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Research Article

Special Fitted Finite Difference Scheme for Delay Differential Equation With Dual Boundary Layers

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Abstract. In this paper, we have proposed a fitted special finite difference method for the solution of delay differential equations with dual boundary layers. The delay differential equation is replaced by an asymptotically equivalent singular perturbation problem using the Taylor's series expansion. Then, a fitted special finite difference scheme is described to get accurate solution to the problem. The method is demonstrated by implementing on several model problems by taking various values for the delay parameter δ and perturbation parameter ε. To show the effect of delay on the boundary layer or oscillatory behaviour of the solution, several numerical problems are carried out in this article. To demonstrate the effect on the layer behaviour, the solution of the problems are shown graphically. We observed that when the order of the coefficient of the delay parameter is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases.

Keywords. Delay differential equations; Boundary layers; Fitted finite differences

Mathematics Subject Classification (2020). 65L11; 65Q10

1. Introduction

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term. In recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical

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modeling of processes in various application fields, for e.g., the first exit time problem in the modeling of the activation of neuronal variability Stein [12], in the study of bistable devices Derstin et al. [2], and variational problems in control theory Glizer [5] where they provide the best and in many cases the only realistic simulation of the observed. Stein [11] gave a differential-difference equation model incorporating stochastic effects due to neuron excitation. Lange and Miura [9, 10] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations. Kadalbajoo and Sharma [6–8], presented a numerical approaches to solve singularly perturbed differential-difference equation, which contains negative shift in the either in the derivative term or the function but not in the derivative term. Analytical discussion on these problems ia available in the books O'Malley [11] and Driver [3].

In this paper, we have proposed a fitted special finite difference method for the solution of delay differential equations with dual boundary layers. The delay differential equation is replaced by an asymptotically equivalent singular perturbation problem using the Taylor's series expansion. Then, a fitted special finite difference scheme is described to get accurate solution to the problem. The method is demonstrated by implementing on several model problems by taking various values for the delay parameter δ and perturbation parameterε. To show the effect of delay on the boundary layer or oscillatory behaviour of the solution, several numerical problems are carried out in this article. To demonstrate the effect on the layer behaviour, the solution of the problems are shown graphically. We observed that when the order of the coefficient of the delay parameter is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases.

2. Description of the Method

Consider the problem:

$$\varepsilon w''(s) + a(s)w(s-\delta) + b(s)w(s) = 0, \quad \text{for } \ 0 < s < 1, \tag{2.1}$$

and subject to the conditions

$$w(s) = \alpha, \ \ -\delta \le s \le 0 \tag{2.2}$$

and

$$w(1) = \beta, \tag{2.3}$$

where $0 < \epsilon \ll 1$, $a(s)$, $b(s)$ and $f(s)$ are bounded continuous function in $(0,1)$ and α, β are finite constants. For $\delta = 0$, the solution of the boundary value problem eqn. (2.1)-eqn. (2.3) exhibits layer or oscillatory behavior depending on the sign of $(a(s)+b(s))$. Further, we assume $(a(s)+b(s)) \le -N < 0$, where N is positive constant, the solution of the problem eqns. (2.1)-(2.3) exhibits layer behavior and $(a(s)+b(s)) \ge N > 0$, it exhibits oscillatory behaviour. These assumptions imply that the boundary layer will be at both end points. i.e., at $s = 0$ and $s = 1$.

By using Taylor series we have

$$w(s-\delta) \approx w(s) - \delta w'(s) \tag{2.4}$$

Substituting eqn. (2.4) in eqn. (2.1), we get a second order differential equation of the form:

$$\varepsilon w''(s)+p(s)w'(s)+q(s)w(s)=0, \quad s\in[0,1] \tag{2.5}$$

where $p(s)=-\delta a(s)$ and $q(s)=a(s)+b(s)$.

This transformation from eqn. (2.1) to eqn. (2.5) is permitted, because of the condition that $0<\delta\ll 1$ and $0<\eta\ll 1$ are sufficiently small Elsgolts and Norkin [4]. Therefore, the solution of eqn. (2.5) will provide a good approximation to the solution of eqn. (2.1).

Now, divide the interval $[0,1]$ into N equal parts with constant mesh length h. Let $0=s_0<s_1<\ldots<s_n=1$ be the mesh points such that $s_i=ih$ for $i=0,1,\ldots,N$.

