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Research Article

About Dirichlet Series and Applications

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Abstract. We will study in this article, some classes of Dirichlet series while binding with the arithmetic functions. Some applications are provided.

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Mathematics Subject Classification (2020). Primary 11K65

1. Introduction

Dirichlet series was introduced by L. Dirichlet in 19th century and it has the form [2, 6, 7]:

$$f(s) := \sum_{n=1}^{\infty} a_n\, e^{\lambda_n s}, \tag{1.1}$$

where $\{a_n\} \in \mathbb{C}$, $0 < \lambda_n \uparrow +\infty$ and $s = \sigma + it$ (σ, t are real variables). It is well known that Dirichlet series are the generalization of Taylor series, because (1.1) can be Taylor series, provided $e^s = z$ and $\lambda_n = n$. In this paper, we do not require λ_n must be integers.

We will study in this paper, some classes of Dirichlet series while binding with the arithmetic functions. Then to give some arithmetic applications.

Indeed, the theory of the Dirichlet series and its generalizations represent an important part in the old and recent mathematical developments. It played a fundamental role for the interest of the prime number theory and arithmetic functions.

2. Definition

Definition 2.1. Let $f:\mathbb{N}^* \to \mathbb{C}$ be an arithmetic function.

We call a Dirichlet series a formal series $D = D(f;s)$ of the form

$$D = D(f;s) := \sum_{n\geq 1} \frac{f(n)}{n^s}, \quad s \in \mathbb{C}.$$

Generally, a Dirichlet series is a formal series $D = D(a_n;s)$ of the form

$$D = D(a_n;s) := \sum_{n\geq 1} a_n \frac{1}{n^s},$$

with coefficients $a_n \in \mathbb{C}$ and variable s in some region of $\mathbb{C}$, or $\mathbb{C}$.

Remark 2.1. We have

(a) the most famous Dirichlet series is $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}$;

(b) Dirichlet series and power series are very much related through the theory of general Dirichlet series, of which both are particular cases (see ([2,4,5]).

3. Abscissas of Convergence

Theorem 3.1 (Fundamental). *Let D be a Dirichlet series*

$$D = D(f;s) := \sum_{n\geq 1} \frac{f(n)}{n^s}, \quad s \in \mathbb{C}.$$

(a) *There exists a $\lambda \in [-\infty, +\infty]$ called abscissa of convergence of the series $D(f;s)$ such that*

- *if $\Re e(s) > \lambda$, the series converges;*
- *if $\Re e(s) < \lambda$, the series diverges;*
- *if $\lambda = -\infty$, the series always converges;*
- *if $\lambda = +\infty$, the series always diverges.*

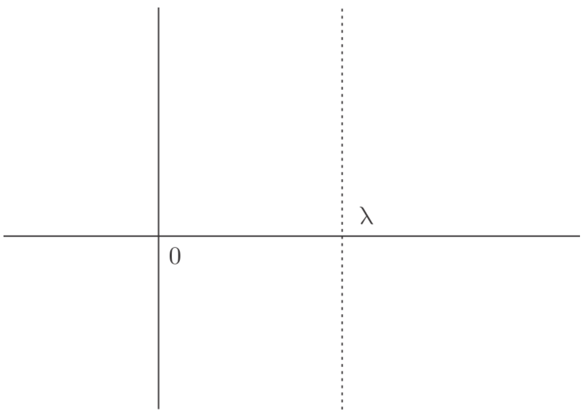

Figure 1

Let us be in the case where λ is finite ($\lambda < +\infty$).

If λ^* is such that $\lambda < \lambda^* < +\infty$, then the series is uniformly convergent in any closed disc contained in the closed half-plane $\Re e(s) \geq \lambda^*$.

The sum of the series $D(f;s)$ thus represents a holomorphic function in any disc of this type thus in any open half-plane $\Re e(s) > \lambda$.

