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Around the Grauert and Remmert Theorem

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Abstract. In this paper we deal with Banach and topological algebras. It is essentially a survey concerning the Grauert and Remmert theorem, its generalizations and connected open problems. It contains also some new observations.

1. Introduction

The present paper is an extended version of the address "Ideals in topological algebras – open problems and recent results", delivered during the *First International Conference on Algebra, Topology and Topological Algebras*. It is concentrated around the Grauert and Remmert theorem (Theorem A below, see [9, Appendix to §5]), its various generalizations and connected, yet unsolved, problems.

All algebras considered in this paper are complex and unital topological algebras. The latter means that the multiplication there is jointly continuous. In the case of a separately continuous multiplication, we call the algebra in question a semi-topological algebra.

Theorem A. *Let A be a commutative complex unital Noetherian Banach algebra. Then A is finitely dimensional.*

The converse result is obviously true. Recall that an algebra A is called right Noetherian if any non-decreasing chain of right ideals is constant beginning of some place on, or, equivalently, if every right ideal I of A is (algebraically) finitely generated, i.e. there are elements $a_1, a_2, \dots, a_n$ in A such that

$$I = a_1A + \cdots + a_nA. \tag{1.1}$$

Similarly we define left or two-sided Noetherian algebras.

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The crucial result on which the proof of Theorem A is based is the following (see [9, Bemerkung 2, p. 51])

Theorem B. *Let A be a commutative complex unital Banach algebra, and let I be an ideal in A. Assume that the closure $\overline{I}$ of I is a finitely generated ideal. Then the ideal I is already closed.*

We sketch shortly the proof of Theorem B as given by Grauert and Remmert. By the assumption, $\overline{I}$ is of the form (1.1), so that

$$(x_1, \ldots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n$$

maps A^n onto $\overline{I}$. The open mapping theorem implies, that for any open ball $B_r = \{x \in A : \|x\| < r\}$, $r > 0$ the set $U_r = a_1B_r + \cdots + a_nB_r$ is an open neighbourhood of zero in $\overline{I}$. The density of I in $\overline{I}$ implies

$$I - U_r = \overline{I} \quad \text{for all positive } r. \tag{1.2}$$

Since the elements a_i are in $\overline{I}$, we can rewrite (1.2) as $u_k - \sum_{i=1}^{n} b_{k,i}a_i = a_k$, or

$$b_{1,k}a_1 + \cdots + (e + b_{k,k})a_k + \cdots + b_{n,k}a_n = u_k, \tag{1.3}$$

where e is the unity of A, u_k and $b_{i,k}$ are suitable elements respectively in I and A. We can treat (1.3) as a system of linear equations with given $b_{i,k}, u_k$. Using the Cramer formulas we can calculate elements a_k. The matrix M of this system is

$$M = \begin{pmatrix} e + b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & e + b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & e + b_{nn} \end{pmatrix} \tag{1.4}$$

The determinant D of this matrix is sum of the product

$$(e + b_{11})(e + b_{22}) \ldots (e + b_{nn}), \tag{1.5}$$

and other products of elements of the matrix M, each containing at least one factor of the form b_{ik}. For sufficiently small r the product (1.5) is invertible and other products are small, so that the determinant D is invertible in A. By the Cramer formula, we have $a_k = D_kD^{-1}$, where D_k is the determinant of the matrix M_k obtained from M by replacing there the k-th column by the column $u_1, u_2, \ldots, u_n$. Since the elements u_k are in I, each product in D_k contains a factor belonging to I. Thus D_k, and so a_k is in I for each k, which implies that $\overline{I} = I$. Consequently, in a Noetherian Banach algebra A all ideals are closed. A next reasoning gives $\dim A < \infty$, but it is not important for further generalizations. This proof can be also extended to a non-commutative case. We cannot use determinants, since they are not well defined, but we can solve equations (1.3) step by step. Such a procedure leads to some non-unique formulas showing that the elements a_k are in I. Some authors ([13], or [7, Proposition 2.6.37]) just write determinants, but it is is formally incorrect.

