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Research Article

Isomorphism Theorems on Intuitionistic Fuzzy Abstract Algebras

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Abstract. The concept of abstract algebra on intuitionistic fuzzy sets were introduced and some basic theorems were proved by authors in 2017. In this study, homomorphism between intuitionistic fuzzy abstract algebras is defined, intuitionistic fuzzy function is examined and then intuitionistic fuzzy congruence relations are defined on intuitionistic fuzzy abstract algebra. First and third isomorphism theorems on intuitionistic abstract algebras are introduced.

Keywords. Intuitionistic fuzzy sets; Intuitionistic fuzzy abstract algebra; Intuitionistic fuzzy function; Intuitionistic fuzzy isomorphism theorems

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1. Introduction

Theory of fuzzy sets was introduced as an extension of crisp sets by Zadeh [21]. As a natural continuation of this study, the generalization of fuzzy set theory called intuitionistic fuzzy set theory was propounded by Atanassov [1]. Intuitionistic fuzzy sets have a widespread application areas [10, 17, 19]. Main applications are; intuitionistic fuzzy expert systems, intuitionistic fuzzy neural networks, intuitionistic fuzzy generalized nets etc.

Algebraic structures firstly generalized to intuitionistic fuzzy sets by Biswas [3]. He defined intuitionistic fuzzy group in 1989 then Zhan and Than [23] introduced intuitionistic M-fuzzy groups. In 2008, the concept of intuitionistic fuzzy rings was propounded by Yan [20]. Palaniappan *et al*. [16] studied intuitionistic L-fuzzy subgroups in 2009. Intuitionistic fuzzy semigroups were examined by Melliani *et al*. [13]. In the following years, several authors had studied on the different intuitionistic fuzzy algebraic structures.

Abstract algebra (or algebra) is a set with finitary operations defined on it. By working on universal algebra, the common properties of algebraic structures can be examined. Thus, it reveals the connection between the concepts that seem to be different in a certain systematic way. The concept of universal algebra on fuzzy set theory was studied by Murali in 1987 [14]. In 2017, authors introduced intuitionistic fuzzy abstract algebra and they obtained some basic properties of intuitionistic fuzzy abstract algebra [9].

In this study, we introduced the fundamental concepts on intuitionistic fuzzy abstract algebras, these are intuitionistic fuzzy homomorphism, intuitionistic fuzzy congruence relation on intuitionistic fuzzy abstract algebra, isomorphism theorems on intuitionistic fuzzy abstract algebras.

2. Preliminaries

Atanassov [1] introduced the intuitionistic fuzzy set theory in 1983 as an extension of fuzzy sets by enlarging the truth value set to the lattice $[0,1]\times[0,1]$ is defined as following.

Definition 2.1. Let $L=[0,1]$ then

$$L^*=\{(x_1,x_2)\in[0,1]^2 : x_1+x_2\le 1\}$$

is a lattice with $(x_1,x_2)\le(y_1,y_2) :\Longleftrightarrow x_1\le y_1$ and $x_2\ge y_2$.
For $(x_1,y_1),(x_2,y_2)\in L^*$, the operators $\wedge$ and $\vee$ on $(L^*,\le)$ are defined as following:

$$(x_1,y_1)\wedge(x_2,y_2)=(\min(x_1,x_2),\max(y_1,y_2)),$$

$$(x_1,y_1)\vee(x_2,y_2)=(\max(x_1,x_2),\min(y_1,y_2)).$$

For each $J\subseteq L^*$

$$\sup J=(\sup\{x:(x,y\in[0,1]),((x,y)\in J)\},\inf\{y:(x,y\in[0,1])((x,y)\in J)\})$$

and

$$\inf J=(\inf\{x:(x,y\in[0,1])((x,y)\in J)\},\sup\{y:(x,y\in[0,1])((x,y)\in J)\}).$$

Definition 2.2 ([1])**.** An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in X\},$$

where $\mu_A(x)$, ($\mu_A : X\to[0,1]$) is called the *degree of membership of x in A*, $\nu_A(x)$, ($\nu_A : X\to[0,1]$) is called the "degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x)+\nu_A(x)\le 1,\quad \text{for all } x\in X.$$

The hesitation degree of x is defined by $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$.

Also, some operations on intuitionistic fuzzy sets were defined by Atanassov [1].

Definition 2.3 ([1]). Let X be a universal and $A, B \in \mathrm{IFS}(X)$. A is said to be contained in an B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 2.4 ([1]). Let X be a universal and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in \mathrm{IFS}(X)$ then the above set is called the complement of A

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

Definition 2.5 ([1]). Let X be a universal and $A, B \in \mathrm{IFS}(X)$. Then

(1) $A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle : x \in X\}$,

(2) $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle : x \in X\}$,

(3) $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$,

(4) $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$.

Level sets have an important role on fuzzy sets and on intuitionistic fuzzy sets. Similarly, in this study the level sets will be used intensively.

Definition 2.6 ([1]). Let $A \in \mathrm{IFS}(X)$. Then (r, s)-cut and strong (r, s)-cut of A are crisp subsets $A_{(r,s)}$ and $A_{\langle r,s\rangle}$ of the X, respectively are given by

$$A_{(r,s)} = \{x : x \in X \text{ such that } \mu_A(x) \geq r,\ \nu_A(x) \leq s\},$$

$$A_{\langle r,s\rangle} = \{x : x \in X \text{ such that } \mu_A(x) > r,\ \nu_A(x) < s\},$$

where $r, s \in [0, 1]$ with $r + s \leq 1$.

In 1995, Burille and Bustince [5] defined the intuitionistic fuzzy relations and they studied some properties of intuitionistic fuzzy relations.