Since the problem exhibits two boundary layers across the interval, we divide the interval $[0,1]$ into two sub intervals, namely: $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$. We choose n such that $s_n=\frac{1}{2}$. Clearly in the interval $\left[0,\frac{1}{2}\right]$ the boundary layer will be at the left end i.e., at $s=0$ and in the interval $\left[\frac{1}{2},1\right]$ the boundary layer will be at the right end i.e., at $s=1$.

2.1 Problem with Left-end Boundary Layer in $\left[0,\frac{1}{2}\right]$

Now we describe the numerical scheme in $\left[0,\frac{1}{2}\right]$ where the boundary layer is at the left end i.e., at $s=0$.

At $s=s_i$ the eqn. (2.5) can be written as

$$\varepsilon w''(s_i)+p(s_i)w'(s_i)+q(s_i)w(s_i)=0\,.$$

We describe the the special second difference scheme for the eqn. (2.5) as follows:

$$\varepsilon\left(\frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\right)+p_i\left(\frac{w_{i+1}-w_i}{h}-\frac{h}{2}w_i''\right)+q_i\left(\frac{w_{i+1}+w_{i-1}}{2}\right)=f_i, \quad \text{for } i=1,2,\ldots,n. \tag{2.6}$$

Now, we introduce a fitting factor σ in the above scheme, eqn. (2.6) as follows:

$$\varepsilon\sigma\left(\frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\right)+p_i\left(\frac{w_{i+1}-w_i}{h}-\frac{h}{2}w_i''\right)+q_i\left(\frac{w_{i+1}+w_{i-1}}{2}\right)=f_i \tag{2.7}$$

$$\varepsilon\sigma\left(\frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\right)+p_i\left(\frac{w_{i+1}-w_i}{h}-\frac{h}{2}\left(\frac{f_i-p_iw_i'-q_iw_i}{\varepsilon}\right)\right)+q_i\left(\frac{w_{i+1}+w_{i-1}}{2}\right)=f_i\,,$$

$$\varepsilon\sigma\left(\frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\right)+p_i\left(\frac{w_{i+1}-w_i}{h}\right)+\frac{h}{2\varepsilon}p_ip_{i+\frac{1}{2}}w_i'+\frac{h}{2\varepsilon}p_iq_{i+\frac{1}{2}}w_i+q_i\left(\frac{w_{i+1}+w_{i-1}}{2}\right)$$
$$=f_i+\frac{h}{2\varepsilon}p_if_{i+\frac{1}{2}}\,. \tag{2.8}$$

Substituting

$$w_i'=\frac{w_{i+1}-w_i}{h}$$

in the above eqn. (2.8) and simplifying, we get

$$\left(\frac{\varepsilon\sigma}{h^2}+\frac{q_i}{2}\right)w_{i-1}-\left(\frac{2\varepsilon\sigma}{h^2}+\frac{p_i}{h}+\frac{p_ip_{i+\frac{1}{2}}}{2\varepsilon}-\frac{h}{2\varepsilon}p_iq_{i+\frac{1}{2}}\right)w_i+\left(\frac{\varepsilon\sigma}{h^2}+\frac{p_i}{h}+\frac{p_ip_{i+\frac{1}{2}}}{2\varepsilon}+\frac{q_i}{2}\right)w_{i+1}$$
$$=f_i+\frac{h}{2\varepsilon}p_if_{i+\frac{1}{2}}\,. \tag{2.9}$$

Using the theory of the singular perturbations, we know that the asymptotic solution of the eqn. (2.1)-(2.3) is of the form (O'Malley [11]):

$$w(s)=w_0(s)+\frac{p(0)}{p(s)}(\alpha-w_0(0))e^{-\frac{p(0)}{\varepsilon}s}\,.$$

Multiply eqn. (2.9) by h and taking as $h \to 0$, we get the fitting factor as:

$$\sigma = \frac{\rho\left(p(0) + \frac{\rho}{2}p^2(0)\right)\left(1 - e^{-\frac{p(0)}{\varepsilon}}\right)}{e^{\frac{p(0)}{\varepsilon}} - 2 + e^{-\frac{p(0)}{\varepsilon}}}.$$

Arranging eqn. (2.9) in three term recurrence relation, we have

$$E_i w_{i-1} - F_i w_i + G_i w_{i+1} = H_i, \quad \text{for } i = 1, 2, \dots, n, \tag{2.10}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2},$$
$$F_i = \frac{2\varepsilon\sigma}{h^2} + \frac{p_i}{h} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon} - \frac{h}{2\varepsilon} p_i q_{i+\frac{1}{2}},$$
$$G_i = \frac{\varepsilon\sigma}{h^2} + \frac{p_i}{h} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon} + \frac{q_i}{2},$$
$$H_i = f_i + \frac{h}{2\varepsilon} p_i f_{i+\frac{1}{2}}.$$

We solve eqn. (2.10) by using the Thomas Algorithm given in Angel and Bellman [1].