For instance, if we have equations

$$ax + by = u, \qquad cx + dy = v,$$

with given, non-commuting a, b, c, d. and invertible a and d in the algebra in question, then, for sufficiently small b and c, we can calculate x from the first equation, substitute it to the second, and then calculate y. In this way we obtain

$$x = a^{-1}\big(u - b(d - ca^{-1}b)^{-1}(v - ca^{-1}u)\big)$$

and

$$y = (d - ca^{-1}b)^{-1}(v - ca^{-1}u).$$

If we first calculate y and later x, we shall obtain different formulas, both hardly similar to the Cramer formulas. Nevertheless, these formulas show that x and y are in the left ideal containing u and v. Note that the similar procedure does not work in the case when I is a two-sided ideal.

In the above proof we used two facts. The first is that the open mapping theorem works for considered algebras. The second is that the group of invertible elements is open in these algebras. The open mapping theorem works for complete metrizable topological vector spaces (F-spaces), see ([12, Theorem 2.3.1]), so we can obtain analogous results to Theorem B, and so to Theorem A, if we can show that considered Noetherian F-algebras must be Q-algebras, i.e. topological algebras with open group of invertible elements. As we shall see, for some non-Banach Noetherian topological algebras the conclusion states that all ideals there are closed, but the algebras do not need to be finitely dimensional.

It is easy to observe that Theorems A and B fail to be true for normed algebras, so the assumption of completeness here is essential.

We close this section by showing that an F-algebra with all ideals closed must be Noetherian. So in order to show that in an F-algebra all ideals are closed if and only if it is Noetherian, only the "only if" part has to be proved.

Proposition 1.1. *Suppose that A is a unital F-algebra with respectively all left, right, or two-sided ideals closed, then A is left, right, or two-sided Noetherian.*

Proof. If not, then there is an infinite sequence of ideals

$$I_1 \subset I_2 \subset \ldots, I_k \neq I_{k+1},$$

Then $J = \bigcup_{k=1}^{\infty} I_k$ is also an ideal (of the same type) which is non-closed as an infinite union ow nowhere dense sets. The contradiction proves our assertion. □

2. Further results and open problems on Banach algebras

In this section we describe further generalizations or improvements of Theorems A and B within the class of Banach algebras, as well as some related open problems.

The first generalization of Theorems A and B within the Banach algebras theory is due to Sinclair and Tullo [13] (see also [7, Theorem 2.6.39]). They extended Theorem A to the non-commutative case. Their result reads as follows.

Theorem 2.1. *Let A be a non-commutative Banach algebra which is left* (*right*) *Noetherian. Then* $\dim A < \infty$.

The proof of the above result is based upon the left (right) version of Theorem B and upon a separate argument showing that the closeness of all ideals implies finite dimensionality of the algebra in question.

It is interesting to note that there is no two-sided version of above result. So we pose

Problem I. Let A be a two-sided Noetherian unital Banach algebra. Does it follow that all its two-sided ideals are closed?

This problem makes a sense also for more general topological algebras, but, perhaps, it should be first solved for the Banach algebra case.

We cannot expect that a two-sided Noetherian Banach algebra is finitely dimensional. Consider the algebra $A = L(H)$ of all bounded operators on a separable Hilbert space. It is well known (see [5]) that A has the unique maximal two-sided ideal M – it consists of all compact operators. Thus the quotient algebra A/M (the Calkin algebra) is a simple infinite dimensional Banach algebra, i.e. its only two-sided ideal is 0. Consequently it is two-sided Noetherian (with all two-sided ideals closed), but the conclusion of Theorem A fails to be true.

Another generalization of Theorem A is the following result, due to Ferreira and Tomassini ([8]), which shows that instead of assuming that all ideals of the algebra in question are finitely generated, it is sufficient to assume that it is so only for maximal ideals (in fact for maximal ideals in the Shilov boundary).

Theorem 2.2. *Let A be a commutative complex unital infinite dimensional Banach algebra. Then A has a maximal ideal which is not* (*algebraically*) *finitely generated.*

Note that Theorem 2.2, as formulated above, does not extend to more general topological algebras. Suitable examples will be given in the next section. However, its another formulation would make a sense (see Problems IV and V).

Theorem 2.1 shows that the left and right versions of Theorem A hold true, and that the left Noetherian Banach algebra is automatically right Noetherian. In the next section (Proposition 3.4) we shall see that the latter statement fails for more general topological algebras. We do not know whether Theorem 2.2 can be extended to a non-commutative case. So we pose the following

Problem II. Is it true the left (right) version of the Theorem 2.2?

