Communications in Mathematics and Applications
Vol. 12, No. 2, pp. 303–314, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i2.1495

Research Article

Common Fixed Point Theorem in a Multiplicative *S*-Metric Space With an Application

Prasad Kanchanapally* and V. Naga Raju

Department of Mathematics, University College of Science, Osmania University, Hyderabad 500007, India

Received: December 8, 2020 **Accepted:** March 1, 2021

Abstract. In this paper, we introduce multiplicative S-metric space as a generalization of multiplicative d-metric space and investigate its some topological properties. Further, we establish a common fixed point theorem for a pair of self maps in the framework of multiplicative S-metric space with an application. This result generalizes some fixed point results in the current literature. Finally, we provide an example in support of the result.

Keywords. Multipicative S-metric; Common fixed point; Multiplicative S-convergent sequence; Multiplicative S-complete metric space; Multiplicative S-continuous

Mathematics Subject Classification (2020). 54H25; 47H10

1. Introduction

In 1906, Frechet introduced the notion of metric space which plays very important role in the fixed point theory. Later, it has been generalized in many ways by several authors such as quasi metric, modular metric and b-metric etc. (see [4, 9, 15]). In 1972, Grossman and Katz [6] developed multiplicative calculus wherein subtraction and addition operations were replaced by division and multiplication to solve multiplicative boundary value problems more efficiently than ordinary Newtonian calculus. In 2008, Bashirov et al. [3] introduced a new concept called multiplicative metric. They studied some properties of multiplicative derivatives, multiplicative integrals and established the fundamental theorem of multiplicative calculus. In 2012, Florack

*Corresponding author: iitm.prasad@gmail.com

and Assen [5] used this concept in biomedical image analysis. In the same year Özavşar and Cevikel [10] studied topological properties of multiplicative d-metric spaces and introduced multiplicative contraction with which they proved some fixed point theorems for such spaces. For more work on multiplicative metric spaces, we refer to ([1, 7, 8, 16]).

Recently, Sedghi et al. [13] introduced the concept of S-metric space as a generalization of G-metric space and metric space. After that many authors proved several fixed and common fixed point results in the setting of S-metric spaces for different contractions (see [11, 12, 14]). The purpose of this paper is to introduce multiplicative S-metric space and to prove a common fixed point theorem for two self maps in the framework of multiplicative S-metric spaces with an application.

2. Preliminaries

In this section, we present some definitions which will be used later in this paper.

Definition 2.1. Let X be a non empty set. Then we say that a function $S : X^3 \to [0,\infty)$ is a multiplicative S-metric on X iff it satisfies the following for all x, y, z and $a \in X$

(P1) $S(x,y,z) \geq 1$;

(P2) $S(x,y,z) = 1$ iff $x = y = z$;

(P3) $S(x,y,z) \leq S(x,x,a)S(y,y,a)S(z,z,a)$.

Here (X,S) is called a multiplicative S-metric space.

Example 2.1. If (X,d) is a multiplicative metric space, then $S : X^3 \to [0,\infty)$ defined by $S(x,y,z) = d(x,z)d(y,z)$ for $x,y,z \in X$ is a multiplicative S-metric on X, called multiplicative S-metric induced by d.

Example 2.2. Consider a metric space (X,d). For fixed $t > 1$, then (X,S) is a multiplicative S-metric space, where $S(x,y,z) := t^{d(x,y)+d(y,z)+d(z,x)}$ for $x,y,z \in X$.

For this, let x,y,z and $a \in X$. Since d is the metric on X and $d(x,y)+d(y,z)+d(z,x) \geq 0$, it follows that $t^{d(x,y)+d(y,z)+d(z,x)} \geq 1$. Thus $S(x,y,z) \geq 1$.
Also,

$$\begin{aligned} S(x,y,z) = 1 \quad &\text{iff} \quad t^{d(x,y)+d(y,z)+d(z,x)} = 1 \\ &\text{iff} \quad d(x,y)+d(y,z)+d(z,x) = 0 \\ &\text{iff} \quad x = y = z. \end{aligned}$$

Consider,

$$\begin{aligned} S(x,y,z) &= t^{d(x,y)+d(y,z)+d(z,x)} \\ &\leq t^{d(x,a)+d(a,y)+d(y,a)+d(a,z)+d(z,a)+d(a,x)} \\ &= S(x,x,a)S(y,y,a)S(z,z,a). \end{aligned}$$

That is,

$$S(x,y,z) \leq S(x,x,a)S(y,y,a)S(z,z,a).$$

Therefore, all the conditions of multiplicative S-metric are satisfied and hence S is a multiplicative S-metric on X.

Example 2.3. Let (X,S') be a S-metric space. Then (X,S) is a multiplicative S-metric space, where $S(x,y,z) := t^{S'(x,y,z)}$ for $x,y,z \in X$ and for fixed $t > 1$.