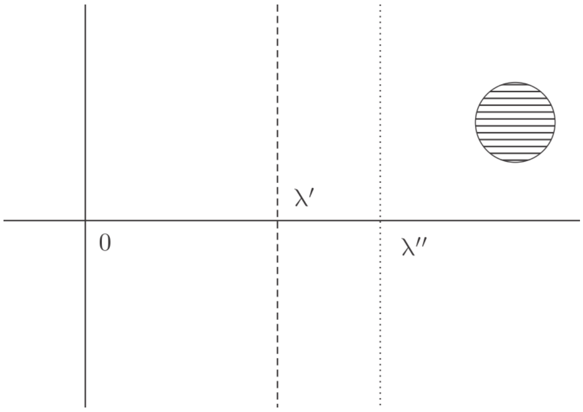

Figure 2

(b) *There exists a number $\ell \in [-\infty, +\infty]$ called abscissa of absolute convergence of the series such that*

- *if $\Re e(s) > \ell$, the series is absolutely convergent;*
- *if $\Re e(s) < \ell$, the series is not absolutely convergent.*

Naturally, one has $\lambda \leq \ell$.

Remark 3.1. It was seen that $\lambda \leq \ell$, in made:

- if $\lambda = -\infty$, then $\ell = -\infty$;
- if $\lambda = +\infty$, then $\ell = +\infty$;
- if $\lambda \in\ ]-\infty, +\infty[$, then $\ell \in\ ]-\infty, +\infty[$ and $\ell - \lambda \leq 1$.

Proof. See ([4]). □

Example 3.1. The series $\sum\limits_{n \geq 1} \frac{n!}{n^s}$ does not converge no share for any s, then $\lambda = +\infty$.

Example 3.2. The series $\sum\limits_{n \geq 1} \frac{1}{n!\, n^s}$ converges everywhere for any s, then $\lambda = -\infty$.

Example 3.3. The series $\sum\limits_{n \geq 1} \frac{1}{n^s}$ converges for all s such that $\Re e(s) > 1$; it diverges for all s such that $\Re e(s) < 1$.

The sum of this series thus represents in the half-plane $\Re e(s) > 1$ a holomorphic function, this function is called $\zeta(s)$.

Example 3.4. The series $\sum_{n\geq 1} \frac{n^{\alpha-1}}{n^s}$, $\alpha \in \mathbb{R}$, converges and $\lambda = \alpha$.

Remark 3.2. A Dirichlet series does not admit necessarily a singularity on the line of convergence.

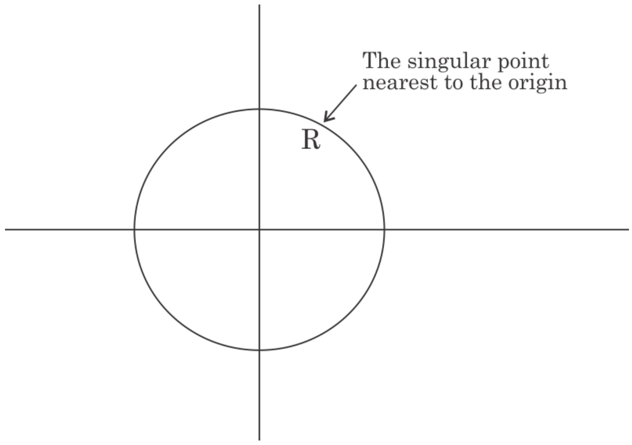

Figure 3

4. Main Theorems

Theorem 4.1 (Term by term differentiation)**.** *Let $\sum_{n\geq 1} \frac{f(n)}{n^s}$, $s \in \mathbb{C}$, be a Dirichlet series of abscissa of convergence $\lambda < +\infty$.*

(a) *The formal series $\sum_{n\geq 1} -\frac{f(n)\log n}{n^s}$ obtained by derivation term by term is again a Dirichlet series which has even the same abscissa of convergence that the initial series.*

(b) *Moreover, in the half-plane of convergence $\Re e(s) > \lambda$, its sum represents the derivative of the sum of the initial series.*