Definition 2.7 ([4]). An intuitionistic fuzzy relation(shortly IFR) is an intuitionistic fuzzy subset of $X \times Y$, that is, is an expression R given by $R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle : x \in X, y \in Y\}$, where $\mu_R : X \times Y \to [0, 1]$, $\nu_R : X \times Y \to [0, 1]$ satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for any $(x, y) \in X \times Y$.

Definition 2.8 ([5]). Let X be a non-empty set and $R \in \mathrm{IFR}(X)$.

ER1: For every $x \in X$,

$$\mu_R(x, x) = 1,$$

$$\nu_R(x, x) = 0$$

then R is called an intuitionistic fuzzy reflexive.

ER2: For every $x, y \in X$,

$$\mu_R(x, y) \leq \mu_R(y, x),$$

$$\nu_R(x, y) \geq \nu_R(y, x)$$

then R is called an intuitionistic fuzzy symmetric.

ER3: For every $x, y, z \in X$,

$$\mu_R(x,y) \wedge \mu_R(y,z) \leq \mu_R(x,z),$$
$$\nu_R(x,y) \vee \nu_R(y,z) \geq \nu_R(x,z)$$

then R is called an intuitionistic fuzzy transitive.

If an intuitionistic fuzzy relation satisfies the previous properties then it is called an intuitionistic fuzzy equivalence relation (IFE(X)). Now, we can mention about intuitionistic fuzzy equivalence classes of R.

Definition 2.9 ([12])**.** Let X be a non-empty set, $R \in \mathrm{IFE}(X)$ and $a \in X$.

$$[a]_R = \{\langle x, \mu_{[a]_R}(x), \nu_{[a]_R}(x)\rangle : x \in X\},$$

where $\mu_{[a]_R}(x) = \mu_R(a,x), \nu_{[a]_R}(x) = \nu_R(a,x)$ is called an intuitionistic fuzzy equivalence class of a with respect to R.

Theorem 2.10 ([12])**.** *Let X be a non-empty set and $R \in \mathrm{IFR}(X)$. Then $R \in \mathrm{IFE}(X)$ if and only if $R_{(r,s)}$ is a equivalence relation on X for each $r, s \in [0,1]$ with $r + s \leq 1$.*

Level sets of intuitionistic fuzzy equivalence relations were studied by different authors. Here, we will use following definition for equivalence classes.

Definition 2.11 ([8])**.** Let X be a non-empty set and $R \in \mathrm{IFE}(X)$. Let $a \in X$ and $r, s \in [0,1]$ and $r + s \leq 1$;

(1) ${}_r[a]_R = \{x \in X : \mu_{[a]_R}(x) = \mu_R(a,x) \geq r\}$,

(2) ${}^s[a]_R = \{x \in X : \nu_{[a]_R}(x) = \nu_R(a,x) \leq s\}$,

(3) ${}^s_r[a]_R = \{x \in X : \mu_{[a]_R}(x) = \mu_R(a,x) \geq r, \nu_{[a]_R}(x) = \nu_R(a,x) \leq s\}$.

For each $a \in X$, ${}^s_r[a]_R$ denotes the crisp equivalence class containing a with respect to $R_{(r,s)}$.

The extension of functions on intuitionistic fuzzy sets is given as follow:

Definition 2.12 ([11])**.** Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let $A \in \mathrm{IFS}(X)$ and $B \in \mathrm{IFS}(Y)$. Then f is extended to a mapping from IFS(X) to IFS(Y) as

$$f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y)),$$

where

$$\mu_{f(A)}(y) = \begin{cases} \bigvee\{\mu_A(x) : x \in f^{-1}(y)\} \\ 0; \quad \text{otherwise} \end{cases}$$

and

$$\nu_{f(A)}(y) = \begin{cases} \bigwedge\{\nu_A(x) : x \in f^{-1}(y)\} \\ 1; \quad \text{otherwise} \end{cases}$$

$f(A)$ is called the image of A under the map f. Also, the pre-image of B under f is denoted by $f^{-1}(B)$ and defined as

$$f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)),$$

where

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \text{ and } \nu_{f^{-1}(B)}(x) = \nu_B(f(x)).$$

The concept of abstract algebra in crisp set theory defined as follow:

Definition 2.13 ([2])**.** An abstract algebra (or an algebra) A is a pair $[S,F]$ where S is a non-empty set and F is a specified set of operations f_α, each mapping a power $S^{n(\alpha)}$ of S into S, for some appropriate nonnegative finite integer $n(\alpha)$.

Otherwise stated, each operation f_α assigns to every $n(\alpha)$-ple $(x_1,\dots,x_{n(\alpha)})$ of elements of S, a value $f_\alpha(x_1,\dots,x_{n(\alpha)})$ in S, the result of performing the operation f_α on the sequence $x_1,\dots,x_{n(\alpha)}$. If $n(\alpha)=1$, the operation f_α is called unary; if $n(\alpha)=2$, it is called binary; if $n(\alpha)=3$, it is called ternary, etc. When $n(\alpha)=0$, the operation f_α is called nullary; it selects a fixed element of S.

$A=[S,F]$ and $B=[T,F']$ are called similar algebras if F and F' for each α the types of f_α and f'_α are same.