Example 2.4. Let $X = (0,\infty)$. Then a function S defined by $S(x,y,z) := \left|\frac{x}{z}\right|^* \left|\frac{y}{z}\right|^*$ is a multiplicative S-metric on X for $x,y,z \in X$, where $|x|^* := \begin{cases} x, & x \geq 1 \\ \frac{1}{x}, & x < 1 \end{cases}$ for $x \in X$. For this, let $x,y,z \in X$. If $\frac{x}{z} \geq 1$ and $\frac{y}{z} \geq 1$, then $S(x,y,z) = \frac{x}{z}\frac{y}{z} \geq 1$. If $\frac{x}{z} \leq 1$ and $\frac{y}{z} \geq 1$, then we have $S(x,y,z) = \frac{z}{x}\frac{y}{z} \geq 1$. If $\frac{x}{z} \leq 1$ and $\frac{y}{z} \leq 1$, then we must have $S(x,y,z) = \frac{z}{x}\frac{z}{y} \geq 1$. If $\frac{x}{z} \geq 1$ and $\frac{y}{z} \leq 1$, then we get that $S(x,y,z) = \frac{x}{z}\frac{z}{y} \geq 1$. From all the cases, we conclude that $S(x,y,z) \geq 1$ for all $x,y,z \in X$.

For $x,y,z \in X$, we have $S(x,y,z) = 1$ iff $\left|\frac{x}{z}\right|^* \left|\frac{y}{z}\right|^* = 1$ iff $\left|\frac{x}{z}\right|^* = 1$ and $\left|\frac{y}{z}\right|^* = 1$. If $\left|\frac{x}{z}\right|^* = \frac{z}{x}$, then we would have $1 = \frac{z}{x}$ and hence $z = x$, but $x < z$-contradiction. Similarly, if $\left|\frac{y}{z}\right|^* = \frac{z}{y}$, we get $z = y < z$-contradiction.

Therefore, $S(x,y,z) = 1$ iff $\frac{x}{z} = 1$ and $\frac{y}{z} = 1$ iff $z = x = y$.

Let x,y,z and $a \in X$. We consider

$$\begin{aligned} S(x,y,z) &= \left|\frac{x}{z}\right|^* \left|\frac{y}{z}\right|^* \\ &= \left|\frac{x}{a}\frac{a}{z}\right|^* \left|\frac{y}{a}\frac{a}{y}\right|^* \\ &\leq \left|\frac{x}{a}\right|^* \left|\frac{a}{z}\right|^* \left|\frac{y}{a}\right|^* \left|\frac{a}{y}\right|^* \\ &\leq \left|\frac{x}{a}\right|^* \left|\frac{x}{a}\right|^* \left|\frac{y}{a}\right|^* \left|\frac{y}{a}\right|^* \left|\frac{z}{a}\right|^* \left|\frac{z}{a}\right|^*. \end{aligned}$$

Therefore $S(x,y,z) \leq S(x,x,a)S(y,y,a)S(z,z,a)$ for all x,y,z and $a \in X$. Hence, S is a multiplicative S-metric on X.

Lemma 2.1 ([2]). *In a multiplicative metric space $((0,\infty),|\cdot|^*)$, we have the following for $x,y \in (0,\infty)$, where* $|x|^* := \begin{cases} x, & \text{for } x \geq 1 \\ \frac{1}{x}, & \text{for } x < 1 \end{cases}$

(i) $|x|^* \geq 1$,

(ii) $\frac{1}{|x|^*} \leq x \leq |x|^*$,

(iii) $|x|^* = \left|\frac{1}{x}\right|^*$,

(iv) $|xy|^* \leq |x|^*|y|^*$.

Definition 2.2. We say that a sequence (x_n) in multiplicative S-metric space (X,S) multiplicative S-converges to some $x \in X$ iff for each $\epsilon > 1$, there exists a $H \in \mathbb{N}$ such that $S(x_n,x_n,x) < \epsilon$, for all $n \geq H$.

Definition 2.3. We say that a sequence (x_n) in multiplicative S-metric space (X,S) is multiplicative S-Cauchy sequence in X iff for each $\epsilon > 1$, there exists a $H \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \epsilon$, for all $n,m \geq H$.

Definition 2.4. We say that a multiplicative S-metric space (X,S) is multiplicative S-complete iff every multiplicative S-Cauchy sequence in X is multiplicative S-convergent in X.

Definition 2.5. Let (X,S) and (Y,S') be two multiplicative S-metric spaces. Then, we say that $f: X \to Y$ is multiplicative S-continuous at some point $x \in X$ iff for every $\epsilon > 1$, there exists $\delta > 1$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$.
Here, we say that f is multiplicative S-continuous on X iff it is multiplicative S-continuous at every point of X.

3. Some Topological Properties

In this section, we present some topological properties of multiplicative S-metric spaces.