We thus have the following theorem:

Theorem 4.2. *In the half-plane $\Re e(s) > \lambda$ of convergence, a Dirichlet series can be derived term by term.*

Proof. See ([4]). □

For the proof of the uniqueness theorem, we need the following lemma:

Lemma 4.1. *Let $f : \mathbb{N}^* \to \mathbb{C}$ be an arithmetic function, λ be the abscissa of convergence of its Dirichlet series and ℓ be its abscissa of absolute convergence. Let us suppose $\lambda < +\infty$. Let us put*

$$\widehat{f}(s) := \sum_{n\geq 1} \frac{f(n)}{n^s}, \quad \text{for } \Re e(s) > \lambda.$$

If for all real $s > \lambda$ one has $\widehat{f}(s) = 0$, then for all $n \geq 1$, $f(n) = 0$.

Proof. See ([4]). □

Theorem 4.3 (Uniqueness Theorem)**.** *Let f and $g : \mathbb{N}^* \to \mathbb{C}$ be two arithmetical functions. Let $\lambda(f)$ and $\lambda(g)$ theirs abscissas of convergence which are $< +\infty$. Let us put*

$$\widehat{f}(s) := \sum_{n\geq 1} \frac{f(n)}{n^s} \quad \text{and} \quad \widehat{g}(s) := \sum_{n\geq 1} \frac{g(n)}{n^s}.$$

If for all real $s > \max(\lambda(f), \lambda(g))$, one has $\widehat{f}(s) = \widehat{g}(s)$ (namely, $D(f;s) = D(g;s)$), then $f(n) = g(n)$ for all $n \geq 1$.

Proof. See ([3]). □

Corollary 4.1. *Let $D(f;s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$ be a Dirichlet series with abscissa of absolute convergence σ_{ac}. Suppose that for some s with $\Re e(s) > \sigma_{ac}$ we have $D(f;s) = 0$. Then there exists a half-plane in which $D(f;s)$ is absolutely convergent and never zero.*

Proof. By the absurdity, let us suppose that if is not, we have an infinite sequence $\{s_k\}$ of complex numbers, with real parts tending to infinity, such that $D(f;s_k) = 0$ for all k. By the Uniqueness Theorem this implies that $f(n) = 0$ for all n and thus $D(f;s)$ is identically zero in its half-plane of absolute convergence, contrary to our assumption. □

5. Dirichlet Algebra $\mathcal{A}$ and Convolution Product

5.1 Dirichlet Algebra

Let us indicate by $\mathcal{A}$ the set of arithmetical functions $f : \mathbb{N}^* \to \mathbb{C}$.

We have the following proposition:

Proposition 5.1. *$\mathcal{A}$ is a vector space, where the addition and the multiplication by a scalar are defined as follows:*

- *Let f, g be two elements of $\mathcal{A}$, then $f + g$ is an element of $\mathcal{A}$ defined by*

  $$(f+g)(n) = f(n) + g(n), \quad \forall\ n \in \mathbb{N}^*;$$

- *Let f be an element of $\mathcal{A}$, α be an element of $\mathbb{C}$, then αf is the element of $\mathcal{A}$ defined by*

  $$(\alpha f)(n) = \alpha f(n), \quad \forall\ n \in \mathbb{N}^*.$$

5.2 Convolution Product

Definition 5.1 (Convolution Product)**.** Let f and g be two elements of $\mathcal{A}$. We call *convolution product* or *Dirichlet product* of f by g and we note it $f * g$, the element $F = f * g$ of $\mathcal{A}$ defined as follows:

$$F(n) := (f * g)(n) = \sum_{k|n} f(k) g\left(\frac{n}{k}\right) = \sum_{kd=n} f(k) g(d),$$

where the last sum is extended to all the couples (k,d) of strictly positive integers such that $kd = n$.

Remark 5.1. If $g = 1$, one obtains the Möbius transform.

