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Research Article

Composite Weiner Hopf Equation with Variational Inequality and Equilibrium Problem

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Abstract. In this paper, we introduce an iteration based on composite Weiner-Hopf equation technique to find the common solution of the set of solution of composite generalized variational inequality, set of equilibrium problem and set of fixed point of non expansive mapping in separable real Hilbert space. As the result, the strong convergence theorem of the suggested iteration has been discussed.

Keywords. Composite Weiner-Hopf equation technique; Convergence analysis; Composite Variational inequality; Monotone operators

Mathematics Subject Classification (2020). 47H10; 49J40; 90C33

1. Introduction

Let $\mathbb{H}$ be a *real separable Hilbert space*, with norm and inner product expressed as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, jointly. Let $\mathbb{K}$ be a non empty *closed convex subset* of $\mathbb{H}$. Let $S,T:\mathbb{K}\to\mathbb{K}$ be non linear mappings. Let $G_1:\mathbb{K}\times\mathbb{K}\to\mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real number and $P_{\mathbb{K}}$ be projection defined from $\mathbb{H}$ onto the *closed convex set* $\mathbb{K}$ and $Q_{\mathbb{K}}=I-P_{\mathbb{K}}$, where I is identity operator.

The *Equilibrium Problem* (EP) for $G_1:\mathbb{K}\times\mathbb{K}\to\mathbb{R}$ is, for finding $v\in\mathbb{K}$ in such a way that

$$G_1(v,u)\geq 0,\quad \forall\ u\in\mathbb{K}. \tag{1.1}$$

The solution set of (1.1) is denoted by $EP(G_1)$. If $G_1(v,u)=\langle Tv,u-v\rangle\ \forall\ u,v\in\mathbb{K}$, then the problem lessen to *Variational Inequality Problem* (VIP), which is solved for $v\in\mathbb{K}$ in such a way that

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$$\langle Tv, u-v\rangle \geq 0, \quad \text{for all } u \in \mathbb{K}. \tag{1.2}$$

i.e. v is solution of *equilibrium problem* if and only if v is solution of variational inequality. This problem (1.2) was given by Stampacchia [10] in 1964. VIP is solved by different method like projection method, auxiliary principle technique, resolvent method, dynamical system technique and Weiner Hopf equation. There are many researcher who work in these different techniques. A mapping $S : \mathbb{K} \to \mathbb{H}$ is nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \quad \text{for all } u, v \in \mathbb{K}. \tag{1.3}$$

In 1991, Shi [9] demonstrate the equivalence of between *Wiener-Hopf Equation* (WHE): $(SP_{\mathbb{K}} + Q_{\mathbb{K}})u = g$ where $g \in \mathbb{K}$ and variational inequality: $\langle Su - g, v - u\rangle \geq 0$, $\forall\ v \in \mathbb{K}$. Later on Noor [7] also established an equivalence between generalized VIP and generalized WHE. After that Al-Shemas and Verma [1,11] worked in the same direction. It shows that WHE techinque is more flexible than projection method.

In 2014, Wang and Zhang [12] work for solving EP, VIP and FP with WHE technique. They are the first one to answer their own question, which is, why the earlier researcher does not consider EP to solve VIP and VIP with FP under the applied WHE technique? They work on EP with VIP and generalized VIP with WHE. Later on, in 2020 Khan *et al*. [3] introduced a composite Wiener-Hopf equation and composite generalized variational inequality in real separable Hilbert space and proved the strong convergence of their iterative algorithm.

Motivated from [3, 12], we introduced a composite iterative algorithm for solving EP, composite generalized variational inequality and fixed point problem with composite Wiener-Hopf equation. We use the equivalence technique of [3] to show the strong convergence of our iterative algorithm.

2. Preliminaries

In this section, we list some fundamental definitions and lemmas which are useful to our result. Let $\mathbb{H}$ be a *real separable Hilbert space*, with norm and inner product expressed as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, jointly and $\mathbb{K}$ be a nonempty *closed convex subset* of $\mathbb{H}$ and $P_{\mathbb{K}}$ is the projection mapping from $\mathbb{K}$ to $\mathbb{H}$.

