}$, there exists $\delta \in (0,1)$ such that $\phi(\|x-y\|) \geq \delta\|x-y\|$ for all $x, y \in K$. Thus, in 1.2, we have that

$$\begin{aligned} \|Tx - Ty\| &\leq \|x-y\| - \delta\|x-y\| \\ &= (1-\delta)\|x-y\|. \end{aligned}$$

It follows that T is a strict contraction in K, and hence has a unique fixed point in K.

Theorem 3.2. *Let K be a closed, convex and bounded subset of a real uniformly smooth Banach space. Let $T : K \to K$ be a d-weakly contractive map. Then T has a unique fixed point in K.*

Proof. Define $Fx = x - \epsilon Ax$, where $A = I - T$, $\epsilon > 0$. Since K is convex, then F maps K into K, and the fixed point of T is also the fixed point of F. That is, T has a fixed point if and only if F has. Using inequality 2.1, valid for uniformly smooth spaces, we have for all $x, y \in K$,

$$\begin{aligned}\|Fx - Fy\|^2 &= \|(x-y) - \epsilon(Ax - Ay)\|^2 \\ &\leq \|x-y\|^2 - 2\epsilon\langle Ax - Ay, j(x-y)\rangle \\ &\quad + \max\{\|x-y\|, 1\}\epsilon\|Ax - Ay\|\beta(\|Ax - Ay\|) \\ &\leq \|x-y\|^2 - 2\epsilon\phi(\|x-y\|^2) \\ &\quad + \max\{\|x-y\|, 1\}\epsilon\|Ax - Ay\|\beta(\|Ax - Ay\|). \qquad (3.1)\end{aligned}$$

Let $d = \sup\{\|x-y\| : x, y \in K\} = \operatorname{diam} K$. Since A is bounded, choose $M_0 > 0$ such that $\|Ax - Ay\| \leq M_0\|x-y\|$. Since β is continuous, it is bounded on bounded sets. Thus, choose M_1 such that $\beta(t) \leq M_1 t$ on a bounded interval. It follows that

$$\beta(\epsilon(\|Ax - Ay\|)) \leq \epsilon M_0\beta(\|x-y\|) \leq \epsilon M_0 M_1\|x-y\| \leq \epsilon M_0 M_1 d. \qquad (3.2)$$

Thus, in equation (3.1), we consider two cases:

Case 1: $\max\{\|x-y\|, 1\} = \|x-y\|$. Then we have:

$$\begin{aligned}\|Fx - Fy\|^2 &\leq \|x-y\|^2 - 2\epsilon\phi(\|x-y\|^2) + \epsilon M_0\|x-y\|\epsilon M_0 M_1 d\|x-y\| \\ &= \|x-y\|^2 - 2\epsilon\phi(\|x-y\|^2) + \epsilon^2 M_0^2 M_1 d\|x-y\|^2. \qquad (3.3)\end{aligned}$$

Case 2: $\max\{\|x-y\|, 1\} = 1$. Then (3.1) becomes

$$\begin{aligned}\|Fx - Fy\|^2 &\leq \|x-y\|^2 - 2\epsilon\phi(\|x-y\|^2) + \epsilon M_0\|x-y\|\epsilon M_0 M_1\|x-y\| \\ &= \|x-y\|^2 - 2\epsilon\phi(\|x-y\|^2) + \epsilon^2 M_0^2 M_1\|x-y\|^2. \qquad (3.4)\end{aligned}$$

Combining (3.3) and (3.4) we get

$$\|Fx - Fy\|^2 \leq \|x-y\|^2 - 2\epsilon\phi(\|x-y\|^2) + \epsilon^2 M_0^2 M_1(1+d)\|x-y\|^2. \qquad (3.5)$$

Choose $M_2 > 0$ such that $\phi(\|x-y\|^2) \geq M_2\|x-y\|^2$ and $\epsilon > 0$ such that $2\epsilon M_2 - \epsilon^2 M_0^2 M_1(1+d) > 0$, i.e. $\epsilon^2 M_0^2 M_1(1+d) < 2\epsilon M_2$ or $\epsilon < \frac{2M_2}{M_0^2 M_1(1+d)}$. Infact, choose $\epsilon > 0$ such that $2\epsilon M_2 - \epsilon^2 M_0^2 M_1(1+d) \geq \delta > 0$, then 3.5 becomes

$$Fx - Fy\|^2 \leq (1-\delta)\|x-y\|^2.$$

Then F is a strict contraction. Hence, T admits a unique fixed point in K. □

As a result of this theorem, the operator T in Theorem 5.7, and many others of Alber and Guerre-Delabriere [1] have unique fixed points. The next theorem, which is on a real Banach space is proved on the condition that the space admits a uniformly continuous duality map.

Theorem 3.3. *Let E be a real Banach space with a uniformly continuous duality map and K a nonempty closed, convex and bounded subset of E. Suppose $T : K \to K$ is a d-weakly contractive map. Then T has a unique fixed point in K.*

Proof. By Lemma 2.1, we have

$$\begin{aligned} \|Fx - Fy\|^2 &= \|x - y - \epsilon(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\epsilon \left\langle Ax - Ay\rangle, j\left(x - y - \epsilon(Ax - Ay)\right)\right\rangle \\ &= \|x - y\|^2 - 2\epsilon\langle Ax - Ay, j(x - y)\rangle \\ &\quad + 2\epsilon\langle Ax - Ay, j(x - y) - j(x - y - \epsilon(Ax - Ay))\rangle \\ &\leq \|x - y\|^2 - 2\epsilon\phi(\|x - y\|^2) \\ &\quad + 2\epsilon\|Ax - Ay\|\,\|j(x - y) - j[(x - y - \epsilon(Ax - Ay)]\|. \end{aligned} \tag{3.6}$$

Since $(x - y) - [(x - y) - \epsilon(Ax - Ay)] = \epsilon(Ax - Ay)$ and $\|Ax - Ay\| \leq M_0\|x - y\|$, then $\|j(x - y) - j[(x - y) - \epsilon(Ax - Ay)]\| \leq \epsilon M^* M_0\|x - y\|$. Hence equation (3.6) becomes

$$\|Fx - Fy\|^2 \leq \|x - y\|^2 - 2\epsilon\phi(\|x - y\|^2) + 2\epsilon^2 M_0^2 M^*\|x - y\|^2. \tag{3.7}$$

The rest of the proof follows as in the proof of Theorem 3.2 to conclude that T has a unique fixed point in K.

Remark. If E is a uniformly smooth Banach space, then E automatically has a uniformly continuous duality map and in this case Theorem 3.3 reduces to that of Theorem 3.2. Thus, Lemma 2.1 could be used in proving Theorem 3.2 without assuming uniform continuity of the duality map. Our theorems and results reveal that most of the spaces considered in Alber and Guerre-Delabriere [1] actually have unique fixed points.

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