It results the following properties in the form of propositions:

Proposition 5.2. *The operation $*$ is an internal law of composition on $\mathcal{A}$, which is associative and commutative.*

Proof. • *Associativity of $*$:*
Associativeness results owing to the fact that each of these expressions

$$(f*g)*h \ \text{ and } \ f*(g*h)$$

has as a value,

$$(f*g*h)(n) = \sum_{k_1k_2k_3=n} f(k_1)g(k_2)h(k_3).$$

• *Commutativity of $*$:*
A commutativity results owing to the fact that if k runs the set of divisors of n, it is the same of $\frac{n}{k}$. □

Proposition 5.3 (Identity element)**.** *The law $*$ admits an identity element and only one, it is the element u of $\mathcal{A}$ defined by*

$$u(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

For all element $f \in \mathcal{A}$, one has then

$$u*f = f*u = f.$$

Proposition 5.4 (Inverse element)**.** *An element f of $\mathcal{A}$ is invertible for the law $*$ if there exists an element g of $\mathcal{A}$ such that*

$$f*g = g*f = u$$

(it is said, also, that f is then a "unit" or a divisor of the identity element u).

Proposition 5.5. *Let f be an element of $\mathcal{A}$. Then, the two following properties are equivalent:*
(p_1): *f is invertible;*
(p_2): $f(1) \neq 0$.

Proof. $(p_1) \to (p_2)$:
Let us suppose f invertible, hence there exists $g \in \mathcal{A}$ such that $f*g = u$. Then

$$(f*g)(1) = u(1) = 1 \Longleftrightarrow f(1)g(1) = 1,$$

what implies $f(1) \neq 0$.

$(p_2) \to (p_1)$:
Let us suppose that $f(1) \neq 0$, show that f is invertible, i.e, it exists an $g \in \mathcal{A}$ such that

$$f * g = u.$$

Indeed, looking at g as an unknown function, the relation

$$(f * g)(n) = u(n), \quad n \geq 1$$

makes it possible to determine by recurrence the numbers: $g(1), g(2), \dots$

$$\begin{cases} (f * g)(1) = u(1) = 1, & \text{if } n = 1, \\ (f * g)(n) = u(n) = 0, & \text{if } n > 1. \end{cases}$$

Namely,

$$\begin{cases} f(1)g(1) = 1 \\ f(1)g(n) + \sum\limits_{k|n,\ k \neq n} f(k)g(\frac{n}{k}) = 0, & \text{for } n > 1, \end{cases}$$

implies

$$\begin{cases} g(1) = \frac{1}{f(1)} \\ g(n) = -\frac{1}{f(1)} \sum\limits_{k|n,\ k \neq n} f(k)g(\frac{n}{k}), & \text{for } n > 1. \end{cases}$$

□

Theorem 5.1. *The set $\mathcal{A}$ provided with the three following laws:*

- *Addition:*$(f+g)(n) = f(n) + g(n), \quad \forall\ n \in \mathbb{N}^*, \quad \forall\ f, g \in \mathcal{A}$.
- *Multiplication by a scalar:*$(\lambda f)(n) = \lambda f(n)$*, for* λ *scalar,* $f \in \mathcal{A}$ *and* $n \in \mathbb{N}^*$.
- *Multiplication Dirichlet product:*$(f * g)(n)$*,* $f, g \in \mathcal{A}$ *and* $n \in \mathbb{N}^*$*, is a commutative algebra with identity element, and this algebra is called* Dirichlet Algebra*.*

Proof. Immediate checking of the three operations. □

6. Norm on $\mathcal{A}$ and Ultrametric Norm on $\mathcal{A}$

6.1 Norm on $\mathcal{A}$

Definition 6.1. Let $f : \mathbb{N}^* \to \mathbb{C}$ be an arithmetical function.