Definition 2.1. An operator $G : \mathbb{K} \to \mathbb{H}$ is called

(i) monotone if,

$$\langle Gv - Gu, v - u\rangle \geq 0, \quad \forall\ v, u \in \mathbb{K};$$

(ii) η-*strongly monotone* with constant $\eta > 0$ such that

$$\langle Gv - Gu, v - u\rangle \geq \eta\|v - u\|^2, \quad \forall\ v, u \in \mathbb{K};$$

(iii) ν-expansive if there exist $\nu > 0$ such that

$$\|Gv - Gu\| \geq \nu\|v - u\|, \quad \forall\ v, u \in \mathbb{K};$$

(iv) α-cocorecive if there exist $\alpha > 0$ such that

$$\langle Gv - Gu, v - u\rangle \geq \alpha\|Gv - Gu\|^2, \quad \forall\ v, u \in \mathbb{K};$$

(v) relaxed γ-cocoervive, if there exist $\gamma \geq 0$ such that

$$\langle Gv - Gu, v-u\rangle \geq (-\gamma)\|Gv - Gu\|^2, \quad \forall\, v,u \in \mathbb{K};$$

(vi) relaxed (γ,t)-cocoercive, if there exist $\gamma, t > 0$ such that

$$\langle Gv - Gu, v-u\rangle \geq (-\gamma)\|Gv - Gu\|^2 + t\|v-u\|^2, \quad \forall\, v,u \in \mathbb{K}.$$

Definition 2.2. The set valued mapping $S:\mathbb{H}\to 2^{\mathbb{H}}$ is called

(i) relaxed monotone operator if, there exists a constant $\xi > 0$ such that

$$\langle w_1 - w_2, u-v\rangle \geq (-\xi)\|u-v\|^2, \quad \forall\, w_1 \in S(u) \text{ and } w_2 \in S(v).$$

(ii) The set-valued mapping $S:\mathbb{H}\to 2^{\mathbb{H}}$ is γ-Lipschitz continuous if, there exists $\gamma > 0$ such that

$$\|w_1 - w_{\|} \leq \gamma\|u-v\|, \quad \forall\, w_1 \in S(u) \text{ and } w_2 \in S(v).$$

Definition 2.3. The single valued mapping $T:\mathbb{K}\to\mathbb{K}$ is called

(i) non expansive if

$$\|Tv - Tu\| \leq \|v-u\|, \quad \forall\, v,u \in \mathbb{K}.$$

(ii) strictly pseudo-contractive, if there exist $l \in [0,1]$ such that

$$\|Tv - Tu\|^2 \leq \|v-u\|^2 + l\|(I-T)v - (I-T)u\|^2, \quad \forall\, v,u \in \mathbb{K}.$$

The fixed point problem is to identify a point $u \in \mathbb{K}$ for the mapping T, in such a way, that

$$Tu = u. \tag{2.1}$$

We represent $F(T)$ by the solution set of (2.1).

The *Composite Generalized Variational Inequality* (CGVIP) [3] for $B,F:\mathbb{K}\to\mathbb{H}$ and $h:\mathbb{K}\to\mathbb{K}$, single valued continuous nonlinear mappings, with $T:\mathbb{H}\to 2^{\mathbb{H}}$ as a set valued mapping, is to find a point $u \in \mathbb{H}$ such that $h(u) \in \mathbb{K}$ and

$$\langle Boh(u) + F(w), h(u) - h(v)\rangle \geq 0, \quad \forall\, h(v) \in \mathbb{K} \text{ and } w \in S(u). \tag{2.2}$$

The solution set of (CGVIP) (2.2) is denoted by $VI(\mathbb{K},B,F,S,h)$.

Special cases:

(i) If $F,h = I$, then (CGVIP) (2.2) is equivalent to finding $u \in \mathbb{K}$ such that

$$\langle Bu + w, u-v\rangle \geq 0, \quad \forall\, v \in \mathbb{K} \text{ and } w \in Su. \tag{2.3}$$

Problem (2.3) introduced by Wu [14].

(ii) If $B = 0$, $F = I$ and S is single valued mapping, then CGVIP (2.2) is identical to find $u \in \mathbb{H}$ such that $h(u) \in \mathbb{K}$

$$\langle Su, h(u) - h(v)\rangle \geq 0, \quad \forall\, h(v) \in \mathbb{K}. \tag{2.4}$$

Problem (2.4) was studied by Noor [8].

(iii) If $F,S = 0$ and $h = I$, then CGVIP (2.2) become equivalent to find a point $u \in \mathbb{K}$ such that

$$\langle Bu, u-v\rangle \geq 0, \quad \forall\, v \in \mathbb{K}. \tag{2.5}$$

Problem (2.5) is classical variational inequality, studied by Stampachia [10].