Theorem B implies that if a closed ideal I of a commutative unital Banach algebra has a dense subideal, then I is not finitely generated. A partial converse result (see [22]) is given in the following

Proposition 2.3. *Let A be a commutative complex unital Banach algebra, and suppose that a closed ideal I in A is not finitely generated. Then I has a dense subideal provided I is separable.*

Problem III. Is Proposition 2.3 true without the separability assumption?[1]

The following result formulated in [22] for Banach algebras, is true also in a much more general situation. It could be helpful in solving the Problem III, and similar, more general problems. Let A be a unital semi-topological algebra, i.e. an algebra which is a topological vector space and the product of two elements is separately continuous. Let I be a closed ideal in A. We say that a subset $S \subset I$ generates I topologically, if I is the smallest closed ideal containing S, i.e. I is the closure of the ideal algebraically generated by S.

Proposition 2.4. *Let A be a unital semitopological algebra and let I be a closed ideal in A. Then I has no dense proper subideal if and only if every set S which generates I topologically generates it also algebraically.*

Proof. If S generates I topologically, but not algebraically, then the ideal J generated by S algebraically is different from I but it is dense in it. On the other hand, if I has a proper dense sub-ideal J, then $S = J$ generates I topologically, but not algebraically. The conclusion follows. □

3. Further results and open problems in F-algebras

A topological vector space is called an F-space, if it is complete and metrizable. Several results true for Banach spaces are also true for F-spaces, for instance the Open Mapping Theorem (see e.g. [12, Theorem 2.3.1]). An F-algebra is a topological algebra which is an F-space. A B_0-algebra A is a locally convex F-algebra. Its topology can be given by means of a sequence

$$\|x\|_1 \le \|x\|_2 \le \dots \tag{3.1}$$

of seminorms, such that

$$\|xy\|_k \le \|x\|_{k+1}\|y\|_{k+1} \quad \text{for all } x, y \in A \text{ and all natural } k, \tag{3.2}$$

and $\|e\|_k = 1$ for all k, where e is the unity of A. A B_0-algebra A is said multiplicatively convex (shortly: m-convex) if the inequalities (3.2) can be replaced by

$$\|xy\|_k \le \|x\|_k \, \|y\|_k \tag{3.3}$$

for all x, y in A. For the properties of algebras of type F or B_0 the reader is referred to [7], [10], [11], [15] and [16].

First we show that the Theorems A and 2.2 fail to be true for the above non-Banach algebras (as we see later, they remain true if we replace there "finite dimensional" by "all ideals closed".

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[1] In [23] the author solved this problem in negative (added in proof).

Example 3.1. Consider the locally convex algebra $A = (s)$ of all formal power series

$$x = \sum_{k=0}^{\infty} a_k(x)t^k,$$

provided with the Cauchy multiplication and the topology given be the increasing sequence of seminorms

$$\|x\|_n = \sum_{k=0}^{n-1} |a_k(x)|. \tag{3.4}$$

It is well known (see e.g. [8], [10], [11] or [16]), that A is an m-convex B_0-algebra. Again, it is well known (see e.g. [15]) that every ideal I of A is of the form

$$I_n = t^n A,$$

so that A is Noetherian, all its ideals are closed, but $\dim A = \infty$

The above example is rather simple. It has only one maximal ideal consisting of all elements s with $\alpha_0(x) = 0$, so it is a unitization of a radical algebra. A much more involved example is given by Carboni and Larotonda [3]. It is an m-convex B_0-algebra and a Q-algebra consisting of complex functions continuous in the closed unit disc of the complex plane **C** and holomorphic in its interior (it is a subalgebra of the disc algebra) with the property that all its ideals are principal (it is a principal ideal domain) and it is a semisimple algebra.

Ferreira and Tomassini ([8], Theorem 2.6) observed first that commutative m-convex Noetherian B_0-algebras have all ideals closed. This result was extended to commutative F-algebras [5], and independently by the author [18], the result can be formulated as follows.

Theorem 3.2. *Let A be a commutative unital F-algebra. Then A is Noetherian if and only if all its ideals are closed.*[2]

As we have seen, Theorem 2.2 fails for F-algebras, however its weaker version could be true, so we pose the following Problem.

Problem IV. Let A be a commutative unital F-algebra with all maximal ideals (algebraically) finitely generated. Does it follow that all its ideals are closed?