(a) We call *order* of f the number $\langle f \rangle \in [0, +\infty]$ defined by

$$\langle f \rangle := \begin{cases} \inf\{n : f(n) \neq 0\}, & \text{if } f \neq 0, \\ +\infty, & \text{if } f = 0. \end{cases}$$

(b) We call *norm* of f the number $\|f\| \in [0, +\infty]$ defined by

$$\|f\| := \frac{1}{\langle f \rangle}$$

and naturally one puts $\frac{1}{\infty} = 0$.

6.2 Space $\mathcal{A}$ as Ultrametric Space Applications

Theorem 6.1. *We have the following properties:*

(p_1): $\|f\| \geq 0$;

(p_2): $\|f\| = 0 \Longleftrightarrow f = 0$;

(p_3): $\|f + g\| \leq \sup(\|f\|, \|g\|)$;

(p_4): $\|f * g\| = \|f\| \cdot \|g\|$.

Proof. We have

(p_3): Let us show that $\|f + g\| \leq \sup(\|f\|, \|g\|)$.

Let us put

$$k_0 = \langle f \rangle,\ \ell_0 = \langle g \rangle,\ n_0 = \langle f + g \rangle.$$

It is enough to show that

$$\langle f + g \rangle \geq \inf(\langle f \rangle, \langle g \rangle).$$

Let us reason by the absurdity, i.e., let us suppose that

$$\langle f + g \rangle < \inf(\langle f \rangle, \langle g \rangle) \text{ is equivalent to } n_0 < \Big|_{\ell_0}^{k_0}.$$

Then one will have

$$0 \neq (f + g)(n_0) = f(n_0) + g(n_0) = 0,$$

(because $f(n_0) = 0$, $g(n_0) = 0$), from where a contradiction and one has the inequality.

(p_4): Let us show that $\|f * g\| = \|f\| \cdot \|g\|$

Let us put

$$k_0 = \langle f \rangle,\ \ell_0 = \langle g \rangle,\ n_0 = \langle f * g \rangle.$$

(a) Let us show that we have $\langle f * g \rangle \geq \langle f \rangle \langle g \rangle$

By the absurdity, let us suppose that this does not take place, i.e.,

$$\langle f * g \rangle < \langle f \rangle \langle g \rangle \text{ is equivalent to } \ n_0 < k_0 \ell_0.$$

Then one has

$$0 \neq (f * g)(n_0) = \sum_{k\ell = n_0} f(k) g(\ell) = 0$$

(n_0 the small value where $*$ of f and g is not cancelled) (had with the fact that $k \mid n_0$, $\ell = \frac{n_0}{k}$ so that $\frac{n_0}{\ell_0} < k_0$) what is contradictory.

(b) It is wanted that $\langle f * g \rangle \leq \langle f \rangle \langle g \rangle$

Indeed, one has

$$(f * g)(k_0 \ell_0) = \sum_{k\ell = k_0 \ell_0} f(k) g(\ell) = f(k_0) g(\ell_0) \neq 0,$$

from where, $k_0\ell_0 \geq n_0$.

(c) The parts (a) and (b) above imply

$$\langle f * g\rangle = \langle f\rangle\langle g\rangle. \qquad \square$$

Remark 6.1. It holds from (p_2) and (p_4) that if $f \neq 0$, $g \neq 0$, then $f * g \neq 0$.

Definition 6.2. The ultrametric norm $\|\cdot\|$ on algebra $\mathcal{A}$ makes it possible to define a metric on $\mathcal{A}$, noted

$$\rho(f,g) := \|f-g\|, \quad f,g \in \mathcal{A}.$$

Proposition 6.1. *The Dirichlet algebra $\mathcal{A}$ is complete for the metric ρ.*

Proof. See ([3]). $\square$

Proposition 6.2. *Let (f_n) be a sequence of elements of $\mathcal{A}$. If $\|f_n\| \to 0$, as $(n \to +\infty)$, then $\sum\limits_{n\geq 1} f_n$ converges compared to ρ.*

Proof. Let us suppose that $\|f_n\| \to 0$, for all $\varepsilon > 0$, it exists $N(\varepsilon) > 0$, such that for all $n \geq N$, then $\|f_n\| < \varepsilon$.