The *mixed equilibrium problem*, denoted by MEP, is to find $u \in \mathbb{K}$ such that

$$G_1(u,v) + \langle Du, u-v \rangle \geq 0, \quad \forall\, v \in \mathbb{K}, \tag{2.6}$$

where $G_1 : \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is a bifunction and $D : \mathbb{K} \to \mathbb{H}$ be a non linear mapping. This problem was imported and calculated by Moudafi and Thera [5] and Moudafi [6]. The solution set of (2.6) is

$$MEP(G_1) = \{u \in \mathbb{K} : G_1(u,v) + \langle Du, u-v \rangle \geq 0,\ \forall\, v \in \mathbb{K}\} \tag{2.7}$$

If $D = 0$, (2.6) reduced to *equilibrium problem*, i.e.

$$G_1(u,v) \geq 0, \quad \forall\, v \in \mathbb{K}. \tag{2.8}$$

Lemma 2.1 ([2]). *Let the function $G_1 : \mathbb{K} \times \mathbb{K} \to R$ satisfy the following conditions*

(i) $G_1(u,u) = 0$ *for all* $u \in \mathbb{K}$.

(ii) G_1 *is monotone, i.e.* $G_1(u,v) + G_1(v,u) \leq 0$ *for all* $u, v \in \mathbb{K}$.

(iii) *for each* $u, v, w \in \mathbb{K}$, $\lim\limits_{t \to 0} G_1(tw + (1-t)u, v) \leq G_1(u,v)$.

(iv) *for each* $u \in \mathbb{K}, G_1(u, \cdot)$ *is convex and lower semicontinuous.*

Then $EP(G_1) \neq \phi$.

Lemma 2.2 ([2]). *Let* $r > 0, u \in \mathbb{H}$, *and* G_1 *satisfy the conditions* (i)-(iv) *in Lemma* 2.1. *Then there exists* $w \in \mathbb{K}$ *such that* $G_1(w,v) + \frac{1}{r}\langle v-w, w-u \rangle \geq 0,\ \forall\, v \in \mathbb{K}$.

Lemma 2.3 ([2]). *Let* $r > 0$, $u \in \mathbb{H}$, *and* G_1 *satisfy the conditions* (i)-(iv) *in Lemma* 2.1. *Define a mapping* $S_r : \mathbb{H} \to \mathbb{K}$ *as* $S_r(u) = \{w \in \mathbb{K} : G_1(w,v) + \frac{1}{r}\langle v-w, w-u \rangle \geq 0, \forall\, v \in \mathbb{K}\}$. *Then the following hold:*

(i) S_r *is single-valued.*

(ii) S_r *is firmly nonexpansive, i.e.* $\|S_r u - S_r v\| \leq \langle S_r u - S_r v, u - v \rangle$ *for all* $u, v \in \mathbb{H}$.

(iii) $EP(G_1) = F(S_r)$, *where* $F(S_r)$ *denotes the sets of fixed point of* S_r.

(iv) $EP(G_1)$ *is closed and convex.*

Lemma 2.4 ([4]). *Given* $w \in \mathbb{H}, u \in \mathbb{K}$ *satisfies the inequality:*

$$\langle u-w, v-u \rangle \geq 0, \quad \forall\, v \in \mathbb{K},$$

if and only if $u = P_{\mathbb{K}} w$, *where* $P_{\mathbb{K}}$ *is the projection of* $\mathbb{H}$ *into* $\mathbb{K}$. *Furthermore, the projection* $P_{\mathbb{K}}$ *is a nonexpansive mapping.*

For the projection mapping $P_{\mathbb{K}}$ from $\mathbb{H}$ into $\mathbb{K}$, consider $Q_{\mathbb{K}} = I - TP_{\mathbb{K}}$, where I is the identity mapping and T is a non-expansive mapping. If h^{-1} exists for (2.2), then we consider the problem of finding $w \in \mathbb{H}$ such that

$$BTP_{\mathbb{K}} w + F(t) + \rho^{-1} Q_{\mathbb{K}} w = 0, \quad \forall\, t \in STP_{\mathbb{K}} w, \tag{2.9}$$

where $\rho > 0$ is constant.

Equation (2.4) is called *composite generalized Wiener-Hopf equation* [3]. The solution set of the problem (2.4) is denoted by $CC_1WE(\mathbb{H}, B, T, F, h)$.