A theorem of Arens [1], states that in a commutative m-convex algebra every finitely generated ideal is non-dense. This result implies that in an m-convex B_0-algebra A satisfying assumptions of the Problem IV, all its maximal ideals must be closed. Thus, by [20], A is a Q-algebra, i.e. the goup of its invertible elements is open. However, the Arens theorem fails for non-m-convex B_0-algebras since these algebras may possess singly generated dense ideals. In view of the theorem of

[2] The author observed that all ideals in the example of Carboni and Laroronda are closed and it led him to the formulation and proof of the Theorem 3.2.

Arens, perhaps attacking the following particular case of the Problem IV would be more easy.

Problem V. Let A be a commutative unital m-convex B_0-algebra with all maximal ideals finitely generated. Does it follow that all ideals of A are closed?

Using the left version of the above Arens theorem Choukri, El Kinani and Oudadess [6] obtained non-commutative version of Theorem 3.2 in the case of an m-convex B_0-algebra.

Theorem 3.3. *Let A be a unital m-convex B_0-algebra. Then A has all left (right) ideals closed if and only if it is left (right) Noetherian.*

Unlike as for the Sinclair-Tullo theorem (Theorem 2.1), it is possible to have a left Noetherian m-convex B_0-algebra which is not right Noetherian. The following result was obtained in [20].

Proposition 3.4. *There exists an m-convex B_0-algebra with all left, but not all right ideals closed.*

In general, the extension of Theorem 3.3 to non-commutative F-algebras, or even to non-commutative B_0-algebras is not known. So we pose the following

Problem VI. Let A be an F-algebra (resp. a B_0-algebras). Is it true that all its left ideals are closed if and only if it is left Noetherian?

A weaker version of this problem has been solved in [19].

Theorem 3.5. *Let A be a unital F-algebra. Then all its one-sided ideals are closed if and only if A is both left and right Noetherian.*

In view of Proposition 3.4 this result does not solve Problem VI.

The proofs of Theorems 3.2 and 3.5 given in [18] and [19] depend upon the following version of Theorem B. This result was stated in [7, Proposition 2.6.37].

Lemma 3.6. *Let A be an F-algebra which is also a Q-algebra. Let I be a left ideal in A, whose closure is finitely generated. Then I is already closed.*

However, without assumption that A is a Q-algebra, the Lemma 3.6 fails to be true even in the case of a commutative m-convex B_0-algebra. Thus Theorem B cannot be extended to F-algebras, or even to the m-convex B_0-algebras, as shows following example.

Example 3.7. Denote by $\mathscr{E}$ the algebra of all entire functions of one complex variable provided with the compact-open topology. This topology is given by means of the sequence of seminorms

$$\|x\|_k = \max_{|\zeta|\le k} |x(\zeta)|, \quad k = 1, 2, \ldots$$

making of $\mathscr{E}$ an m-convex B_0-algebra. Since $\mathscr{E}$ is not a Q-algebra, it has a dense maximal ideal M_∞ (see e.g. [21]). It is well known that every closed maximal ideal of $\mathscr{E}$ is of the form

$$M_{\zeta_0} = (z(\zeta) - \zeta_0 e)\mathscr{E},$$

where $z(\zeta) = \zeta$ and e (the identity of $\mathscr{E}$) is the constant entire function equal to one. Thus each closed maximal ideal in $\mathscr{E}$ is singly generated. Consider now the ideal

$$J = M_0 \cap M_\infty.$$

It is different from M_0 and contains the product $M_0 M_\infty$. Since M_∞ is dense in $\mathscr{E}$, it contains a sequence (x_k) with $\lim_k x_k = e$. For an element y in M_0 the product $x_k y$ is in J and $\lim_k x_k y = y$, so that the closure of J is M_0. It is finitely generated, but different from J.

Observe now that the Lemma 3.6 and Proposition 1.1 imply immediately the following proposition.

Proposition 3.8. *Let A be a unital F-algebra. Then all left ideals of A are closed if and only if A is a Noetherian Q-algebra.*

If we could omit in the above the condition that A is a Q-algebra, we would have solution of the Problem VI.

In the case of a commutative F-algebra, or in the case of all one-sided ideals in an F-algebra, the reasonings showing that considered Noetherian algebras must be Q-algebras (an important step towards obtaining Theorems 3.2 and 3.5) was connected with a concept of a topologically invertible, or left topologically invertible element. An element x of an F-algebra A is said *properly left invertible*, if it is not left invertible and there is a sequence $(y_k) \subset A$ with $\lim_k y_j x = e$. Clearly, if x is such an element, the the left ideal Ax is dense. So it has to be shown also that the left-Noetherian algebra in question has no topologically left invertible elements. By showing that, it was possible to show also that the algebra in question is a Q-algebra. We have therefore

Proposition 3.9. *Let A be a unital F-algebra. Then A has all left ideals closed if and only if it is Noetherian and has no proper topologically left invertible element.*

The Propositions 3.8 and 3.9 give a sort of weak solution of the Problem VI. They were explicitly stated in [6].