2.2 Problem with Right-end Boundary Layer in $[\frac{1}{2}, 1]$

Now we describe the numerical scheme in $[\frac{1}{2}, 1]$ where the boundary layer is at the right end i.e., at $s = 1$.

At $s = s_i$ the eqn. (2.5) can be written as

$$\varepsilon w''(s_i) + p(s_i) w'(s_i) + q(s_i) w(s_i) = 0.$$

We extended the special second difference scheme for the eqn. (2.5) as follows:

$$\varepsilon\left(\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) + p_i\left(\frac{w_i - w_{i-1}}{h} + \frac{h}{2} w''_{ii}\right) + q_i\left(\frac{w_{i+1} + w_{i-1}}{2}\right) = f_i, \quad \text{for } i = n+1, n+2, \dots, N-1. \tag{2.11}$$

Now, we introduce a fitting factor σ in this scheme, eqn. (2.11) as follows:

$$\varepsilon\sigma\left(\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) + p_i\left(\frac{w_i - w_{i-1}}{h} + \frac{h}{2} w''_i\right) + q_i\left(\frac{w_{i+1} + w_{i-1}}{2}\right) = f_i, \tag{2.12}$$

$$\varepsilon\sigma\left(\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) + p_i\left(\frac{w_i - w_{i-1}}{h} + \frac{h}{2}\left(\frac{f_i - p_i w'_{ii} - q_i w_i}{\varepsilon}\right)\right) + q_i\left(\frac{w_{i+1} + w_{i-1}}{2}\right) = f_i, \tag{2.13}$$

$$\varepsilon\sigma\left(\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) + p_i\left(\frac{w_i - w_{i-1}}{h}\right) - \frac{h}{2\varepsilon} p_i p_{i-\frac{1}{2}} w'_{ii} - \frac{h}{2\varepsilon} p_i q_{i-\frac{1}{2}} w_i + q_i\left(\frac{w_{i+1} + w_{i-1}}{2}\right) = f_i + \frac{h}{2\varepsilon} p_i f_{i-\frac{1}{2}}. \tag{2.14}$$

Substituting

$$w'_{ii} = \frac{w_i - w_{i-1}}{h}$$

in the above eqn. (2.14) and simplifying, we get

$$\left(\frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2} - \frac{p_i}{h} + \frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon}\right) w_{i-1} - \left(\frac{2\varepsilon\sigma}{h^2} - \frac{p_i}{h} + \frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon} - \frac{h}{2\varepsilon} p_i q_{i-\frac{1}{2}}\right) w_i + \left(\frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2}\right) w_{i+1} = f_i - \frac{h}{2\varepsilon} p_i f_{i-\frac{1}{2}}. \tag{2.15}$$

Again we have from the theory of the singular perturbations, the solution of eqn. (2.1)-(2.3) is of the form (O'Malley [11])

$$w(s) = w_0(s) + \frac{p(1)}{p(s)}(\alpha - w_0(0))e^{-\left(1-\frac{p(1)}{\varepsilon}\right)s}.$$

Multiply eqn. (2.15) by h and taking as $h \to 0$, we get the fitting factor as:

$$\sigma = \frac{\rho\left(p(1) - \frac{\rho}{2}p^2(1)\right)\left(e^{-\frac{p(1)}{\varepsilon}} - 1\right)}{e^{\frac{p(1)}{\varepsilon}} - 2 + e^{-\frac{p(1)}{\varepsilon}}}.$$

The three term recurrence relation of eqn. (2.15) is given by

$$E_i w_{i-1} - F_i w_i + G_i w_{i+1} = H_i, \quad \text{for } i = n+1, n+2, \ldots, N+1, \tag{2.16}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2} - \frac{p_i}{h} + \frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon},$$

$$F_i = \frac{2\varepsilon\sigma}{h^2} - \frac{p_i}{h} + \frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon} + \frac{h}{2\varepsilon} p_i q_{i-\frac{1}{2}},$$

$$G_i = \frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2},$$

$$H_i = f_i - \frac{h}{2\varepsilon} p_i f_{i-\frac{1}{2}}.$$

The system of eqn. (2.16) is solved by the Thomas Algorithm given in Angel and Bellman [1].