Lemma 2.5 ([3]). *The element $u \in \mathbb{K}$ is a common solution of $VI(\mathbb{K}, B, F, S, h) \cap F(Toh)$ if and only if the composite Wiener-Hopf equation* (2.4) *has a solution $w \in \mathbb{H}$, where*

$$w = h(u) - \rho[Boh(u) + F(t)], \tag{2.10}$$

$$h(u) = TP_{\mathbb{K}}(w), \tag{2.11}$$

where $P_{\mathbb{K}}$ is the projection of $\mathbb{H}$ into $\mathbb{K}$ and $\rho > 0$ is constant.

Lemma 2.6 ([13]). *Consider $\{a_n\}$ be a sequence of non negative real numbers such that*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n,$$

with $\lambda_n \in [0,1]$, $\sum_{i=1}^{\infty} \lambda_n = \infty, b_n = o(\lambda_n)$, then $\lim_{n\to\infty}(a_n) = 0$.

3. Convergence Analysis of EP with CC_1WE

First we define iterative algorithm based on Lemma 2.5 for finding the solution of CGVIP (2.2) then prove strong convergence of our iteration.

Algorithm 3.1. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$h(u_n) = (\beta I + (1-\beta)T)P_{\mathbb{K}}a_n,$$

$$G_1(h(v_n), h(y)) + \frac{1}{r}\langle h(y) - h(v_n), h(v_n) - h(u_n)\rangle \ge 0, \quad \forall\ y \in \mathbb{K},$$

$$a_{n+1} = (1-\beta_n)a_n + \beta_n[h(v_n) - \rho(Boh(v_n) + F(w_n))], \tag{3.1}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$, $r > 0$ and T is strictly contractive mapping.

(I) If $F = h = I$, Algorithm 3.1, reduces to:

Algorithm 3.2. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$u_n = (\beta I + (1-\beta)T)P_{\mathbb{K}}a_n,$$

$$G_1(v_n, y) + \frac{1}{r}\langle y - v_n, v_n - u_n\rangle \ge 0, \quad \forall\ y \in \mathbb{K},$$

$$a_{n+1} = (1-\beta_n)a_n + \beta_n[v_n - \rho(B(v_n) + w_n)], \tag{3.2}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$, $r > 0$ and T is strictly contractive mapping.

(II) If $h, F, T = I$, then Algorithm 3.1 reduced to:

Algorithm 3.3. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$u_n = P_{\mathbb{K}}a_n,$$

$$G_1(v_n, y) + \frac{1}{r}\langle y - v_n, v_n - u_n\rangle \ge 0, \quad \forall\ y \in \mathbb{K},$$

$$a_{n+1} = (1-\beta_n)a_n + \beta_n[v_n - \rho(B(v_n) + w_n)], \tag{3.3}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$ and studied by Wang [12] in respective Algorithm 5.3.

(III) If $F = h = T = I$ and $\beta_n = 1$, $\forall\ n$ Algorithm 3.1 become:

Algorithm 3.4. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$\begin{aligned} &u_n = P_{\mathbb{K}} a_n, \\ &G_1(v_n, y) + \frac{1}{r}\langle y - v_n, v_n - u_n\rangle \geq 0, \quad \forall\ y \in \mathbb{K}, \\ &a_{n+1} = v_n - \rho(B(v_n) + w_n), \end{aligned} \tag{3.4}$$

was studied by Wang [12], in respective Algorithm 5.4.

Theorem 3.1. *Let $\mathbb{K}$ be a closed convex subset of real separable Hilbert space $\mathbb{H}$ and bifunction G_1 satisfy the conditions* (i)-(iv) *of Lemma* 2.3. *Let $B, F : \mathbb{K} \to \mathbb{H}$ and $T, h : \mathbb{K} \to \mathbb{K}$ be the single-valued nonlinear mappings such that B is relaxed (γ, t)-cocoercive mapping and η-Lipschitz continuous, F is ξ Lischitz continuous and T is $\mathbb{K}$ strictly pseudocontractive mapping such that $F(Toh) \cap VI(\mathbb{K}, B, F, S, h) \cap EP(G_1 oh) \neq \phi$, respectively. Let $S : \mathbb{H} \to 2^{\mathbb{H}}$ be a set valued Lipschitz continuous operator and relaxed monotone with corresponding constants $m > 0$ and $k > 0$, respectively. Let $\{a_n\}$ and $\{u_n\}$ be the sequences provoked by Algorithm* 3.1 *and let β_n be a sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\beta \in [k, 1)$,

(iii) $0 < \rho < \frac{2(t - \gamma\eta - k)}{(\eta + \xi m)^2}$, $t > \gamma\eta + k$,

then the sequences $\{u_n\}$, $\{v_n\}$ strongly converges to $u^* \in F(Toh) \cap VI(\mathbb{K}, B, F, S, h) \cap EP(G_1 oh)$ and $\{a_n\}$ strongly converges to $a^* \in CC_1WE(\mathbb{H}, B, T, F, h)$.