4. General topological algebras and final remarks

In this section we shall comment on the possibility of extending the results of previous sections to non-metrizable topological algebras. First we discuss the commutative examples.

Example 4.1. Neither the Theorem 3.2, nor even each of the contained there two implications can be extended to non-metrizable complete locally convex algebras. Let A_1 be the unitization of the commutative free algebra in countably many variables $z_1, z_2 \ldots,$ provided with the maximal locally convex topology. This topology is given by means of all seminorms on A and makes of A a complete locally convex algebra (the multiplication there is jointly continuous, for details see [17]).

Under the maximal locally convex topology all linear subspaces, and so all ideals are closed. On the other hand, A is not a Noetherian algebra. To see it, consider the ideal I in A consisting of (finite) sums of monomials, each of them containg a factor of the form $z_i z_j$ for natural i and j. Since each element a of A can contain only finitely many variables z_i, any finite n-tuple $a_1, \ldots, a_k$ can contain only finitely many variables z_i. If z_j is not one of these variables, the element z_j^2 is in I, but not in the ideal generated by $a_1, \ldots, a_k$.

On the other hand there is a complete Noetherian locally convex algebra A_2 possessing proper dense ideals. For instance, let A be the algebra of all complex polynomials with one of topologies constructed in [14]. It was shown in [2], that A is a proper TQ-algebra, i.e. a topological algebra which has open set of topologically invertible elements but is not a Q-algebra. Thus A has proper dense (singly generated) ideals. More specifically: every multiplicative linear functional f on A is of the form $f_\zeta(x) = x(\zeta)$, where ζ is a fixed complex number. In the algebra in question, all functionals f_ζ are continuous for $\|\zeta\| \leq 1$ and discontinuous for $\|\zeta| > 1$. In the latter case their kernels are non-closed and dense. Our claim follows. The example A_2 shows also that the answer to Problem IV is in negative for complete non-metrizable locally convex algebras.

We give now some non-commutative examples (modifications of A_1 and A_2).

Example 4.2. Denote by A_3 the free algebra in countably many variables z_i (the non-commutative version A_1). It is a complete locally convex topological algebra. The two-sided ideal I generated by all products $z_i z_j$ is neither two-sided, nor left or right Noetherian, but it is closed as a vector subspace of A_3. It is also not a Q-algebra. This example shows that Proposition 1.1, as well as Propositions 3.8 and 3.9 fail to be true for non-metrizable complete locally convex algebras.

The second example is the Cartesian product A_4 of A_2 with the algebra M_n of all complex $n \times n$, matrices $(n > 1)$, provided with the product topology. It is a complete non-metrizable locally convex algebra which is Noetherian, but has non-closed left, right and two-sided ideals. This example shows that Problem VI has a negative solution for non-metrizable algebras and the Theorem 3.5 fails to be true there.

However the following two Problems (implications taken from propositions 3.8 and 3.9) seem to be open even in the commutative case. Here by an ideal we shall mean either left or right, or two-sided ideal, the latter case makes a sense only in Problem VI. Similarly, by a proper topological divisor of zero we mean, in the non-commutative case a left (right) divisor if there is involved a left (right) ideal.

Problem VII. Let A be a complete Noetherian topological Q-algebra. Does it follow that A has all ideals closed?

Problem VIII. Let A be a complete Noetherian topological algebra without proper topologically invertible elements. Does it follow that A has all ideals closed?

The proofs that the considered in this paper results fail to be true for incomplete algebras are left to the reader.

We close this paper with an example which is not directly connected with our topic. However the result, which was a surprise for the author, was obtained during the conference and was included into the author's talk. It says that in an m-convex B_0-algebra it is possible to have a minimal (in fact, the smallest) dense ideal.

Example 4.3. Let $A = C[0,\infty)$ be the algebra of all continuous functions on the closed half-line $[0,\infty)$ with pointwise algebra operations, provided with the compact-open topology. This topology can be given by means of the seminorms

$$\|x\|_k = \max_{0\leq t\leq k}(|x(t)|).$$

Clearly A is an m-convex B_0-algebra. Denote by I the ideal of all compactly supported elements in A. Clearly it is a dense ideal. We claim that I has the following properties

(i) The ideal I is not finitely generated,

(ii) I is the intersection of all dense ideals in A.