Let us put

$$S_n = \sum_{k=1}^{n} f_k.$$

One has for all N, for all $k > 0$,

$$\begin{aligned}\|S_{N+k} - S_N\| &= \|f_{N+1} + \ldots + f_{N+k}\| \\ &\leq \sup(\|f_{N+1}\|, \ldots, \|f_{N+k}\|) \\ &\underset{\text{norm}}{\overset{\text{ulta.}}{\leq}} \varepsilon.\end{aligned}$$

(This inequality is not true with the ordinary norms). From where (S_n) is of Cauchy, as space $\mathcal{A}$ is complete then (S_n) has a limit. $\square$

Example 6.1. One has

$$\sum_{k|n} \mu(k) = u(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

By introducing the arithmetic function "1" (the mapping $(\mathbb{N}^* \to 1)$), one seen that,

$$1 * \mu = \mu * 1 = u.$$

The arithmetic functions 1 and μ are inverse one of the other for the law $*$ in the Dirichlet algebra.

As application we have the following proposition:

Proposition 6.3 (Möbius reciprocity formula)**.** *Let $f:\mathbb{N}^*\to\mathbb{C}$ be such that*

$$F(n)=\sum_{k|n} f(k).$$

The Möbius transform of f can be written

$$F=1*f \tag{6.1}$$

and

$$f=\mu*F. \tag{6.2}$$

Proof. To show the expression (6.2) let us multiply the expression (6.1) on the left by μ inverse of 1 then

$$\mu*F=\mu*1*f=u*f=f,$$

from where

$$f=\mu*F$$

and reciprocally from of deduced the expression (6.1) starting from the expression (6.2) while multiplying on the left by 1.

It results the Möbius reciprocity formulas:

$$\begin{cases} F=1*f, \\ f=\mu*F \end{cases} \Longleftrightarrow \begin{cases} F(n)=\sum\limits_{k|n} f(k), \\ f(n)=\sum\limits_{k|n}\mu(k)F\left(\frac{n}{k}\right). \end{cases}$$

□

7. Dirichlet Transform

Let $f:\mathbb{N}^*\to\mathbb{C}$ be an arithmetical function. Associate to this last a Dirichlet series

$$\sum_{n\geq 1}\frac{f(n)}{n^s}, \quad s\in\mathbb{C}$$

whose we will denote the abscissa of convergence $\lambda(f)$ and the abscissa of absolute convergence $\ell(f)$.

Introduce then $\mathcal{A}$ a class of arithmetical functions such that $\lambda(f)<+\infty$. So for $f\in\mathcal{A}$, a Dirichlet series

$$\sum_{n\geq 1}\frac{f(n)}{n^s}, \quad s\in\mathbb{C}$$

is convergent for $\Re e(s)>\lambda(f)$, and divergent for $\Re e(s)>\lambda(f)$. It represents a holomorphic function of a complex variable s in the a half-plane $\Re e(s)>\lambda(f)$ like that $\mathcal{A}$ equipped with addition process, multiplication by a scalar and convolution product $*$, ($\mathcal{A}$, $+$, *multiplication by a scalar*, $*$) is an algebra of arithmetical functions and which is a sub-algebra of Dirichlet's algebra.

Next we introduce a class denoted $\mathfrak{C}$ of functions of complex variable s, defined on a half-plane $\Re e(s) > a$ where $a \in [-\infty, +\infty]$.

$\mathfrak{C}$ equipped with operations $+$, multiplication by a scalar, ordinary product, ($\mathfrak{C}$, $+$, *multiplication by a scalar*, $\cdot$) is an algebra called functions algebra.

Definition 7.1 (Dirichlet Transform)**.** We call Dirichlet transform a mapping $\wedge : \mathcal{A} \to \mathfrak{C}$ which to an element $f \in \mathcal{A}$ associates a function $\widehat{f} \in \mathfrak{C}$ defined by

$$\widehat{f}(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}, \quad \Re e(s) > \lambda(f).$$

The function $\widehat{f} \in \mathfrak{C}$ is said *Dirichlet transform* of f.