Proof. Let $R = \beta I + (I - \beta)T$. From the restriction (ii) we have that R is nonexpansive with $F(R) = F(T)$. Let $h(u) \in \mathbb{K}$ be the common element of $F(T) \cap VI(\mathbb{K}, B, F, S, h)$, then by Lemma 2.5 we have $h(u^*) = RP_{\mathbb{K}} a^*, a^* = (1 - \beta_n)a^* + \beta_n[h(u^*) - \rho(Boh(u^*) + F(w^*))]$ where $w^* \in Sh(u^*)$ and $a^* \in CC_1WE(\mathbb{H}, B, T, F, h)$. From Algorithm 3.1, we have

$$\begin{aligned} \|a_{n+1} - a^*\| &= \|(1 - \beta_n)a_n + \beta_n[h(v_n) - \rho(Boh(v_n) + F(w_n))] \\ &\quad - [(1 - \beta_n)a^* + \beta_n[h(u^*) - \rho(Boh(u^*) + F(w^*))]]\| \\ &\leq (1 - \beta_n)\|a_n - a^*\| + \beta_n\|h(v_n) - h(u^*) \\ &\quad - \rho[(Boh(v_n) + F(w_n)) - (Boh(u^*) + F(w^*))]\|. \end{aligned} \tag{3.5}$$

On solving second term of right side of (3.5).

$$\begin{aligned} &\|h(v_n) - h(u^*) - \rho[(Boh(v_n) + F(w_n)) - (Boh(u^*) + F(w^*))]\|^2 \\ &\quad = \|h(v_n) - h(u^*)\|^2 - 2\rho\langle (Boh(v_n) + F(w_n)) - (Boh(u^*) + F(w^*)), (h(v_n) - h(u^*))\rangle \\ &\qquad + \rho^2\|(Boh(v_n) + F(w_n)) - (Boh(u^*) + F(w^*))\|^2 \\ &\quad = \|h(v_n) - h(u^*)\|^2 - 2\rho\langle (Boh(v_n) - Boh(u^*)), h(v_n) - h(u^*)\rangle \\ &\qquad - 2\rho\langle F(w_n) - F(w^*), h(v_n) - h(u^*)\rangle + \rho^2\|(Boh(v_n) + F(w_n)) - (Boh(u^*) + F(w^*))\|^2 \\ &\quad \leq \|h(v_n) - h(u^*)\|^2 - 2\rho(-\gamma\|Boh(v_n) - Boh(u^*)\|^2 + t\|h(v_n) - h(u^*)\|^2) \end{aligned}$$

$$+2\rho k\|h(v_n)-h(u^*)\|^2+\rho^2\|(Boh(v_n)+F(w_n))-(Boh(u^*)+F(w^*))\|^2$$
$$\leq \|h(v_n)-h(u^*)\|^2+2\rho(\gamma\eta-t+k)\|h(v_n)-h(u^*)\|^2$$
$$+\rho^2\|(Boh(v_n)+F(w_n))-(Boh(u^*)+F(w^*))\|^2. \tag{3.6}$$

Now consider the third term of right side of (3.6)

$$\begin{aligned}\|(Boh(v_n)+F(w_n))-(Boh(u^*)+F(w^*))\| &= \|(Boh(v_n)-Boh(u^*))+(F(w_n)-F(w^*))\|\\ &\leq \|(Boh(v_n)-Boh(u^*))\|+\|(F(w_n)-F(w^*))\|\\ &\leq (\eta+\xi m)\|(h(v_n)-h(u^*))\|. \end{aligned}\tag{3.7}$$

Use (3.7) in (3.6), we get

$$\begin{aligned}&\|h(v_n)-h(u^*)-\rho[(Boh(v_n)+F(w_n))-(Boh(u^*)+F(w^*))]\|^2\\ &\quad\leq \|h(v_n)-h(u^*)\|^2+2\rho(\gamma\eta-t+k)\|h(v_n)-h(u^*)\|^2+\rho^2(\eta+\xi m)^2\|(h(v_n)-h(u^*))\|^2\\ &\quad= [1+2\rho(\gamma\eta-t+k)+\rho^2(\eta+\xi m)^2]\|(h(v_n)-h(u^*))\|^2\\ &\quad= \zeta^2\|h(v_n)-h(u^*)\|^2,\end{aligned}\tag{3.8}$$

where $\zeta=\sqrt{1+2\rho(\gamma\eta-t+k)+\rho^2(\eta+\xi m)^2}$.