3. Convergence Analysis

Now we consider the convergence analysis of left end boundary layer described in Section 2. Incorporating the boundary conditions we obtain the system of equations in the matrix form as

$$AW + Q + T(h) = 0. \tag{3.1}$$

In which $A = (a_{i,j})$, $1 \le i, j \le N-1$ is a tridiagonal matrix of order $N-1$, where $a_{i,j}$ are the values of E_i, F_i and G_i in eqn. (2.10) and eqn. (2.16) of left and right end boundary layer respectively with truncation error $T(h) = kh^4 + O(h^5)$, where $k = \left(\frac{\varepsilon\sigma}{12} w^{iv} + \frac{a\delta}{12} w''' + \frac{(a+b)}{2} w'' + \frac{a\delta(a+b)}{4\varepsilon} w'\right)$.

Let $W = [W_1, W_2, W_3, \ldots, W_{N-1}]^T, T(h) = [T_1, T_2, \ldots, T_{N-1}]^T O = [0, 0, \ldots, 0]^T$ are associated vectors of eqn. (3.1). Let $w = [w_1, w_2, \ldots, w_{N-1}]^T \cong W$ which satisfies the equation

$$Aw + Q = 0. \tag{3.2}$$

Let $e_i = w_i - W_i$, $i = 1, 2, \ldots, N-1$ be the discretization error so that $E = [e_1, e_2, \ldots, e_{N-1}]^T = w - W$. Subtracting eqn. (3.1) from eqn. (3.2), we obtain the error equation

$$AE = T(h). \tag{3.3}$$

Let $\bar{S}_i$ be the sum of the elements of the ith row of the matrix A, then we have

$$S_i = -\varepsilon\sigma + \frac{h^2 q_i}{2} + \frac{h^3 p_i q_{i+1/2}}{2\varepsilon}, \quad \text{for } i = 1,$$

$$S_i = h^2 q_i + \frac{h^3 p_i q_{i+1/2}}{2\varepsilon}, \quad \text{for } i = 2, 3, \ldots, N-2,$$

$$S_i = -\varepsilon\sigma - hp_i + h^2\left(\frac{q_i}{2} - \frac{p_i p_{i+1/2}}{2\varepsilon}\right) + \frac{h^3 p_i q_{i+1/2}}{2\varepsilon}, \quad \text{for } i = N-1.$$

We can choose h sufficiently small so that the matrix A is irreducible and monotone.

It follows that A^{-1} exists. Hence from (3.3), we have

$$\|E\| \leq \left\|(A)^{-1}\right\| \|T\| . \tag{3.4}$$

Also from theory of matrices we have

$$\sum_{k=1}^{N-1} (A)^{-1}_{i,k} \bar{S}_k = 1, \quad \text{for } i = 1,2,\dots,N-1. \tag{3.5}$$

Let $A^{-1}_{i,k}$ and $C_1 = \max|p(s_i)|$, $C_2 = \max|q(s_i)|$.

We define

$$\|(A)^{-1}\| = \max_{1\leq i\leq N-1} \sum_{k=1}^{N-1} (A)^{-1}_{i,k} \quad \text{and} \quad \|T(h)\| = \max_{1\leq i\leq N-1} |T(h_i)|. \tag{3.6}$$

Hence

$$(A)^{-1}_{i,1} \leq \frac{1}{\bar{S}_1} < \frac{2}{h^2 C_2}, \tag{3.7}$$

$$(A)^{-1}_{i,N-1} \leq \frac{1}{\bar{S}_{N-1}} < \frac{2}{h^2 C_2}. \tag{3.8}$$

Furthermore,

$$\sum_{k=2}^{N-2} (A)^{-1}_{i,k} \leq \frac{1}{\min\limits_{2\leq k\leq N-2} \bar{S}_k} < \frac{1}{h^2 C_2}, \quad i = 2,3,\dots,N-2. \tag{3.9}$$

By the help of eqns. (3.7)-(3.9), using (3.4), we obtain

$$\|E\| \leq O(h^2). \tag{3.10}$$

Hence the proposed method is second order convergent.