Lemma 3.1. *In multiplicative S-metric space (X,S), we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Proof. For $x,y \in X$, we have $S(x,x,y) \leq S(x,x,x)S(x,x,x)S(y,y,x) = S(y,y,x)$.
Thus, $S(x,x,y) \leq S(y,y,x)$. By interchanging x and y, we get $S(y,y,x) \leq S(x,x,y)$ and hence the result proved. □

Lemma 3.2. *In multiplicative S-metric space (X,S), $x_n \to x$ iff $S(x_n,x_n,x) \to 1$, as $n \to \infty$.*

Proof. (⇒): Then for every $\epsilon > 1$, there exists $H \in \mathbb{N}$ such that $S(x_n,x_n,x) < \epsilon$, for all $n \geq H$. In particular, we have $S(x_n,x_n,x) < 1 + \frac{1}{n}$ for every $n \geq H$. Now, letting $n \to \infty$, we obtain that $\lim_{n\to\infty} S(x_n,x_n,x) \leq 1$. Note that $S(x_n,x_n,x) \geq 1$ for every $n \in \mathbb{N}$ and thus, $\lim_{n\to\infty} S(x_n,x_n,x) \geq 1$ and the results follows.

(⇐): Let $\epsilon > 1$. As $S(x_n,x_n,x) \to 1$, then there exists $H' \in \mathbb{N}$ such that

$$|S(x_n,x_n,x) - 1| < \epsilon - 1, \quad \text{for all } n \geq H'.$$

This will imply that

$$S(x_n,x_n,x) < \epsilon, \quad \text{for all } n \geq H'$$

and hence the result proved. □

Lemma 3.3. *In multiplicative S-metric space (X,S), if there exist two sequences (x_n) and (y_n) in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} S(x_n,x_n,x_n) = S(x,x,y)$.*

Proof. For $n \in \mathbb{N}$, we have

$$\begin{aligned} S(x_n,x_n,y_n) &\leq S(x_n,x_n,x)S(x_n,x_n,x)S(y_n,y_n,x) \\ &\leq S(x_n,x_n,x)S(x_n,x_n,x)S(y_n,y_n,y)S(y_n,y_n,y)S(x,x,y). \end{aligned}$$

Therefore

$$S(x_n,x_n,y_n) \le S(x_n,x_n,x)S(x_n,x_n,x)S(y_n,y_n,y)S(y_n,y_n,y)S(x,x,y), \quad \text{for every } n \in \mathbb{N}.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} S(x_n,x_n,y_n) \le S(x,x,y).$$

Consider

$$\begin{aligned} S(x,x,y) &\le S(x,x,x_n)S(x,x,x_n)S(y,y,x_n) \\ &\le S(x,x,x_n)S(x,x,x_n)S(y,y,y_n)S(y,y,y_n)S(x_n,x_n,y_n) \\ &= S(x_n,x_n,x)S(x_n,x_n,x)S(y_n,y_n,y)S(y_n,y_n,y)S(x_n,x_n,y_n). \end{aligned}$$

Thus,

$$S(x,x,y) \le S(x_n,x_n,x)S(x_n,x_n,x)S(y_n,y_n,y)S(y_n,y_n,y)S(x_n,x_n,y_n), \quad \text{for every } n \in \mathbb{N}.$$

As $n \to \infty$, we have $S(x,x,y) \le \lim_{n\to\infty} S(x_n,x_n,y_n)$ and hence the result follows. □

Lemma 3.4. *In multiplicative S-metric space (X,S), (x_n) is a multiplicative S-Cauchy sequence in X iff $S(x_n,x_n,x_m) \to 1$, as $n,m \to \infty$.*

Proof. The proof is similar to the proof of Lemma 3.2. □

Lemma 3.5. *Let (X,S) and (Y,S') be two multiplicative S-metric spaces. Then $f : X \to Y$ is continuous at point $x \in X$ iff $fx_n \to fx$ for every sequence (x_n) in X with $x_n \to x$ in X.*

Proof. ($\Rightarrow$): Let $x_n \to x$ in X. Now we show that $fx_n \to fx$ in Y. For this, let $\epsilon > 1$. Since f is continuous at point $x \in X$, there exists $\delta > 1$ such that $f(B(x,\delta)) \subset B(fx,\epsilon)$. Now $x_n \to x$ implies that there exists $H \in \mathbb{N}$ such that $S(x_n,x_n,x) < \delta$, for all $n \ge H$. For $n \ge H$, we have $x_n \in B(x,\delta))$. It follows that $fx_n \in B(fx,\epsilon))$ for $n \ge H$ and hence $fx_n \to fx$ in Y.

($\Leftarrow$): Suppose that f is not continuous at $x \in X$. Then for any $\delta > 1$, there exists a $\epsilon > 1$ such that $f(B(x,\delta)) \nsubseteq B(fx,\epsilon)$. In particular, for every $n \in \mathbb{N}$ there exists $x_n \in B\left(x, 1+\frac{1}{n}\right)$ such that $fx_n \notin B(fx,\epsilon)$. Thus $S(x_n,x_n,x) < 1+\frac{1}{n}$, but $S(fx_n,fx_n,fx) \ge \epsilon$ for every $n \in \mathbb{N}$. By letting $n \to \infty$, we have $\lim_{n\to\infty} S(x_n,x_n,x) = 1$, but (fx_n) does not converge to fx in Y-contradiction and hence the result follows. □

Lemma 3.6. *Suppose in a multiplicative metric space $((0,\infty),|\cdot|^*)$, there is a sequence (x_n) in $(0,\infty)$ such that $x_n \to x$, where* $|x|^* := \begin{cases} x, & x \ge 1 \\ \frac{1}{x}, & x < 1 \end{cases}$ *for $x \in X$.*

Then $|x_n|^* \to |x|^*$.