From condition (iii), we get $\zeta<1$. Now use (3.8) in (3.5), we get

$$\|a_{n+1}-a^*\|\leq (1-\beta_n)\|a_n-a^*\|+\beta_n\zeta\|h(v_n)-h(u^*)\|. \tag{3.9}$$

Since $u^*\in EP(G_1oh)$ implies

$$G_1(h(u^*),h(y))\geq 0, \quad \forall\ y\in\mathbb{K}. \tag{3.10}$$

Put $y=v_n$ in (3.10) and $y=u^*$ in Algorithm 3.1, we obtain

$$G_1(h(u^*),h(v_n))\geq 0 \ \text{ and } \ G_1(h(v_n),h(u^*))+\frac{1}{r}\langle h(u^*)-h(v_n),h(v_n)-h(u_n)\rangle\geq 0. \tag{3.11}$$

From the monotonicity of G_1, we have

$$G_1(h(u^*),h(v_n))\geq 0 \implies G_1(h(v_n),h(u^*))\leq 0. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\langle h(u^*)-h(v_n),h(v_n)-h(u_n)\rangle\geq 0.$$

It follows that

$$\begin{aligned}&\langle h(u^*)-h(v_n),h(v_n)-h(u^*)+h(u^*)-h(u_n)\rangle\geq 0\\ \implies\ &\langle h(u^*)-h(v_n),h(v_n)-h(u^*)\rangle+\langle h(u^*)-h(v_n),h(u^*)-h(u_n)\rangle\geq 0\\ \implies\ &\|h(u^*)-h(v_n)\|^2\leq \langle h(u^*)-h(v_n),h(u^*)-h(u_n)\rangle\\ &\qquad\leq \|h(u^*)-h(v_n)\|\cdot\|h(u^*)-h(u_n)\|\\ \implies\ &\|h(u^*)-h(v_n)\|\leq \|h(u^*)-h(u_n)\|\\ \implies\ &\|h(v_n)-h(u^*)\|\leq \|h(u_n)-h(u^*)\|.\end{aligned}\tag{3.13}$$

Since R is non expansive, we get

$$\begin{aligned}\|h(u_n)-h(u^*)\| &= \|RP_{\mathbb{K}}a_n-RP_{\mathbb{K}}a^*\|\\ &\leq \|a_n-a^*\|.\end{aligned}\tag{3.14}$$

From (3.9), (3.13) and (3.14), we obtain

$$\|a_{n+1}-a^*\| \le (1-\beta_n)\|a_n-a^*\| + \beta_n\zeta\|h(v_n)-h(u^*)\|$$
$$\le [1-\beta_n(1-\zeta)]\|a_n-a^*\|. \tag{3.15}$$

From condition (i) and Lemma 2.6 into equation (3.15), we have

$$\lim_{n\to 0}\|a_n-a^*\| \to 0.$$

On the other hand, from (3.13) and (3.14), we obtain

$$\lim_{n\to 0}\|u_n-u^*\| \to 0 \text{ and } \lim_{n\to 0}\|v_n-u^*\| \to 0.$$

Therefore the sequence $\{u_n\}$ and $\{v_n\}$ strongly converges to $u^* \in F(Toh)\cap VI(\mathbb{K},B,F,S,h)\cap EP(G_1oh)$ and $\{a_n\}$ strongly converges to $a^* \in CC_1WE(\mathbb{H},B,T,F,h)$. □

Remark 3.1. If $G_1 = 0$ we obtain Theorem 4.1 of Wang [12].