Consequently I is the smallest dense ideal in A.

The Property (i) follows immediately from the Arens theorem [1] stating that in a commutative m-convex algebra every finitely generated ideal is non-dense. We have only to show (ii), i.e. to prove that if J is a dense ideal in A, then $I \subseteq J$. To this end consider an interval $[0,m]$, $m \in \mathbf{N}$. Observe first that there is an element x_m in J which does not vanish on $[0,m]$, otherwise, there would be a (fixed) t_0 in $[0,m]$ with $x(t_0) = 0$ for all x in J, and J would not be a dense ideal. Thus we can find a y_m in A with $x_m y_m = 1$ for $0 \leq t \leq m$. Consequently every element x supported by $[0,m]$ can be written as $x = x_m y_m x$, which belongs to J. Thus all compactly supported element of A are in J. The conclusion follows.

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References

[1] R.F. Arens, Dense inverse limit rings, *Michigan Math. J.* **5** (1958), 169–182.

[2] H. Arizmendi, A. Carillo and L. Palacios, On Q_t-algebras, in *Topological Algebras and Applications*, Contemporary Mathematics, **427** (2007), 49–55.
[3] J.W. Calkin, Two-sided ideals and congruence in the ring of bounded operators in Hilbert space, *Ann. of Math.* **42** (1941), 839–873.
[4] G. Carboni and A. Larotonda, An example of Fréchet algebra which is a principal ideal domain, *Studia Math.* **138** (2000), 265–275.
[5] R. Choukri and A. El Kinani, Topological algebras with ascending and descending chain condition, *Arch. Math.* (*Basel*) **72** (1999), 438–443.
[6] R. Choukri, A. El Kinani et M, Oudadess, Algèbres topologiques à idéaux à gauche fermés, *Studia Math.* **168** (2005), 159–164.
[7] H.G. Dales, Banach algebras and automatic continuity, *London Math. Soc. Monographs* **24**, Oxford University Press, 2000.
[8] A.V. Ferreira and G. Tomassini, Finiteness properties of topological algebras, *Ann. Scuola Norm. Sup. Pisa* **5** (1978), 471–488.
[9] H. Grauert and R. Remmert, *Analytische Stellenalgebren*, Springer-Verlag, Berlin — Heidelberg — New York, 1971.
[10] A. Mallios, *Topological Algebras. Selected Topics*, North Holland, 1986.
[11] E. Michael, Locally multiplicatively-convex topological algebras, *Mem. Amer. Math. Soc.* **11**, 1952.
[12] S. Rolewicz, *Metric Linear Spaces*, PWN Warszawa, 1972.
[13] A.M. Sinclair and A.W. Tullo, Noetherian Banach algebras are finite dimensional, *Math. Ann.* **211** (1974), 151–153.
[14] M. Wojciechowski and W. Żelazko, Non-uniqueness of topology for algebras of polynomials, *Coll. Math.* **72** (1997), 111–121.
[15] W. Żelazko, Metric generalizations of Banach algebras, *Dissertationes Math.* (*Rozprawy Mat.*), 1965.
[16] W. Żelazko, Selected topics in topological algebras, *Aahrus Univ. Lecture Notes* Series No. **31**, 1971.
[17] W. Żelazko, On topologization of countably generated algebras, *Studia Math.* **112** (1994), 83–88.
[18] W. Żelazko, A characterization of commutative Fréchet algebras with all ideals closed, *Studia Math.* **138** (2000), 293–300.
[19] W. Żelazko, A characterization of F-algebras with all one-sided ideals closed, *ibidem* **168** (2005), 135–145.
[20] W. Żelazko, An m-convex B_0-algebra with all left but not all right ideals closed, *Coll. Math.* **104** (2006), 317–324.
[21] W. Żelazko, When a unital F-algebra has all maximal left (right) ideals closed?, *Studia Math.* **175** (2006), 279–284.
[22] W. Żelazko, When a closed ideal of a commutative Banach algebra has a dense subideal, *Funct. Approx. Comm. Math.* **44** (2011), 285–287.
[23] W. Żelazko, Concerning dense subideals in commutative Banach algebras, *submitted*.

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