Proof. Note that $\left|\frac{|x_n|^*}{|x|^*}\right|^* \le \left|\frac{x_n}{x}\right|^*$ for all $n \in \mathbb{N}$. Now letting $n \to \infty$, we have $\lim_{n\to\infty} \left|\frac{|x_n|^*}{|x|^*}\right|^* = 1$ and hence the proof completed.

Let $C^*(I)$ be a collection of all multiplicative S-continuous functions from I to $(0,\infty)$, where $I = [0,T]$ for a fixed $T > 1$. □

Example 3.1. $(C^*(I),S)$ is multiplicative complete S-metric space, where

$$S(x,y,z) := \max_{t\in I}\left\{\left|\frac{x(t)}{z(t)}\right|^* \left|\frac{y(t)}{z(t)}\right|^*\right\}, \quad \text{for } x,y,z \in X.$$

To show this, let (x_n) be a multiplicative S-Cauchy sequence in $(C^*(I),S)$ and $\epsilon > 1$. Then there exists $H \in \mathbb{N}$ such that $S(x_n,x_n,y) < \epsilon^{\frac{1}{3}}$ for $n,m \geq H$. Thus,

$$\max_{t\in I}\left\{\left|\frac{x_n(t)}{x_m(t)}\right|^* \left|\frac{x_n(t)}{x_m(t)}\right|^*\right\} < \epsilon^{\frac{1}{3}}, \quad \text{for } n,m \geq H.$$

For each $t \in I$, we have

$$\begin{aligned}\left|\frac{x_n(t)}{x_m(t)}\right|^* &\leq \left|\frac{x_n(t)}{x_m(t)}\right|^* \left|\frac{x_n(t)}{x_m(t)}\right|^* \\ &\leq \max_{t\in I}\left\{\left|\frac{x_n(t)}{x_m(t)}\right|^* \left|\frac{x_n(t)}{x_m(t)}\right|^*\right\} \\ &< \epsilon^{\frac{1}{3}}, \quad \text{for } n,m \geq H.\end{aligned}$$

This shows that $(x_n(t))$ is a Cauchy sequence in $((0,\infty),|\cdot|^*)$ for each fixed $t \in I$. Since $((0,\infty),|\cdot|^*)$ is multiplicative S-complete,there exists $x(t) \in (0,\infty)$ such that $x_n(t) \to x(t)$ for each fixed $t \in I$. By letting $m \to \infty$, we have

$$\left|\frac{x_n(t)}{x(t)}\right|^* \left|\frac{x_n(t)}{x(t)}\right|^* < \epsilon^{\frac{1}{3}}, \quad \text{for } n \geq H.$$

Let us show that $x \in C^*(I)$. It is enough to verify that x is multiplicative S-continuous on I. For this, let $a \in I$. Since x_H is multiplicative S-continuous at a, there exists a $\delta > 1$ such that $S(x_Ht, x_Ht, x_Ha) < \epsilon^{\frac{1}{3}}$, whenever $t \in B(a,\delta)$. Thus,

$$\left|\frac{x_H(t)}{x_H(a)}\right|^* \left|\frac{x_H(t)}{x_H(a)}\right|^* < \epsilon^{\frac{1}{3}}, \quad \text{for } t \in B(a,\delta).$$

For $t \in B(a,\delta)$, we have

$$\begin{aligned}S(xt,xt,xa) &= \left|\frac{xt}{xa}\right|^* \left|\frac{xt}{xa}\right|^* \\ &= \left|\frac{xt}{x_Ht}\frac{x_Ht}{x_Ha}\frac{x_Ha}{xa}\right|^* \left|\frac{xt}{\alpha_Ht}\frac{x_Ht}{x_Ha}\frac{x_Ha}{xa}\right|^* \\ &\leq \left|\frac{xt}{\alpha_Ht}\right|^* \left|\frac{x_Ht}{x_Ha}\right|^* \left|\frac{x_Ha}{xa}\right|^* \left|\frac{\alpha t}{\alpha_Ht}\right|^* \left|\frac{x_Ht}{x_Ha}\right|^* \left|\frac{x_Ha}{xa}\right|^* \\ &= \left|\frac{x_Ht}{xt}\right|^* \left|\frac{x_Ht}{x_Ha}\right|^* \left|\frac{x_Ha}{xa}\right|^* \left|\frac{x_Ht}{xt}\right|^* \left|\frac{x_Ht}{x_Ha}\right|^* \left|\frac{x_Ha}{xa}\right|^* \\ &=< \epsilon^{\frac{1}{3}}\epsilon^{\frac{1}{3}}\epsilon^{\frac{1}{3}} = \epsilon.\end{aligned}$$

Therefore, $S(xt,xt,xa) < \epsilon$ for $t \in B(a,\delta)$. It shows that $x \in C^*(I)$. Since $\left|\frac{x_n(t)}{x(t)}\right|^* \left|\frac{x_n(t)}{x(t)}\right|^* < \epsilon^{\frac{1}{3}}$ for $n \geq H$ and for all $t \in I$, we have $x_n \to x$ in $(C^*(I),S)$, completing the proof.