We have the following property:

Proposition 7.1. *A mapping $\wedge$ is injective if, and only if, f and $g \in \mathcal{A}$ and if $\widehat{f} = \widehat{g}$ then $f = g$.*

Proof. The proof is a consequence of uniqueness theorem of Dirichlet series (see ([4, Theorem 3.3])). □

8. Dirichlet Transform as a Homomorphism of Algebra

We have

Theorem 8.1. *The Dirichlet transform of convolution product of two elements of $\mathcal{A}$ is equal to the ordinary product of Dirichlet transform of these two elements. More precisely, let $f, g \in \mathcal{A}$, let us put $h = f * g$, then we have*

(a) $\ell(h) \leq \max\{\ell(f), \ell(g)\} < +\infty$, *hence $h \in \mathcal{A}$;*

(b) $\widehat{h}(s) = \widehat{f}(s)\widehat{g}(s)$, *for $\Re e(s) > \max\{\ell(f), \ell(g)\}$.*

In short cut, one has

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

Proof. Formally, we have (one does not deal with question of convergence)

$$\widehat{f}(s) \cdot \widehat{g}(s) = \left(\sum_{k \geq 1} \frac{f(k)}{k^s} \right) \left(\sum_{m \geq 1} \frac{g(m)}{m^s} \right) = \sum_{k,m \geq 1} \frac{f(k)g(m)}{(k\ m)^s}.$$

Hence looking terms of same denominator, namely, in fact summing at km constant, we have

$$\widehat{f}(s) \cdot \widehat{g}(s) = \sum_{n \geq 1} \frac{1}{n^s} \left(\sum_{km=n} f(k)g(m) \right) = \sum_{n \geq 1} \frac{h(n)}{n^s} = \widehat{h}(s).$$

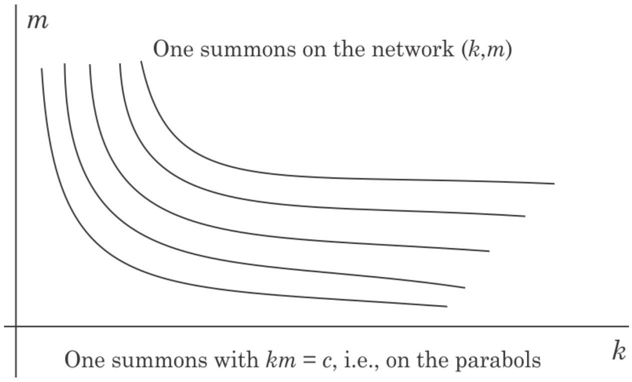

Figure 4

If everyone of series is convergent (absolutely convergent), namely, $h < +\infty$, hence the theorem results. □

Theorem 8.2. *The Dirichlet transform* $\wedge : \mathcal{A} \to \mathfrak{C}$ *is an homomorphism of algebra from algebra* $(\mathcal{A}, +,$ *multiplication by a scalar,* $*)$ *into algebra* $(\mathfrak{C}, +,$ *multiplication by a scalar,* $\cdot)$*. One has*

(a) $\widehat{f+g} = \widehat{f} + \widehat{g}$;

(b) $\widehat{\alpha f} = \alpha \widehat{f}$, $\alpha \in \mathbb{N}$;

(c) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

9. Dirichlet Transform of Möbius Transform

We have

Theorem 9.1. *Let* $f \in \mathcal{A}$ *and* F *be its Möbius transform* $F = 1 * f$*. Then*

$$\widehat{F}(s) = \zeta(s) \cdot \widehat{f}(s), \quad \Re e(s) > \max\{\ell(f), 1\}. \tag{9.1}$$

Proof. Apply Theorem 8.1, then we have

$$\widehat{1}(s) = \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s), \quad \ell(1) = 1$$

and

$$\widehat{F} = \widehat{1} \cdot \widehat{f}.$$

□

10. Probabilistic Interpretation

Interpret the expression (9.1) above in probabilistic meaning.