4. Numerical Experiments

The proposed method is implemented on model problems for the type given by eqns. (2.1)-(2.3). The exact solution of the differential-difference equation

$$\varepsilon w''(s) + a(s)w(s-\delta) + b(s)w(s) = f(s)$$

subject to conditions $w(0) = \alpha$ and $w(1) = \beta$ with constant coefficients is given by Lange and Miura [9, 10] as:

$$w(s) = [((1-a-b)\exp(m_2)-1)\exp(m_1 s) - (((1-a-b)\exp(m_1)-1))\exp(m_2 s)]/$$
$$[(a+b)(\exp(m_1)-\exp(m_2))] + 1/(a+b)$$

where

$$m_1 = \frac{\left(a\delta + \sqrt{a^2\delta^2 - 4\varepsilon^2(a+b)}\right)}{2\varepsilon^2}, \quad m_2 = \frac{\left(a\delta - \sqrt{a^2\delta^2 - 4\varepsilon^2(a+b)}\right)}{2\varepsilon^2}.$$

Problem 4.1. Consider the problem with the dual layer behaviour:

$$\varepsilon^2 w''(x) - 2w(s-\delta) - w(s) = 1$$

with boundaray conditions are $w(0) = 1$, $-\delta \leq s \leq 0$, $w(1) = 0$.

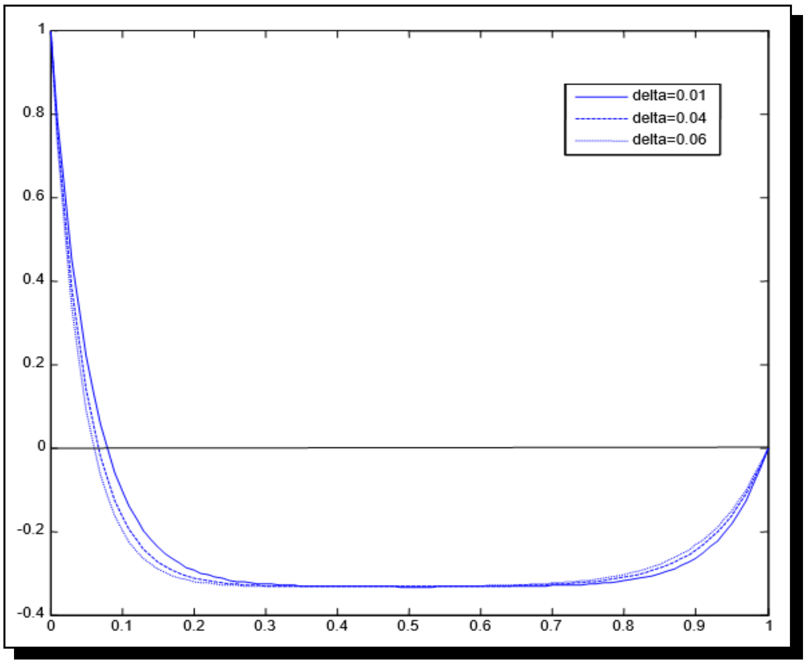
Maximum absolute errors are presented in Tables 1 and 2, for $\varepsilon = 0.1$ and 0.01 and different values of δ, respectively. The effect of δ on the boundary layer solutions has been presented in Graphs 1 and 3.

Table 1. The maximum errors of Problem 4.1 for different values of δ for $\varepsilon = 0.1$