4. Main Results

In this section, we establish a common fixed point theorem for two self maps in the frame work of multiplicative S-metric spaces and provide an example in support of the theorem.

Theorem 4.1. *Suppose in a multiplicative complete S-metric space (X,S), there are two self maps f and T on X and $\gamma\in\Gamma$, where $\Gamma:=\{\gamma:[1,\infty)\to[1,3):\gamma(k_n)\to 3\Rightarrow k_n\to 1\}$, such that*

(i) $fX\subset TX$,

(ii) *T is multiplicative S-continuous, and*

(iii) $S(fx,fx,fy)\le S(Tx,Tx,Ty)^{\frac{\gamma(S(Tx,Tx,Ty))}{3}}$ *for all $x,y\in X$.*

Then f and T have a unique common fixed point.

Proof. Define a sequence (y_n) in X by $y_n:=fy_{2n}=Ty_{2n+2}$ for $n=0,1,2,\dots$.
For $n\in\mathbb{N}\cup\{0\}$, we consider

$$\begin{aligned}S(y_{n+1},\beta_{n+1},y_{n+2})&=S(fy_{2n+2},fy_{2n+2},fy_{2n+4})\\&\le S(Ty_{2n+2},Ty_{2n+2},Ty_{2n+4})^{\frac{\gamma(S(Ty_{2n+2},Ty_{2n+2},Ty_{2n+4}))}{3}}\\&=S(y_n,y_n,y_{n+1})^{\frac{\gamma(S(y_n,y_n,y_{n+1}))}{3}}\\&<S(y_n,y_n,y_{n+1}).\end{aligned}$$

Thus,

$$S(y_{n+1},y_{n+1},y_{n+2})<S(y_n,y_n,y_{n+1}),\quad \text{for all } n\in\mathbb{N}\cup\{0\}.$$

This shows that $\{S(y_n,y_n,y_{n+1})\}$ is a decreasing sequence in $\mathbb{R}$ and bounded below by 1 and hence it converges in $\mathbb{R}$, say $p\ge 1$. Clearly, $\left[1-\frac{\lim_{n\to\infty}\gamma(S(y_n,y_n,y_{n+1}))}{3}\right]\log p\le 0$. Since $p\ge 1$, we have $\left[1-\frac{\lim_{n\to\infty}\gamma(S(y_n,y_n,y_{n+1}))}{3}\right]\le 0$. This implies that $\lim_{n\to\infty}\gamma(S(y_n,y_n,y_{n+1}))=3$ and hence $\lim_{n\to\infty}S(y_n,y_n,y_{n+1}))=1$, since $\gamma\in\Gamma$. Now, we show that $\lim_{n\to\infty}S(y_m,y_m,y_n)=1$ for $m>n$. If not, there exists $\epsilon>1$ and two subsequences (m_k) and (n_k) of $\mathbb{N}$ with $n_k>m_k>k$ such that

$$S(y_{m_k},y_{m_k},y_{n_k})\ge\epsilon,\quad \text{for } k=1,2,\dots.$$

For $k\in\mathbb{N}$, we consider

$$\begin{aligned}S(y_{m_k},y_{m_k},y_{n_k})&\le S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{n_k},y_{n_k},y_{m_k+1})\\&\le S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{n_k},y_{n_k},y_{n_k+1})\\&\quad\cdot S(y_{n_k},y_{n_k},y_{n_k+1})S(y_{m_k+1},y_{m_k+1},y_{n_k+1}).\end{aligned}$$

Therefore

$$\begin{aligned}S(y_{m_k},y_{m_k},y_{n_k})&\le S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{n_k},y_{n_k},y_{n_k+1})\\&\quad\cdot S(y_{n_k},y_{n_k},y_{n_{k+1}})S(y_{m_k},y_{m_k},y_{n_k})^{\frac{\gamma(S(y_{m_k},y_{m_k},y_{n_k}))}{3}},\quad \text{for } k\in\mathbb{N}.\end{aligned}$$

By taking log on both sides, we have

$$\begin{aligned}\left[1-\frac{\gamma(S(y_{m_k},y_{m_k},y_{n_k}))}{3}\right]\log S(y_{m_k},y_{m_k},y_{n_k})&\le\log[S(y_{m_k},y_{m_k},y_{m_k+1})S(y_{m_k},y_{m_k},y_{m_k+1})\\&\quad\cdot S(y_{n_k},y_{n_k},y_{n_k+1})S(y_{n_k},y_{n_k},y_{n_k+1})],\\&\quad\text{for every } k\in\mathbb{N}.\end{aligned}$$

Letting $k\to\infty$, we have $\left[1-\frac{\lim_{k\to\infty}\gamma(S(y_{m_k},y_{m_k},y_{n_k}))}{3}\right]\le\frac{0}{\log\epsilon}=0$ and hence $\lim_{k\to\infty}\gamma(S(y_{m_k},y_{m_k},y_{n_k}))=3$.