Definition 2.14 ([2])**.** Let $A=[S,F]$ and $B=[T,F']$ be two similar algebras. A function $\varphi: S\to T$ is called a homomorphism of A into B if and only if for all $f_\alpha \in F$ and $x_i \in S$, $i=1,2,\dots,n(\alpha)$,

$$f'_\alpha(\varphi(x_1),\varphi(x_2),\dots,\varphi(x_{n(\alpha)})) = \varphi(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})).$$

Fuzzy abstract algebra introduced by Murali [14,15] using Zadeh's extension principle [22]. Fuzzy subalgebras and homomorphism between fuzzy algebras defined by same author. The extension of the concept of abstract algebra to intuitionistic fuzzy sets defined by authors [9] as follows:

Definition 2.15 ([9])**.** Let $S=[X,F]$ be an algebra where X is a non-empty set and F is a specified set of finite operations f_α, each mapping a power $X^{n(\alpha)}$ of X into X, for some appropriate nonnegative finite integer $n(\alpha)$. For each f_α, a corresponding operation ω_{f_α} on IFS(X) as follows:

$$\omega_{f_\alpha}: \ \mathrm{IFS}(X)\times\mathrm{IFS}(X)\times\dots\times\mathrm{IFS}(X)\to\mathrm{IFS}(X),\ \omega_{f_\alpha}(A_1,A_2,\dots,A_{n(\alpha)})=A$$

such that

$$A(x)=\begin{cases}\sup(A_1(x_1)\wedge A_2(x_2)\wedge\dots\wedge A_{n(\alpha)}(x_{n(\alpha)})), & f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})=x\\ \Theta=(0,1), & \text{otherwise.}\end{cases}$$

Shortly, it will be shown $A=\omega_{f_\alpha}(A_1,A_2,\dots,A_{n(\alpha)})$. Let $\Omega=\{\omega_{f_\alpha}$: corresponding operation for each $f_\alpha\in F\}$ then $L=[(I\times I)^X,\Omega]$ is called intuitionistic fuzzy abstract algebra (or intuitionistic fuzzy algebra).

If $n(\alpha)=0$ then $f_\alpha(x)=e$ that e is a fixed element of X. So, ω_{f_α} is defined as following:

$$\omega_{f_\alpha}(A)=A_e,$$

$$A_e(x)=\begin{cases}\sup\limits_{x\in X}A(x), & x=e\\ (0,1), & x\neq e.\end{cases}$$

Example 2.16. A group $S=[G,F]$ is an algebra where $F=\{\cdot,e\}$ include one binary operation and one nullary operation respectively. Let $L=[\mathrm{IFS}(G),\Omega]$ and $A_1,A_2\in\mathrm{IFS}(G)$, $x,x_1,x_2\in G$ then with corresponding operations defined as follow:

$$A_1A_2(x)=(\mu_{A_1}\mu_{A_2}(x),\nu_{A_1}\nu_{A_2}(x))$$

such that

$$\mu_{A_1}\mu_{A_2}(x)=\sup_{x=x_1x_2}(\mu_{A_1}(x_1)\wedge\mu_{A_2}(x_2)),$$

$$\nu_{A_1}\nu_{A_2}(x)=\inf_{x=x_1x_2}(\nu_{A_1}(x_1)\vee\nu_{A_2}(x_2))$$

L is intuitionistic fuzzy algebra.

Definition 2.17 ([9])**.** Let X be a non-empty set and $A\in\mathrm{IFS}(X)$. A is called an intuitionistic fuzzy subalgebra (IF-subalgebra) of $L=[\mathrm{IFS}(X),\Omega]$ intuitionistic fuzzy algebra if and only if for nonnegative finite integer $n(\alpha),\omega_{f_\alpha}(A,A,\dots,A)\le A$, for every ω_{f_α}.

Theorem 2.18 ([9])**.** *Let $S=[X,F]$ be an algebra, $f_\alpha\in F$ and $A,A_1,A_2,\dots,A_{n(\alpha)}$ be* IF-*subalgebras.* $\omega_{f_\alpha}(A_1,A_2,\dots,A_{n(\alpha)})\sqsubseteq A$ *if and only if* $A(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}))\ge\min_{1\le i\le n(\alpha)}A_i(x_i)$ *is true for every* $(x_1,x_2,\dots,x_{n(\alpha)})\in X^{n(\alpha)}$.

Example 2.19. Let G be a group. $A\in\mathrm{IFS}(G)$ intuitionistic fuzzy subgroup defined as follow: for all $x,y\in G$,

$$A(xy)\ge A(x)\wedge A(y),$$

$$A(x^{-1})\ge A(x)$$

that is,

$$\mu_A(xy)\ge\mu_A(x)\wedge\mu_A(y)\ \text{ and }\ \nu_A(xy)\le\nu_A(x)\vee\nu_A(y),$$

$$\mu_A(x^{-1})\ge\mu_A(x)\ \text{ and }\ \nu_A(x^{-1})\le\nu_A(x).$$

This definition coincides with [16].

Theorem 2.20 ([9])**.** *Let $L=[\mathrm{IFS}(X),\Omega]$ be an IF-algebra. If $\{A_i\}_{i\in\Lambda}$ is a family of* IF-*subalgebras of L then*

$$A=\bigcap_{i\in\Lambda}A_i$$

is a IF-*subalgebra of L.*

3. Main Results

In this section, firstly, the generalization of homomorphisms between given algebras to intuitionistic fuzzy algebras are defined. Intuitionistic fuzzy algebras are examined under these homomorphisms, intuitionistic fuzzy function and intuitionistic fuzzy homomorphism are introduced by intuitionistic fuzzy relations satisfying certain properties. In the last part, intuitionistic fuzzy isomorphism theorems are introduced.