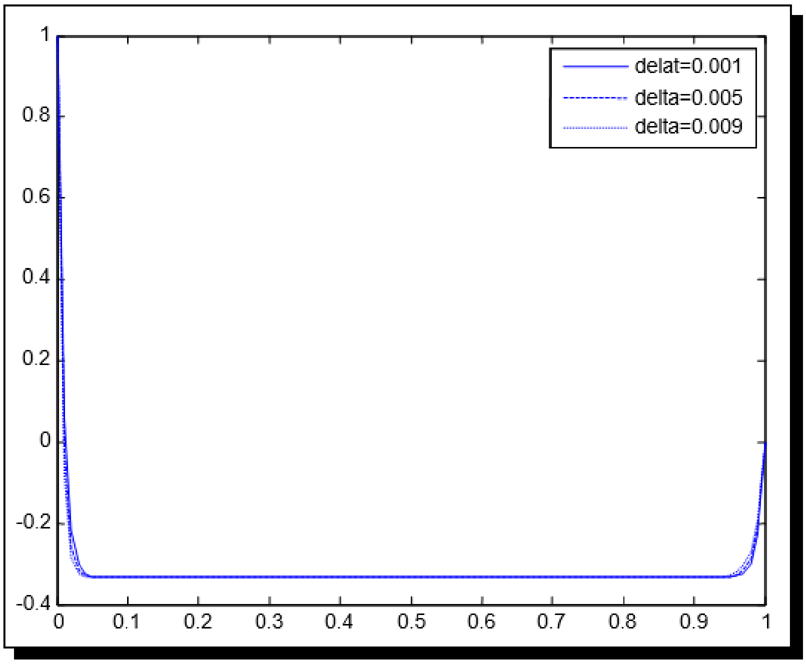
$\delta\downarrow$ \ $N\rightarrow$	100	200	300	400	500
Present method					
0.03	2.5190e-04	3057e-05	2.8040e-05	1.5778e-05	1.0100e-05
0.05	2.6071e-04	5343e-05	2.9069e-05	1.6360e-05	1.0474e-05
0.09	2.7398e-04	8833e-05	3.0646e-05	1.7254e-05	1.1049e-05
Results by [7, 8]					
0.03	3.2676e-003	1.6475e-003	1.1015e-003	8.2735e-004	6245e-004
0.05	3.2657e-003	1.6526e-003	1.1062e-003	8.3136e-004	6593e-004
0.09	3.5460e-003	1.7987e-003	1.2051e-003	9.0609e-004	7.2594e-004

Table 2. The maximum errors of Problem 4.1 for different values of δ for $\varepsilon = 0.3$

$\delta\downarrow$ \ $N\rightarrow$	100	200	300	400	500
Present method					
2^{-1}	4.6890e-05	1.1723e-05	5.2104e-06	2.9309e-06	1.8755e-06
2^{-2}	1.0266e-04	2.5674e-05	1.1413e-05	4202e-06	4.1092e-06
2^{-3}	2.0268e-04	5.0724e-05	2.2554e-05	1.2689e-05	8.1222e-06
2^{-4}	4.0178e-04	1.0070e-04	4.4795e-05	2.5208e-05	1.6139e-05
2^{-5}	8.1594e-04	2.0488e-04	9.1243e-05	5.1373e-05	3.2897e-05
Results by [7, 8]					
2^{-1}	9.2363e-004	4.6407e-004	3.0991e-004	2.3263e-004	1.8619e-004
2^{-2}	1.6390e-003	8.2516e-004	5.5141e-004	4.1404e-004	3.3146e-004
2^{-3}	2.7044e-003	1.3653e-003	9.1315e-004	8602e-004	5.4937e-004
2^{-4}	4.1751e-003	2.1168e-003	1.4178e-003	1.0658e-003	8.5380e-004
2^{-5}	2518e-003	3.1866e-003	2.1382e-003	1.6088e-003	1.2895e-003



Graph 1. Effect of Layer behavior in Problem 4.1 with $\varepsilon = 0.1$



Graph 2. Effect of Layer behavior in Problem 4.1 with $\varepsilon = 0.01$

Problem 4.2. Consider the problem with the dual layer behaviour:

$$\varepsilon^2 w''(s) + 0.25w(s-\delta) - w(s) = 1$$

with boundaray conditions are $w(0) = 1$, $-\delta \le s \le 0$, $w(1) = 0$.

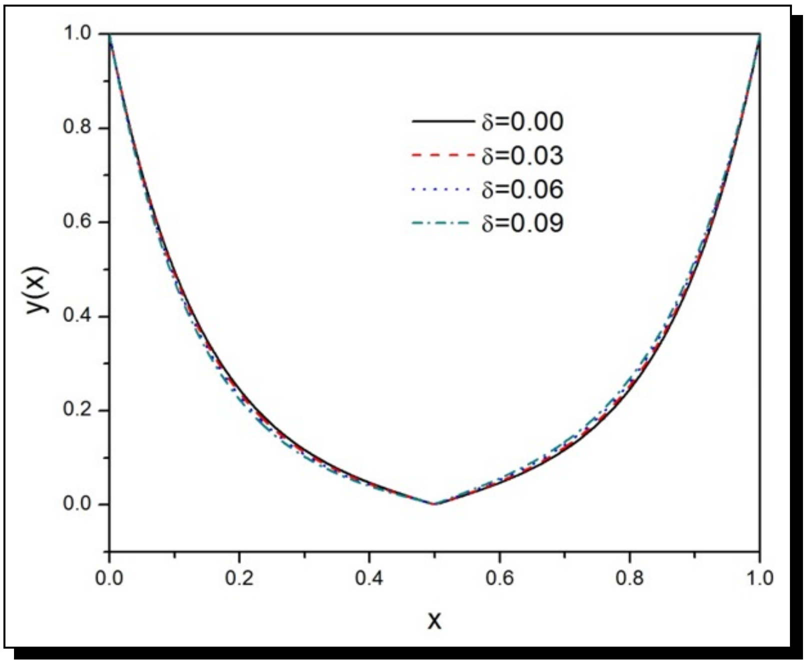
Maximum absolute errors are presented in Table 3 and 4 for different values of δ, respectively. The effect of δ on the boundary layer solutions has been presented in Graphs 3 and 4.