It follows that $\lim_{k\to\infty} S(y_{m_k}, y_{m_k}, y_{n_k}) = 1$-contradiction, since $\gamma \in \Gamma$.
Therefore, (y_n) is a Cauchy sequence in X. Since (X,S) is multiplicative S-complete, there exists $u \in X$ such that $\lim_{n\to\infty} S(y_n, y_n, u) = 1$. This forces that $y_{2n+2} \to u$ and hence $\lim_{k\to\infty} Ty_{2n+2} = Tu$, since T is multiplicative S-continuous. Thus, $Tu = u$.
Consider

$$\begin{aligned} S(y_{n+1}, y_{n+1}, fu) &= S(fy_{2n+2}, fy_{2n+2}, fu) \\ &\le S(Ty_{2n+2}, Ty_{2n+2}, Tu)^{\frac{\gamma(S(Ty_{2n+2}, Ty_{2n+2}, Tu))}{3}} \\ &= S(y_n, y_n, u)^{\frac{\gamma(S(y_n, y_n, u))}{3}} \\ &< S(y_n, y_n, u). \end{aligned}$$

Thus,

$$S(y_{n+1}, y_{n+1}, fu) < S(y_n, y_n, u), \quad \text{for every } n \in \mathbb{N}.$$

Applying limit $n \to \infty$ in the inequality, we have $\lim_{n\to\infty} S(y_{n+1}, y_{n+1}, fu) \le 1$ and hence $\lim_{n\to\infty} S(y_{n+1}, y_{n+1}, fu) = 1$. Clearly, we have

$$S(y_n, y_n, fu) \le S(y_n, y_n, y_{n+1}) S(y_n, y_n, y_{n+1}) S(fu, fu, y_{n+1}), \quad \text{for all } n \in \mathbb{N}.$$

Now letting $n \to \infty$, we have $\lim_{n\to\infty} S(y_n, y_n, fu) = 1$. Hence $fu = \lim_{n\to\infty} y_n = u$. Let $v \in X$ be another common fixed point of f and T. Then $fv = Tv = v$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} S(y_{n+1}, y_{n+1}, v) &= S(fy_{2n+2}, fy_{2n+2}, fv) \\ &\le S(Ty_{2n+2}, Ty_{2n+2}, Tv)^{\frac{\gamma(S(Ty_{2n+2}, Ty_{2n+2}, Tv))}{3}} \\ &= S(y_n, y_n, v)^{\frac{\gamma(S(y_n, y_n, v))}{3}} \\ &< S(y_n, y_n, v). \end{aligned}$$

Therefore,

$$S(y_{n+1}, y_{n+1}, fv) < S(y_n, y_n, v), \quad \text{for every } n \in \mathbb{N}.$$

Thus, $(S(y_n, y_n, v))$ is a decreasing sequence of real numbers. So it converges, say $p' \ge 1$. Clearly, we have

$$\log S(y_{n+1}, y_{n+1}, v) \le \frac{\gamma(S(y_n, y_n, v))}{3} \log S(y_n, y_n, v), \quad \text{for every } n \in \mathbb{N}.$$

Letting $n \to \infty$, we have

$$\left[1 - \frac{\lim_{n\to\infty} \gamma(S(y_n, y_n, v))}{3}\right] \log p' \le 0.$$

This will imply that $\lim_{n\to\infty} \gamma(S(y_n, y_n, v) = 3$ and hence $\lim_{n\to\infty} S(y_n, y_n, v) = 1$, since $\gamma \in \Gamma$. Therefore, $\lim_{n\to\infty} y_n = v$ and $u = v$. □

Example 4.1. Consider a multiplicative complete S-metric space $(C^*(I), S)$, where

$$S(x, y, z) := \max_{t\in I} \left\{ \left| \frac{x(t)}{z(t)} \right|^* \left| \frac{y(t)}{z(t)} \right|^* \right\}, \quad \text{for } x, y, z \in X.$$

Define f and $T:X \to X$ by $fx = \left(\frac{x}{2}\right)^{\frac{1}{8}}$ and $Tx = \frac{x}{2}$ for $x \in X$. We also define $\gamma:[1,\infty) \to [1,3)$ by $\gamma k = 1$ for $k \in [1,\infty)$. Clearly, $fX \subset TX$ and $\gamma \in \Gamma$.

Case (i): Firstly, we show that T is multiplicative S-continuous on X.
For this, let $\epsilon > 1$ and $a \in X$. Choose $\delta = \epsilon > 1$. Clearly,

$$S(Tx,Tx,Ta) = \left|\frac{Tx}{Ta}\right|^* \left|\frac{Tx}{Ta}\right|^* = \left|\frac{x}{a}\right|^* \left|\frac{x}{a}\right|^*.$$

If $x \in B(a,\delta)$, then $S(x,x,a) < \delta = \epsilon$. It follows that $\left|\frac{x}{a}\right|^* \left|\frac{x}{a}\right|^* < \epsilon$. That is $\left|\frac{Tx}{Ta}\right|^* \left|\frac{Tx}{Ta}\right|^* < \epsilon$. So $S(Tx,Tx,Ta) < \epsilon$ whenever $x \in B(a,\delta)$. Therefore, T is multiplicative S-continuous on X.