Proposition 3.1. *Let $S = [X,F]$ be an algebra. If A is an* IF*-subalgebra of $L = [\mathrm{IFS}(X),\Omega]$ then so are $\Box A$ and $\Diamond A$.*

Proof. If A is an IF-subalgebra then $\omega_{f_\alpha}(A,A,\dots,A) \sqsubseteq A$ for all ω_{f_α}. So,

$$\sup_{f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})=x}\left(\min_{1\le i\le n(\alpha)}\mu_A(x_i)\right) \le \mu_A(x)$$

and

$$\inf_{f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})=x}\left(\max_{1\le i\le n(\alpha)} 1-\mu_A(x_i)\right) = 1-\left(\sup_{f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})=x}\left(\min_{1\le i\le n(\alpha)}\mu_A(x_i)\right)\right) \ge 1-\mu_A(x),$$

$$\omega_{f_\alpha}(\Box A,\Box A,\dots,\Box A)(x) = \begin{pmatrix} \sup\limits_{f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})=x}\left(\min\limits_{1\le i\le n(\alpha)}\mu_A(x_i)\right), \\ \inf\limits_{f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})=x}\left(\max\limits_{1\le i\le n(\alpha)} 1-\mu_A(x_i)\right) \end{pmatrix} \le \Box A.$$

$\Diamond A$ can be proved similarly. □

Proposition 3.2. *Let $S = [X,F]$, $T = [Y,F']$ be two similar algebras and φ be a homomorphism of S into T. The extension of φ from* IF*-algebra $L = [\mathrm{IFS}(X),\Omega]$ to* IF*-algebra $K = [\mathrm{IFS}(Y),\Omega']$ is a homomorphism of intuitionistic fuzzy algebras L to K.*

Proof. Let $Z = \{\varphi(x) : x \in X\}$. It is clear that $[Z,F']$ is a subalgebra of T.

Now, we can suppose that

$$\omega_{f_\alpha}(A_1,A_2,\dots,A_{n(\alpha)}) = A, \quad \text{where } A_1,A_2,\dots,A_{n(\alpha)},A \in \mathrm{IFS}(X)$$

and

$$\omega_{f_\alpha}(\varphi(A_1),\varphi(A_2),\dots,\varphi(A_{n(\alpha)})) = B, \quad B \in \mathrm{IFS}(Y).$$

We will prove that $\varphi(A)(y) = B(y)$ for all $y \in Y$.

(I): If $y \notin Z$ then $\varphi(A)(y) = \Theta = (0,1)$. That is, if $y = f_\alpha(y_1,y_2,\dots,y_{n(\alpha)})$ then $\exists\ y_i$, $i = 1,2,\dots,n(\alpha)$ such that $y_i \notin Z$.

Let for some j with $1 \le j \le n(\alpha)$, $y_j \notin Z$. So, $\varphi(A)(y_j) = \Theta$.

Furthermore,

$$\varphi(A_1)(y_1) \wedge \varphi(A_2)(y_2) \wedge \dots \wedge \varphi(A_{n(\alpha)})(y_{n(\alpha)}) = \Theta$$

and

$$\begin{aligned} B(y) &= \omega_{f_\alpha}(\varphi(A_1),\varphi(A_2),\dots,\varphi(A_{n(\alpha)}))(y) \\ &= \Big(\sup_{y=f_\alpha(y_1,y_2,\dots,y_{n(\alpha)})}\{\mu_{\varphi(A_1)}(y_1) \wedge \mu_{\varphi(A_2)}(y_2) \wedge \dots \wedge \mu_{\varphi(A_{n(\alpha)})}(y_{n(\alpha)})\}, \\ &\qquad \inf_{y=f_\alpha(y_1,y_2,\dots,y_{n(\alpha)})}\{\nu_{\varphi(A_1)}(y_1) \wedge \nu_{\varphi(A_2)}(y_2) \wedge \dots \wedge \nu_{\varphi(A_{n(\alpha)})}(y_{n(\alpha)})\}\Big) \\ &= \Theta. \end{aligned}$$

So, $\varphi(A)(y) = B(y)$.

(II): If $y \in Z$ then $\varphi(A)(y) = \sup\limits_{y=\varphi(x)} A(x) = (a_1,a_2)$. Without losing generality, let $(a_1,a_2) > \Theta$.

Given small enough $\epsilon, \epsilon' > 0$ such that $(a_1-\epsilon, a_2+\epsilon') < (a_1, a_2)$, there exist $x \in X$ with $\varphi(x) = y$ and $A(x) > (a_1-\epsilon, a_2+\epsilon')$. So,

$$\omega_{f_\alpha}(A_1, A_2, \ldots, A_{n(\alpha)})(x) = A(x) > (a_1-\epsilon, a_2+\epsilon').$$

Therefore, $x_1, \ldots, x_{n(\alpha)} \in X$ such that $f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) = x$ and

$$A_1(x_1) \wedge A_2(x_2) \wedge \ldots \wedge A_{n(\alpha)}(x_{n(\alpha)}) > (a_1-\epsilon, a_2+\epsilon').$$

Because φ is a homomorphism,

$$\begin{aligned} y &= \varphi(x) = \varphi(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})) \\ &= f_\alpha(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{n(\alpha)})). \end{aligned}$$

Hence,

$$\begin{aligned} B(y) &= \omega_{f_\alpha}(\varphi(A_1), \varphi(A_2), \ldots, \varphi(A_{n(\alpha)}))(y) \\ &\geq \varphi(A_1)(\varphi(x_1)) \wedge \varphi(A_2)(\varphi(x_2)) \wedge \ldots \wedge \varphi(A_{n(\alpha)})(\varphi(x_{n(\alpha)})) \\ &\geq A_1(x_1) \wedge A_2(x_2) \wedge \ldots \wedge A_{n(\alpha)}(x_{n(\alpha)}) \\ &> (a_1-\epsilon, a_2+\epsilon'). \end{aligned}$$

Since ϵ, ϵ' are small enough, we obtain that $B(y) \geq (a_1, a_2) = \varphi(A)(y)$.

Let $y_1, y_2, \ldots, y_{n(\alpha)} \in Y$ with $y = f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)})$ and $B(y) = (c_1, c_2)$. Then, given small enough $\delta, \delta' > 0$ such that $(c_1-\delta, c_2+\delta') < (c_1, c_2)$. Now,

$$(c_1-\delta, c_2+\delta') < \varphi(A_1)(y_1) \wedge \varphi(A_2)(y_2) \wedge \ldots \wedge \varphi(A_{n(\alpha)})(y_{n(\alpha)}).$$

If $B(y) = 0$ then $\varphi(A)(y) \geq B(y)$.