Table 3. The maximum errors of Problem 4.2 for different values of δ for $\varepsilon = 0.1$

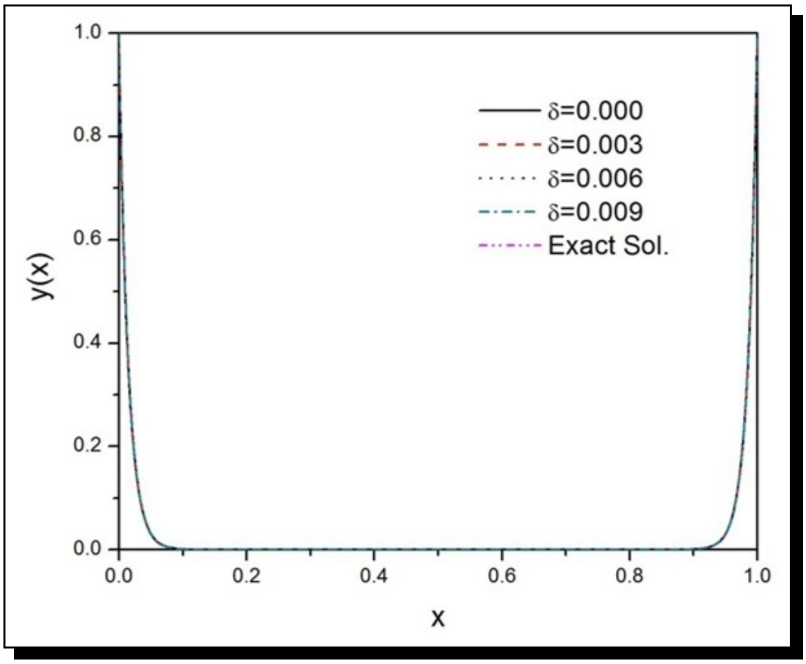
$\delta\downarrow$ \ $N\rightarrow$	100	200	300	400	500
Present method					
0.03	1.2412e-04	3.1024e-05	1.3788e-05	7.7538e-06	4.9705e-06
0.05	1.2331e-04	3.0817e-05	1.3695e-05	7.7034e-06	4.9306e-06
0.09	1.2160e-04	3.0389e-05	1.3505e-05	7.5959e-06	4.8614e-06
Results by [7, 8]					
0.03	2.1226e-003	1.0639e-003	7.0985e-004	5.3259e-004	4.2617e-004
0.05	2.1099e-003	1.0574e-003	7.0543e-004	5.2928e-004	4.2351e-004
0.09	2.0816e-003	1.0426e-003	9547e-004	5.2178e-004	4.1750e-004

Table 4. The maximum errors of Problem 4.2 for different values of δ for $\varepsilon = 0.3$

$\delta\downarrow$ \ $N\rightarrow$	100	200	300	400	500
Present method					
2^{-1}	1.2011e-05	2.9974e-06	1.3245e-06	7.3349e-07	4.5415e-07
2^{-2}	3.7160e-05	9.2960e-06	4.1286e-06	2.3159e-06	1.5165e-06
2^{-3}	9.6015e-05	2.3999e-05	1.0668e-05	0068e-06	3.8365e-06
2^{-4}	1.9621e-04	4.9032e-05	2.1789e-05	1.2256e-05	7.8426e-06
2^{-5}	3.4198e-04	8.5403e-05	3.7946e-05	2.1341e-05	1.3657e-05
Results by [7, 8]					
2^{-1}	5.0597e-004	2.5321e-004	1.6886e-004	1.2667e-004	1.0134e-004
2^{-2}	9.6556e-004	4.8357e-004	3.2250e-004	2.4194e-004	1.9357e-004
2^{-3}	1.7698e-003	8.8657e-004	5.9149e-004	4.4377e-004	3.5508e-004
2^{-4}	3.0307e-003	1.5201e-003	1.0145e-003	7.6132e-004	0926e-004
2^{-5}	4.7379e-003	2.3810e-003	1.5901e-003	1.1937e-003	9.5544e-004