Corollary 3.1. *Let $\mathbb{K}$ be a closed convex subset of real separable Hilbert space $\mathbb{H}$ and bifunction G_1 satisfy the condition* (i)-(iv) *of Lemma* 2.3*. Let $B,F:\mathbb{K}\to\mathbb{H}$ and $T,h:\mathbb{K}\to\mathbb{K}$ be the single-valued nonlinear mappings such that B is relaxed (γ,t)-cocoercive mapping and η-Lipschitz continuous, F is ξ Lipschitz continuous and T is non expansive mapping such that $F(Toh)\cap VI(\mathbb{K},B,F,S,h)\cap EP(G_1oh) \neq \phi$, respectively. Let $S:\mathbb{H}\to 2^{\mathbb{H}}$ be a multi valued Lipschitz continuous and relaxed monotone operator with respective constants $m>0$ and $k>0$, respectively. Let $\{a_n\}$ and $\{u_n\}$ be the sequences provoked by Algorithm* 3.1 *and for the sequence β_n in* $[0,1]$ *which satisfying the successive conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\beta \in [k,1)$,

(iii) $0<\rho<\frac{2(t-\gamma*\eta-k)}{(\eta+\xi m)^2}$, $t>\gamma\eta+k$,

then the sequence $\{u_n\}$, $\{v_n\}$ strongly converges to $u^* \in F(Toh)\cap VI(\mathbb{K},B,F,S,h)\cap EP(G_1oh)$ and $\{a_n\}$ strongly converges to $a^* \in CC_1WE(\mathbb{H},B,T,F,h)$.

4. Convergence Analysis of MEP with CC_1WE

In this section we consider *Mixed Equilibrium Problem* (MEP) and define several iterative algorithm and prove its convergence theorem for solving $MEP(G_1oh)\cap F(Toh)\cap VI(\mathbb{K},B,F,S,h)$

Algorithm 4.1. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$h(u_n) = (\beta I + (1-\beta)T)P_{\mathbb{K}}a_n,$$
$$G_1(h(v_n),h(y)) + \langle Boh(v_n), y-v_n\rangle + \frac{1}{r}\langle h(y)-h(v_n), h(v_n)-h(u_n)\rangle \ge 0, \quad \forall\, y\in\mathbb{K},$$
$$a_{n+1} = (1-\beta_n)a_n + \beta_n[h(v_n)-\rho(Boh(v_n)+F(w_n))], \tag{4.1}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$, $r>0$ and T is strictly contractive mapping.

(I) If $F = h = I$, Algorithm 4.1, reduces to algorithm:

Algorithm 4.2. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$\begin{aligned} &u_n = (\beta I + (1-\beta)T)P_{\mathbb{K}}a_n, \\ &G_1(v_n, y) + \langle B(v_n), y - v_n\rangle + \frac{1}{r}\langle y - v_n, v_n - u_n\rangle \geq 0, \quad \forall\ y \in \mathbb{K}, \\ &a_{n+1} = (1-\beta_n)a_n + \beta_n[v_n - \rho(B(v_n) + w_n)], \end{aligned} \tag{4.2}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$, $r > 0$ and T is strictly contractive mapping.

(II) If $h, F, T = I$, then Algorithm 4.1 reduces to:

Algorithm 4.3. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$\begin{aligned} &u_n = P_{\mathbb{K}}a_n, \\ &G_1(v_n, y) + \langle B(v_n), y - v_n\rangle + \frac{1}{r}\langle y - v_n, v_n - u_n\rangle \geq 0, \quad \forall\ y \in \mathbb{K}, \\ &a_{n+1} = (1-\beta_n)a_n + \beta_n[v_n - \rho(B(v_n) + w_n)], \end{aligned} \tag{4.3}$$

where $\{\beta_n\}$ is a sequence in $[0,1]$.

(III) If $F = h = T = I$ and $\beta_n = 1$, $\forall\ n$ Algorithm 4.1 become:

Algorithm 4.4. For any $a_0 \in \mathbb{H}$, calculate the sequence $\{a_n\}$ by the iterative process

$$\begin{aligned} &u_n = P_{\mathbb{K}}a_n, \\ &G_1(v_n, y) + \langle B(v_n), y - v_n\rangle + \frac{1}{r}\langle y - v_n, v_n - u_n\rangle \geq 0, \quad \forall\ y \in \mathbb{K}, \\ &a_{n+1} = v_n - \rho(B(v_n) + w_n), \end{aligned} \tag{4.4}$$