Let $0 < B(y)$.

$$(c_1-\delta, c_2+\delta') < A_1(x_1) \wedge A_2(x_2) \wedge \ldots \wedge A_{n(\alpha)}(x_{n(\alpha)}).$$

If we use $\varphi(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})) = f_\alpha(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{n(\alpha)}))$ and $\varphi(A)(y) > (c_1-\delta, c_2+\delta')$ then $\varphi(A)(y) \geq (c_1, c_2)$ since δ, δ' are small enough. Therefore, $\varphi(A)(y) = B(y)$, for all $y \in Z$. □

Theorem 3.3. *Let $S = [X, F]$, $T = [Y, F']$ be two similar algebras and φ be a homomorphism of S into T. Then φ extends to a homomorphism between intuitionistic fuzzy algebras $L = [\mathrm{IFS}(X), \Omega]$ and $K = [\mathrm{IFS}(Y), \Omega']$. If A is an intuitionistic fuzzy subalgebra of L then $\varphi(A)$ is an intuitionistic fuzzy subalgebra of K. On the other hand, if B is an intuitionistic fuzzy subalgebra of K then $\varphi^{-1}(B)$ is an intuitionistic fuzzy subalgebra of L.*

Proof. Since A is an intuitionistic fuzzy subalgebra of L,

$$\omega_{f_\alpha}(A, A, \ldots, A) \sqsubseteq A, \quad \text{for all } f_\alpha \in F.$$

Also, we know that $\varphi(\omega_{f_\alpha}(A, A, \ldots, A)) \leq \varphi(A)$. So we obtain that

$$\begin{aligned} \varphi(\omega_{f_\alpha}(A, A, \ldots, A)) &= \omega_{f_\alpha}(\varphi(A), \varphi(A), \ldots, \varphi(A)) \\ &= \sup_{f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) = x} (\varphi(A)(x_1) \wedge \varphi(A)(x_2) \wedge \ldots \wedge \varphi(A)(x_{n(\alpha)})) \\ &\leq \varphi(A) \end{aligned}$$

and that is $\varphi(A)$ is an intuitionistic fuzzy subalgebra of K.

Here, $f_\alpha \in F$ and for every $n(\alpha)$-tuples $(x_1, x_2, \dots, x_{n(\alpha)}) \in X^{n(\alpha)}$

$$\begin{aligned}
\varphi^{-1}(B)(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)})) &= B(\varphi(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}))) \\
&= B(f_\alpha(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_{n(\alpha)}))) \\
&\geq B(\varphi(x_1)) \wedge B(\varphi(x_2)) \wedge \dots \wedge B(\varphi(x_{n(\alpha)})) \\
&= \varphi^{-1}(B)(x_1) \wedge \varphi^{-1}(B)(x_2) \dots \wedge \varphi^{-1}(B)(x_{n(\alpha)})
\end{aligned}$$

$$\Rightarrow \quad \omega_{f_\alpha}(\varphi^{-1}(B), \varphi^{-1}(B), \dots, \varphi^{-1}(B)) \leq \varphi^{-1}(B).$$

$\varphi^{-1}(B)$ is an intuitionistic fuzzy subalgebra of L. □

Proposition 3.4. *Let $S = [X, F]$ be an algebra and A is an* IF*-subalgebra of $L = [\mathrm{IFS}(X), \Omega]$. For any $r, s \in [0,1]$ with $r + s \leq 1$, $A_{(r,s)}$ is a crisp subalgebra of S.*

Proof. Let $f_\alpha \in F$. If $x_1, x_2, \dots, x_{n(\alpha)} \in A_{(r,s)}$ then $A(x_i) \geq (r,s)$ for each $i = 1, 2, \dots, n(\alpha)$. Since A is an IF-subalgebra,

$$A(f_\alpha(x_1, x_2, \dots, x_{n(\alpha)})) \geq \min_{1 \leq i \leq n(\alpha)} A(x_i) \geq (r,s)$$

$$\Rightarrow \quad f_\alpha(x_1, x_2, \dots, x_{n(\alpha)}) \in A_{(r,s)}.$$

Hence, $A_{(r,s)}$ is a crisp subalgebra of S. □

Before introduce the definition of intuitionistic fuzzy function we need to prove following propositions. We will use ${}^0_1[a]_A = \bar{a}$ presentation for $A \in \mathrm{IFE}(X)$ and define an intuitionistic fuzzy subset $A_{\bar{a}} : X \to I \times I$ as follows:

$$A_{\bar{a}}(z) = A(a, z), \quad \text{for all } z \in X.$$

Proposition 3.5. *$A_{\bar{a}} : X \to I \times I$ defined as above is well-defined.*

Proof. For all $x \in X$, $0 \leq \mu_{A_{\bar{a}}}(x) + \nu_{A_{\bar{a}}}(x) \leq 1$. If $y \in \bar{a}$ then $A(a, y) = (1, 0)$.

$$A_{\bar{a}}(x) = A(y, x) \leq (1,0) \text{ and } A_{\bar{a}}(x) = A(a, x) \leq (1,0).$$

So, $A(a, x) = A(y, x)$. That is $A_{\bar{a}}$ is well defined.

Proposition 3.6. *For each fixed (r,s) such that $r, s \in [0,1]$ with $r + s \leq 1$, $\{(A_{\bar{a}})_{(r,s)} : a \in X\}$ the set of (r,s)-cuts of A is a crisp partition of X.*

Proof. (i): For all $a \in X$ there is at least one $r, s \in [0,1]$ with $r + s \leq 1$ such that $a \in (A_{\bar{a}})_{(r,s)}$. So, $\bigcup_{a \in X} (A_{\bar{a}})_{(r,s)} = X$.