Graph 3. Effect of Layer behavior in Problem 4.2 with $\varepsilon = 0.01$



Graph 4. Effect of Layer behavior in Problem 4.2 with $\varepsilon = 0.01$

Problem 4.3. Consider the problem with the dual layer behaviour:

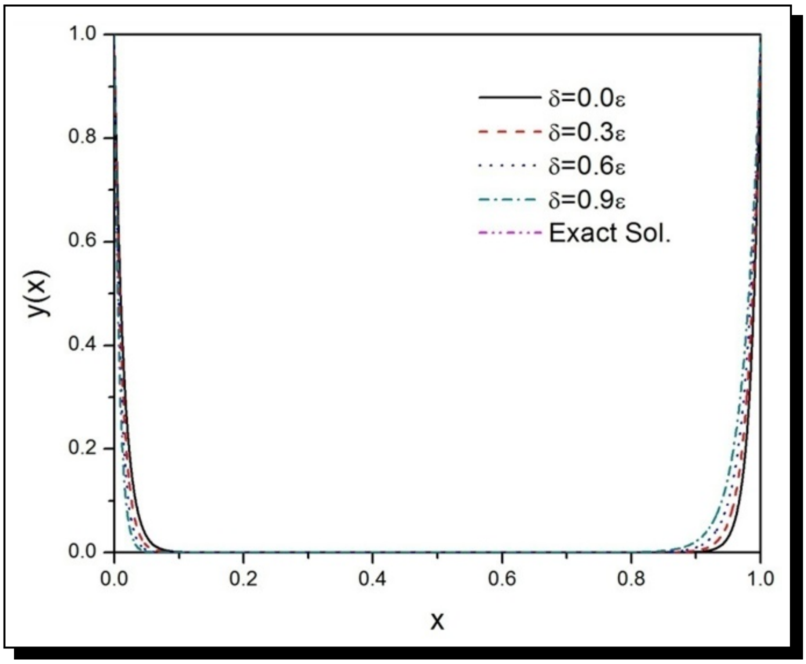
$$\varepsilon^2 w''(s) - w(s-\delta) + 0.5w(s) = 0.$$

with boundaray conditions are $w(0) = 1$, $-\delta \le s \le 0$, $w(1) = 1$.

Maximum absolute errors are presented in Table 5 for $\varepsilon = 0.01$ and different values of δ. The effect of δ on the boundary layer solutions has been presented in Graph 5.

Table 5. The maximum errors of Problem 4.3 for different values of δ for $\varepsilon = 0.1$

$\delta\downarrow$ \ $N\rightarrow$	100	200	300	400	500
Present method					
0.03	2.6141e-03	5232e-04	2.8981e-04	1.6302e-04	1.0422e-04
0.05	2.5840e-03	4482e-04	2.8650e-04	1.6113e-04	1.0311e-04
0.09	2.5303e-03	3139e-04	2.8052e-04	1.5777e-04	1.0097e-04
Results by [7, 8]					
0.03	2.4582e-003	1.2196e-003	8.1096e-004	0742e-004	4.8554e-004
0.05	2.5127e-003	1.2472e-003	8.2948e-004	2134e-004	4.9669e-004
0.09	2.6198e-003	1.3016e-003	8.6589e-004	4872e-004	5.1863e-004



Graph 5. Effect of Layer behavior in Problem 4.3 for $\varepsilon = 0.01$ with different δ

5. Discussion and Conclusion

We have presented a numerical scheme for using the fitted special finite difference method for the solution of delay differential equation with dual boundary layer behaviour. We derived the scheme by extending the idea given by Van Veldhuizen to the second order boundary value problem for ordinary differential equations. The delay differential equation is replaced by an asymptotically equivalent singular perturbation problem using Taylor's series expansion. Then, the derived scheme is implemented to get accurate solution to the problem. The method is demonstrated by implementing several model problems by taking various values for the delay parameter and perturbation parameter. This method is very easy to implement. To show the effect of delay on the boundary layer or oscillatory behaviour of the solution, several numerical problems are carried out in Section 3. We observed that when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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