Let us suppose s real $> \max\{\ell(f), 1\}$, we have

$$\frac{1}{\zeta(s)} F(s) = \widehat{f}(s),$$

namely

$$\frac{1}{\zeta(s)}\sum_{n\geq 1}\frac{F(n)}{n^s}=\widehat{f}(s)$$

and the mathematical expectation $E_s(F)$ of F is

$$E_s(F)=\widehat{f}(s).$$

For the remainder of interpretation (see [1]).

11. Comments

(a) Let us take again Theorem 3.1, then one has the following comments:

(c_1) Let us consider the case $-\infty<\lambda<\ell<+\infty$.

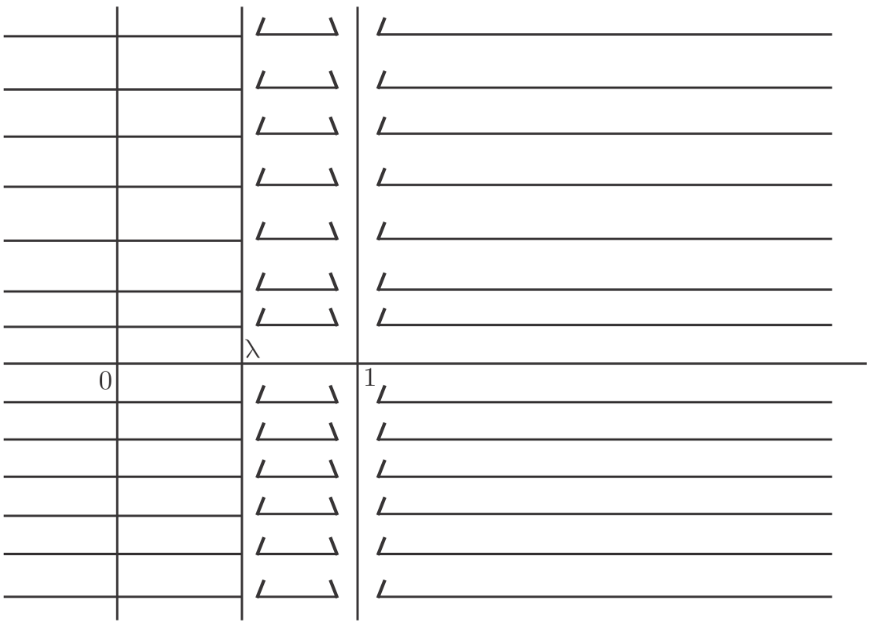

Figure 5

To the half-plane of divergence $\Re e(s)>1$ succeeds the band of semi-convergence $\lambda<\Re e(s)<\ell$ whose width is with most equal to 1 and it even followed by the band of absolute convergence $\Re e(s)>1$.

(c_2) The width $\ell-\lambda$ of the band of semi-convergence can take any value understood in the interval $[0,1]$.

(1) If f is with positive values and if λ is finite then $\ell=\lambda$, namely, $\ell-\lambda=0$.

(2) Let us consider

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^s}.$$

(a) $\left.\begin{array}{l}\text{This series converges for any } s \text{ real strictly positive (alternate series)}\\ \text{In addition, the series diverges for any } s \text{ real} < 0.\end{array}\right\} \to \lambda = 0;$

(b) $\left.\begin{array}{l}\text{This series is absolutely convergent for any } s \text{ real} > 1.\\ \text{It is not absolutely convergent for } s \text{ real} < 1.\end{array}\right\} \to \ell = 1.$

The band of semi-convergence is of width 1 (thus is maximum here), $\ell - \lambda = 1$.

(c_3) λ (and ℓ) can take any value $\in [-\infty, +\infty]$.

(c_4) The (fundamental) Theorem 3.1 does not affirm anything on the behavior series on the line of convergence $\Re e(s) = \lambda$.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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