Theorem 4.1. *Let $\mathbb{K}$ be a closed convex subset of separable real Hilbert space $\mathbb{H}$ and bifunction G_1 satisfy the condition* (i)-(iv) *of Lemma* 2.3. *Let $B, F : \mathbb{K} \to \mathbb{H}$ and $T, h : \mathbb{K} \to \mathbb{K}$ be the single-valued nonlinear mappings such that B is relaxed (γ, t)-cocoercive mapping and η-Lipschitz continuous, F is ξ Lischitz continuous and T is $\mathbb{K}$ strictly pseudocontractive mapping such that $F(Toh) \cap VI(\mathbb{K}, B, F, S, h) \cap MEP(G_1oh) \neq \phi$, respectively. Let $S : \mathbb{H} \to 2^{\mathbb{H}}$ be a set valued Lipschitz continuous operator and relaxed monotone with corresponding constants $m > 0$ and $k > 0$, respectively. Let $\{a_n\}$ and $\{u_n\}$ be the sequences provoked by Algorithm* 4.1 *and β_n be the sequence in* $[0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\beta \in [k, 1)$,

(iii) $0 < \rho < \frac{2(t - \gamma * \eta - k)}{(\eta + \xi m)^2}$, $t > \gamma\eta + k$,

then the sequence $\{u_n\}$, $\{v_n\}$ strongly converges to $u^* \in F(Toh) \cap VI(\mathbb{K}, B, F, S, h) \cap EP(G_1oh)$ and $\{a_n\}$ strongly converges to $a^* \in CC_1WE(\mathbb{H}, B, T, F, h)$.

Proof. By the same technique as in Theorem 3.1, we have

$$u^* \in MEP(G_1oh) \text{ implies } G_1(h(u^*), h(y)) + \langle Boh(u^*), y - u^*\rangle \geq 0. \tag{4.5}$$

Put $y = v_n$ in (4.5) and $y = u^*$ in Algorithm 4.1, we obtain

$$G_1(h(u^*),h(v_n)) + \langle Boh(u^*), v_n - u^* \rangle \geq 0 \tag{4.6}$$

and

$$G_1(h(v_n),h(u^*)) + \langle Boh(v_n), u^* - v_n \rangle + \frac{1}{r}\langle h(u^*) - h(v_n), h(v_n) - h(u_n) \rangle \geq 0\,. \tag{4.7}$$

From the monotonicity of G_1 and (4.7), we get

$$G_1(h(u^*),h(v_n)) + \langle Boh(u^*), v_n - u^* \rangle \geq 0,$$

$$G_1(h(v_n),h(u^*)) \leq \langle Boh(u^*), v_n - u^* \rangle,$$

$$\begin{aligned}
0 &\leq G_1(h(v_n),h(u^*)) + \langle Boh(v_n), u^* - v_n \rangle + \frac{1}{r}\langle h(u^*) - h(v_n), h(v_n) - h(u_n) \rangle \\
&\leq \langle Boh(u^*), v_n - u^* \rangle + \langle Boh(v_n), u^* - v_n \rangle + \frac{1}{r}\langle h(u^*) - h(v_n), h(v_n) - h(u_n) \rangle \\
&\leq -\langle Boh(u^*) - Boh(v_n), u^* - v_n \rangle + \frac{1}{r}\langle h(u^*) - h(v_n), h(v_n) - h(u_n) \rangle\,.
\end{aligned}$$

As B is relaxed (γ,t)-cocoercive mapping, then

$$\begin{aligned}
&\leq -(-\gamma\|Boh(u^*) - Boh(v_n)\|^2 + t\|h(u^*) - h(v_n)\|^2) + \frac{1}{r}\langle h(u^*) - h(v_n), h(v_n) - h(u_n) \rangle \\
&\leq r(\gamma\eta - t)\|h(u^*) - h(v_n)\|^2 - \|h(u^*) - h(v_n)\|^2 + \|h(u^*) - h(v_n)\| \cdot \|h(u^*) - h(u_n)\|
\end{aligned}$$

$$\begin{aligned}
\|h(u^*) - h(v_n)\| &\leq r(\gamma\eta - t) + \|h(u^*) - h(u_n)\| \\
&\leq \|h(u^*) - h(u_n)\| \qquad (\text{because } (\gamma\eta - t) < 0 \text{ and } r > 0).
\end{aligned}$$

Again proceeding in the same manner, we get

$$\lim_{n\to 0}\|a_n - a^*\| \to 0.$$

Also we have

$$\lim_{n\to 0}\|u_n - u^*\| \to 0 \quad \text{and} \quad \lim_{n\to 0}\|v_n - u^*\| \to 0.$$

Therefore the sequence $\{u_n\}$ and $\{v_n\}$ strongly converges to $u^* \in F(Toh) \cap VI(\mathbb{K},B,F,S,h) \cap MEP(G_1oh)$ and $\{a_n\}$ strongly converges to $a^* \in CC_1WE(\mathbb{H},B,T,F,h)$. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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