(ii): Let $a, b \in X$ and $a \in \bar{a}$, $b \in \bar{b}$. Suppose that $\bar{a} \neq \bar{b}$. If $z \in (A_{\bar{a}})_{(r,s)} \cap (A_{\bar{b}})_{(r,s)}$ then

$$\begin{aligned}
& A_{\bar{a}}(z) \geq (r,s) \text{ and } A_{\bar{b}}(z) \geq (r,s) \\
\Rightarrow \quad & A(a, z) \geq (r,s) \text{ and } A(b, z) \geq (r,s) \\
\Rightarrow \quad & A(a, b) \geq (r,s).
\end{aligned}$$

If $x \in (A_{\bar{a}})_{(r,s)}$ then $A(a, x) \geq (r,s)$. It is clear by symmetry that $A(x, a) \geq (r,s)$. So, $A(b, x) \geq (r,s)$ and $x \in (A_{\bar{b}})_{(r,s)}$. We obtain that $(A_{\bar{a}})_{(r,s)} \subseteq (A_{\bar{b}})_{(r,s)}$. Similarly, we can show that $(A_{\bar{b}})_{(r,s)} \subseteq (A_{\bar{a}})_{(r,s)}$. Therefore $(A_{\bar{a}})_{(r,s)} \cap (A_{\bar{b}})_{(r,s)} = \varnothing$ or $(A_{\bar{a}})_{(r,s)} = (A_{\bar{b}})_{(r,s)}$. □

3.1 Intuitionistic Fuzzy Functions

Definition 3.7. Let X and Y be non-empty sets and f be an intuitionistic fuzzy relation from X to Y. Each $y \in Y$ determines an intuitionistic fuzzy subset of X as follows:

$$A_y : X \to I \times I, A_y(x) = f(x,y), \quad \text{for all } x \in X.$$

The intuitionistic fuzzy relation is called an intuitionistic fuzzy function from X to Y if

(i) For each $x \in X$, $\exists!\ y \in Y$ such that $f(x,y) = (1,0)$.

(ii) For each (r,s) such that $r,s \in [0,1]$ with $r+s \leq 1$ the set $\{(A_y)_{(r,s)} : y \in Y\}$ is a crisp partition of X.

If for each $y \in Y$, $\exists\ x \in X$ such that $f(x,y) = (1,0)$ then f called onto function.

If each pair $x_1, x_2 \in X$ such that $f(x_1,y) = f(x_2,y) = (1,0) \Rightarrow x_1 = x_2$ then f called one-to-one function.

After this definition we can give the following concepts.

Let $f : X \times Y \to I \times I$ be an intuitionistic fuzzy function then converse of f defined as $\widetilde{f} : Y \times X \to I \times I$, $\widetilde{f}(y,x) = f(x,y)$ for $x \in X, y \in Y$.

Let $A : X \to I \times I$ and $B : Y \to I \times I$ be two intuitionistic fuzzy sets then image and preimage under f as follows:

$$f(A)(y) = \sup_{x \in X}(A(x) \wedge f(x,y)), \quad y \in Y \text{ and}$$

$$f^{-1}(B)(x) = \sup_{x \in X}(B(y) \wedge \widetilde{f}(y,x)), \quad x \in X.$$

Example 3.8. Let $X = \{a,b,c\}$ and $Y = \{d,e\}$ be universal sets then the intuitionistic fuzzy relation f defined as follows;

$$f = \{((a,d),1,0),((a,e),0.6,0.3),((b,d),0.6,0.3),((b,e),1,0),((c,d),1,0),((c,e),0.6,0.3)\}.$$

So, we can examine $(A_d)_{(r,s)}$ and $(A_e)_{(r,s)}$ for all (r,s) such that $r,s \in [0,1]$ with $r+s \leq 1$. If $(r,s) \leq (0.6,0.3)$ then $(A_d)_{(r,s)} = (A_e)_{(r,s)} = X$. For other situations of (r,s), it is clear that $(A_d)_{(r,s)} = \{a,c\}$ and $(A_e)_{(r,s)} = \{b\}$. As a result, for all (r,s) such that $r,s \in [0,1]$ with $r+s \leq 1$, $\{(A_y)_{(r,s)} : y \in Y\}$ is a crisp partition of X. Hence, f is an intuitionistic fuzzy function from X to Y.

Proposition 3.9. *Let X and Y be non-empty sets and $f : X \times Y \to I \times I$ be an intuitionistic fuzzy function. The composition of $f : X \to Y$ and $\widetilde{f} : Y \to X$ such that*

$$\widetilde{f} \circ f(x_1,x_2) = \sup_{y}(f(x_1,y) \wedge \widetilde{f}(y,x_2)), \quad \text{for all } x_1,x_2 \in X$$

is an intuitionistic fuzzy equivalence relation on X.

Proof. Let A be the relation $\widetilde{f} \circ f$. For all $x \in X$, $A(x,x) = (1,0)$ so, A is intuitionistic fuzzy reflexive. From the definition A is intuitionistic fuzzy symmetric. Now, let for $x_1,x_2 \in X$,

$$(\alpha_1,\beta_1) = (A \circ A)(x_1,x_2) > \Theta$$

and let given small enough $\epsilon,\epsilon' > 0$ such that $(\alpha_1 - \epsilon, \beta_1 + \epsilon') < (\alpha_1,\beta_1)$.

There exist $z \in X$ such that $A(x_1,z) > (\alpha_1 - \epsilon, \beta_1 + \epsilon')$ and $A(z,x_2) > (\alpha_1 - \epsilon, \beta_1 + \epsilon')$.