Case (ii): Let x and y be in X such that $y \le x$. Consider

$$\begin{aligned}
S(fx,fx,fy) &= \left|\frac{fx}{fy}\right|^* \left|\frac{fx}{fy}\right|^* \\
&= \left|\frac{(\frac{x}{2})^{\frac{1}{8}}}{(\frac{y}{2})^{\frac{1}{8}}}\right|^* \left|\frac{(\frac{x}{2})^{\frac{1}{8}}}{(\frac{y}{2})^{\frac{1}{8}}}\right|^* \\
&= \left|\left(\frac{x}{y}\right)^{\frac{1}{8}}\right|^* \left|\left(\frac{x}{y}\right)^{\frac{1}{8}}\right|^* \\
&= \left(\frac{x}{y}\right)^{\frac{1}{8}} \left(\frac{x}{y}\right)^{\frac{1}{8}} \qquad \text{(since } y \le x\text{)} \\
&= \left(\frac{x}{y}\right)^{\frac{1}{4}} \\
&\le S(Tx,Tx,Ty)^{\frac{\gamma(S(Tx,Tx,Ty))}{3}} \\
&= \left(\left|\frac{x}{y}\right|^* \left|\frac{x}{y}\right|^*\right)^{\frac{1}{3}} \\
&= \left(\frac{x}{y}\frac{x}{y}\right)^{\frac{1}{3}} = \left(\frac{x}{y}\right)^{\frac{2}{3}}.
\end{aligned}$$

Therefore,

$$S(fx,fx,fy) \le S(Tx,Tx,Ty)^{\frac{\gamma(S(Tx,Tx,Ty))}{3}}, \quad \text{for } y \le x.$$

Let x and y be in X such that $x < y$. Then, we have

$$\begin{aligned}
S(fx,fx,fy) &= \frac{1}{(\frac{x}{y})^{\frac{1}{8}}} \frac{1}{(\frac{x}{y})^{\frac{1}{8}}} \\
&= \left(\frac{y}{x}\right)^{\frac{1}{8}} \left(\frac{y}{x}\right)^{\frac{1}{8}} \\
&= \left(\frac{y}{x}\right)^{\frac{1}{4}} \\
&\le \left(\frac{y}{x}\right)^{\frac{2}{3}} \\
&= S(Tx,Tx,Ty)^{\frac{\gamma(S(Tx,Tx,Ty))}{3}}.
\end{aligned}$$

From both cases, we conclude that $S(fx,fx,fy) \leq S(Tx,Tx,Ty)^{\frac{\gamma(S(Tx,Tx,Ty))}{3}}$ for all $y,x \in X$. Therefore all the conditions of Theorem 4.1 are satisfied and f and T have a unique common fixed point, namely 2.

Corollary 4.1. *Suppose in a multiplicative complete S-metric space (X,S), there is a self map f on X and $\gamma \in \Gamma$ such that*

(iii) $S(fx,fx,fy) \leq S(x,x,y)^{\frac{\gamma(S(x,x,y))}{3}}$ *for all $x,y \in X$.*

Then f have a unique fixed point.

Proof. Follows from Theorem 4.1 by letting $T = I$, where I is the identity map on X. □

5. Application to Integral Equation

Consider an integral equation $u(t) = u_0 \int_{t_0}^{t} f(s,u(s))^{ds}$ for $t \in I$, where $f : I \times C^*(I) \to (0,\infty)$ is a multiplicative continuous function satisfying $\left[\left|\frac{f(t,x(t))}{f(t,x(t))}\right|^*\right]^2 \leq \left[\phi\left(\left|\frac{x(t)}{u(t)}\right|^*\right)^2\right]^{\lambda}$ with $u(t_0) = u_0$ for some constant $\lambda \geq 1$ and $\phi : [1,\infty) \to [1,\infty)$ is an increasing function such that $1 < \phi(r) < r$ for $r > 1$. Then given integral equation has a unique solution in some interval $|t - t_0| \leq k$ for sufficiently $k > 0$ with $k\lambda < \frac{1}{3}$. For this, we define $G : C^*(I) \to C^*(I)$ by $G(u(t)) = u_0 \int_1^t f(s,u(s))^{ds}$, $t \in I$. For $x,u \in C^*(I)$, we consider

$$
\begin{aligned}
S(Gx,Gx,Gu) &= \max_{t\in I}\left\{\left|\frac{G(x(t))}{G(u(t))}\right|^* \left|\frac{G(u(t))}{G(u(t))}\right|^*\right\} \\
&= \max_{t\in I}\left\{\left|\frac{\int_{t_0}^t f(s,x(s))^{ds}}{\int_{t_0}^t f(s,u(s))^{ds}}\right|^* \left|\frac{\int_{t_0}^t f(s,x(s))^{ds}}{\int_{t_0}^t f(s,u(s))^{ds}}\right|^*\right\} \\
&= \max_{t\in I}\left\{\left[\left|\int_{t_0}^t \left(\frac{f(s,x(s))}{f(s,u(s))}\right)^{ds}\right|^*\right]^2\right\} \\
&\leq \max_{t\in I}\left\{\int_{t_0}^t \left[\left|\frac{f(s,x(s))}{f(s,u(s))}\right|^* \left|\frac{f(s,x(s))}{f(s,u(s))}\right|^*\right]^{ds}\right\} \\
&\leq \max_{t\in I}\left\{\int_{t_0}^t \left[\left(\phi\left(\left|\frac{x(t)}{u(t)}\right|^*\right)^2\right)^{\lambda}\right]^{ds}\right\} \\
&= \max_{t\in I}\left\{\int_{t_0}^t \left[\phi(S(x,x,u))^{\lambda}\right]^{ds}\right\} \\
&\leq (\phi(S(x,x,u))^{k\lambda} \\
&\leq (\phi(S(x,x,u))^{\frac{1}{3}} \\
&\leq (\phi(S(x,x,u))^{\frac{\gamma(S(x,x,u))}{3}}.
\end{aligned}
$$