Hence, there are $y,w \in Y$ with

$$f(x_1,y) \wedge f(z,y) > (\alpha_1 - \epsilon, \beta_1 + \epsilon') \text{ and } f(z,w) \wedge f(x_2,w) > (\alpha_1 - \epsilon, \beta_1 + \epsilon').$$

So,

$$\begin{aligned} & z \in (A_y)_{\langle \alpha_1-\epsilon,\beta_1+\epsilon'\rangle} \cap (A_w)_{\langle \alpha_1-\epsilon,\beta_1+\epsilon'\rangle} \\ \Rightarrow\ & (A_y)_{\langle \alpha_1-\epsilon,\beta_1+\epsilon'\rangle} = (A_w)_{\langle \alpha_1-\epsilon,\beta_1+\epsilon'\rangle} \\ \Rightarrow\ & x_2 \in (A_w)_{\langle \alpha_1-\epsilon,\beta_1+\epsilon'\rangle} \text{ and } f(x_2,y) > (\alpha_1-\epsilon,\beta_1+\epsilon'). \end{aligned}$$

So, $A(x_1,x_2) \geq f(x_1,y) \wedge f(x_2,y) > (\alpha_1 - \epsilon, \beta_1 + \epsilon')$. Since ϵ, ϵ' are small enough, $A(x_1,x_2) \geq (\alpha_1,\beta_1)$.
The intuitionistic fuzzy equivalence relation A, as defined above, is called *kernel* of f. □

Theorem 3.10. *Let X and Y be non-empty sets, $f : X \times Y \to I \times I$ be an intuitionistic fuzzy function and A be the kernel of f. Then there is a decomposition of f given by following diagram where X/A is the class of intuitionistic fuzzy subsets $(A_{\overline{x}})_{x\in X}$, $f(X) = \{y \in Y : \exists\, x \in X$ with $f(x,y) = (1,0)\}$ and ε is a natural mapping given by $\varepsilon(x, A_{\overline{x}}) = A_{\overline{x}}(x)$. Also, f' and i are crisp mappings defined by $f'(A_{\overline{x}}) = y$ where $\overline{x}$ is the class given by x such that $f(x,y) = (1,0)$ and i is given by inclusion, $i : X_f \hookrightarrow Y$, $i(y) = y$ and $g(x) = y$ is given by $f(x,y) = (1,0)$.*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\varepsilon"'] & Y \arrow[d, "i"] \\ X/A \arrow[r, "f'"'] & f(X) \end{tikzcd}$$

Figure 1

Proof. It is enough to show that f' is well-defined. Let $A(x,x') = (1,0)$ and $y, y' \in T$ such that $f(x,y) = f(x',y') = (1,0)$.

$$\begin{aligned} & A(x,x') = (1,0) \\ \Rightarrow\ & \overline{x} = \overline{x'} \\ \Rightarrow\ & f(x,y) = f(x,y') = (1,0) \\ \Rightarrow\ & y = y'. \end{aligned}$$

□

Now, as a preliminary to the isomorphism theorems, we will define the intuitionistic fuzzy congruence relations and we will examine the intuitionistic fuzzy homomorphisms on intuitionistic fuzzy similar algebras.

Definition 3.11. Let $S = [X,F]$ be an algebra and $f_\alpha \in F$. For any $(A_1, A_2, \ldots, A_{n(\alpha)}) \in \mathrm{IFR}(x)^{n(\alpha)}$ and for any $x, y \in S, \varpi_{f_\alpha}(A_1, A_2, \ldots, A_{n(\alpha)})$ to be an element of $\mathrm{IFR}(X)$ defined by

$$\varpi_{f_\alpha}(A_1, A_2, \ldots, A_{n(\alpha)})(x,y) = \sup_{x,y}\Big(\min_{1\leq i\leq n(\alpha)} A_i(x_i,y_i)\Big)$$

such that the supremum is taken over all representations of $f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) = x$ and $f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)}) = y$.

Hence $[\mathrm{IFR}(x), \Omega]$ is an intuitionistic fuzzy algebra.

Definition 3.12. Let $S = [X,F]$ be an algebra. $A \in (\text{IFE}(x),\Omega)$ is said to be an intuitionistic fuzzy congruence relation on S if and only if, for each $f_\alpha \in F$,

$$\varpi_{f_\alpha}(A,A,\dots,A) \sqsubseteq A.$$

Proposition 3.13. *Let A be an intuitionistic fuzzy congruence relation on algebra $S = [X,F]$. $A_{(r,s)}$(shortly $\sim_{A_{(r,s)}}$) is a crisp congruence relation on S for each $r,s \in [0,1]$ with $r+s \le 1$.*

Proof. (i): For all $x \in X$, $A(x,x) = (1,0) \ge (r,s)$. So, $x \sim_{A_{(r,s)}} x$.
(ii): For $x,y \in X$, if $x \sim_{A_{(r,s)}} y$ then

$$\begin{aligned} & A(x,y) \ge (r,s) \\ \Rightarrow\ & A(y,x) \ge (r,s) \\ \Rightarrow\ & y \sim_{A_{(r,s)}} x \end{aligned}$$

(iii): For $x,y,z \in X$, if $x \sim_{A_{(r,s)}} y$ and $y \sim_{A_{(r,s)}} z$ then

$$\begin{aligned} A(x,z) &= (A \circ A)(x,z) \\ &= \sup_{t \in X}(A(x,t) \wedge A(t,z)) \\ &\ge A(x,y) \wedge A(y,z) \ge (r,s) \end{aligned}$$