Thus $S(Gx,Gx,Gu) \leq (\phi(S(x,x,u))^{\frac{\gamma(S(x,x,u))}{3}}$ for all $x,u \in X$. Therefore all the conditions of the Corollary 4.1 are satisfied and G has a unique fixed point say $u \in C^*(I)$, infact unique solution of the integral equation.

6. Conclusion

In this paper, we introduced multiplicative S-metric space and studied its properties. We also obtained a common fixed point result with an application.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas, M. De La Sen and T. Nazir, Common fixed points of generalized rational type cocyclic mappings in multiplicative metric spaces, *Discrete Dynamic in Nature and Society* **2015** (2015), Article ID 532725, DOI: 10.1155/2015/532725.

[2] M. Abbas, B. Ali and Y.I. Suleiman, Common fixed points of locally contractive mappings in multiplicative metric spaces with application, *International Journal of Mathematics and Mathematical Sciences* **2015** (2015), Article ID 218683, DOI: 10.1155/2015/218683.

[3] A.E. Bashirov, E.M. Kurplnara and A. Ozyapici, Multiplicative calculus and its applications, *Journal of Mathematical Analysis and Applications* **337** (2008), 36 – 48, DOI: 10.1016/j.jmaa.2007.03.081.

[4] S. Czerwik, Contraction mappings in b-metric spaces, *Communications in Mathematics* **1** (1993), 5 – 11, `http://dml.cz/dmlcz/120469`.

[5] L. Florack and H.V. Assen, Multiplicative calculus in biomedical image analysis, *Journal of Mathematical Imaging and Vision* **42**(1) (2012), 64 – 75, DOI: 10.1007/s10851-011-0275-1.

[6] M. Grossman and R. Katz, *Non-Newtonian Calculus*, Pigeon Cove, Lee Press, Massachusats (1972).

[7] S.M. Kang, P. Kumar, S. Kumar, P. Nagpal and S.K. Garg, Common fixed points for compatible mappings and its variants in multiplicative metric spaces, *International Journal of Pure and Applied Mathematics* **42**(1) 102(2), (2015), 383 – 406, DOI: 10.12732/ijpam.v102i2.14.

[8] C. Mongkolkeha and W. Sintunavarat, Best proximity points for multiplicative proximal contraction mapping on multiplicative metric spaces, *Journal of Nonlinear Sciences and Applications* **8** (2015), 1134 – 1140, DOI: 10.22436/jnsa.008.06.22.

[9] H. Nakano, *Modular Semi-Ordered Linear Spaces*, Maruzen, Tokyo (1950).

[10] M. Ozavsar and A.C. Cevikel, Fixed point of multiplicative contraction mappings on multiplicative metric spaces, (2012), `URL:https://arxiv.org/abs/1205.5131`.

[11] N.Y. Ozgur and N. Tas, Some generalizations of fixed point theorems on S-metric spaces, in *Essays in Mathematics and Its Applications, in Honor of Vladimir Arnold*, pp. 229 – 261, New York, Springer (2016), DOI: 10.1007/978-3-319-31338-2_11.

[12] S. Sedghi, N. Shobe and H. Zhou, A common fixed point theorem in D^*-metric space, *Fixed Point Theory and Applications* **2007** (2007), Article number: 027906, DOI: 10.1155/2007/27906.

[13] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorem in S-metric spaces, *Matematički Vesnik* **64** (2012), 258 – 266, `https://emis.de/journals/MV/123/mv12309.pdf`.

[14] S. Sedghi, A. Gholidahneh, T. Dosenović, J. Esfahani and S. Radenovi, Common fixed point of four maps in S_b-metric spaces, *Journal of Linear and Topological Algebra* **5**(2) (2016), 93 – 104, `http://jlta.iauctb.ac.ir/article_523028_f5e63d18f0cde338db78527fb4edf152.pdf`.

[15] W.A. Wilson, On quasi-metric spaces, *American Journal of Mathematics* **53**(3) (1931), 675 – 684, DOI: 10.2307/2371174.

[16] O. Yamaod and W. Sintunavarat, Some fixed point results for generalized contraction mappings with cyclic (α, β)-admissible mappings in multiplicative metric space, *Journal of Inequalities and Applications* **2014** (2014), Article number: 488, DOI: 10.1186/1029-242X-2014-488.