$A_{(r,s)}$ is a crisp equivalence relation.
(iv): Let $f_\alpha \in F$ and $f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}) = x$, $f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}) = y$ such that $x,y \in X$ and $(x_1,x_2,\dots,x_{n(\alpha)}),(y_1,y_2,\dots,y_{n(\alpha)}) \in X^{n(\alpha)}$.
If $x_i \sim_{A_{(r,s)}} y_i$, $i = 1,2,\dots,n(\alpha)$ then $A(x_i,y_i) \ge (r,s)$. If $\varpi_{f_\alpha}(A,A,\dots,A) \sqsubseteq A$ then

$$\begin{aligned} & A(x,y) \ge \sup_{x,y}\Big(\min_{1\le i\le n(\alpha)} A_i(x_i,y_i)\Big) \\ \Rightarrow\ & A(x,y) \ge (r,s) \\ \Rightarrow\ & A(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}), f_\alpha(y_1,y_2,\dots,y_{n(\alpha)})) \ge (r,s). \end{aligned}$$

So,

$$f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}) \sim_{A_{(r,s)}} f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}).$$ □

Definition 3.14. Let $S = [X,F]$ and $T = [Y,F]$ be two similar algebras. The intuitionistic fuzzy function φ from S to T is called an intuitionistic fuzzy homomorphism if and only if for each $f_\alpha \in F$

$$\varpi_{f_\alpha}(\varphi,\varphi,\dots,\varphi) \sqsubseteq \varphi.$$

That is, φ is an intuitionistic fuzzy algebra of $S \times T$. If $S = T$ then φ called an intuitionistic fuzzy endomorphism.

It is clear that this definition is definitely different from the extensions of ordinary homomorphisms to intuitionistic fuzzy operations.

Proposition 3.15. *Let $S = [X,F]$, $T = [Y,F]$ be two similar algebras and φ is an intuitionistic fuzzy homomorphism from S to T. If A is an intuitionistic fuzzy subalgebra of S then $\varphi(A)$ is intuitionistic fuzzy subalgebra of T.*

Proof. Let $f_\alpha \in F$ and $f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}) = x$, $f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}) = y$. With notation $\varphi(A) = B$ and φ is an intuitionistic fuzzy homomorphism then

$$\varphi(x,y) \geq \min_{1\leq i\leq n(\alpha)} \varphi(x_i,y_i) \quad \text{and} \quad A(x) \geq \min_{1\leq i\leq n(\alpha)} A(x_i).$$

Let $(\theta_1,\theta_2) < \min_i B(y_i)$. There are $x_i \in X$ such that

$$\begin{aligned}
& A(x_i) \wedge \varphi(x_i,y_i) > (\theta_1,\theta_2) \\
\Rightarrow \quad & B(y) \geq A(x) \wedge \varphi(x,y) \\
& \geq \left(\min_i A(x_i)\right) \wedge \left(\min_i \varphi(x_i,y_i)\right) > (\theta_1,\theta_2) \\
\Rightarrow \quad & B(y) \geq \min_i B(y_i).
\end{aligned}$$

So, $\omega_\alpha(B,B,\dots,B) \leq B$. $\varphi(A) = B$ is intuitionistic fuzzy subalgebra of T.

Proposition 3.16. *Let $S = [X,F]$, $T = [Y,F]$ be two similar algebras and φ is an intuitionistic fuzzy homomorphism from S to T. If B is an intuitionistic fuzzy subalgebra of T then $\varphi^{-1}(B)$ is intuitionistic fuzzy subalgebra of S.*

Proof. It is clear. □

3.2 Isomorphism Theorems on Intuitionistic Fuzzy Universal Algebras

Isomorphic theorems proved in algebraic structures such as group theory and ring theory have also been studied on universal algebras. Murali examined these theorems on fuzzy algebra [14] and our aim is extend these theorems to intuitionistic fuzzy algebra.

Let $S = [X,F]$ be an algebra and A be an intuitionistic fuzzy congruence relation on S. X/A is the class of intuitionistic fuzzy subsets $(A_{\overline{x}})_{x\in X}$ such that ${}^0_1[x]_A = \overline{x}$, $x \in X$. Now, for $f_\alpha \in F$ let $A_{\overline{x_1}}, A_{\overline{x_2}},\dots,A_{\overline{x_{n(\alpha)}}} \in X/A$ then f'_α on X/A is defined as follow:

$$f'_\alpha(A_{\overline{x_1}}, A_{\overline{x_2}},\dots,A_{\overline{x_{n(\alpha)}}}) = A_{\overline{z}}, \quad A_{\overline{z}}(t) = A(z,t),$$

where $f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}) = z$, $x_i \in \overline{x_i}$ for each $i = 1,2,\dots,n(\alpha)$. Suppose that $y_i \in \overline{x_i}$ for each $i = 1,2,\dots,n(\alpha)$. Then $x_i \sim_{A_{(1,0)}} y_i$. It is clear that,

$$f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}) \sim_{A_{(1,0)}} f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}) \;\Rightarrow\; z \sim_{A_{(1,0)}} z',$$

where $f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}) = z'$. So,

$$z' \in^0_1 [z]_A \quad \text{and} \quad A_{\overline{z'}}(t) = A(z',t) = A(z,t) = A_{\overline{z}}(t).$$

We obtain that, $S/A = [X/A, F']$ is an algebra similar to $S = [X,F]$.

Theorem 3.17. *Let ε be a mapping from S to S/A, $\varepsilon(x) = A_{\overline{x}}$ then ε is a homomorphism.*

Proof. If $\overline{z} =^0_1 [f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})]_A$ then

$$\begin{aligned}
\varepsilon(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})) &= A_{\overline{z}} \\
&= f_\alpha(A_{\overline{x_1}}, A_{\overline{x_2}},\dots,A_{\overline{x_{n(\alpha)}}}) \\
&= f_\alpha(\varepsilon(x_1),\varepsilon(x_2),\dots,\varepsilon(x_{n(\alpha)})).
\end{aligned}$$

So, ε is a homomorphism. □

Proposition 3.18. *Let $S=[X,F]$, $T=[Y,F]$ be two similar algebras and f is an intuitionistic fuzzy homomorphism from S to T. The kernel of f is an intuitionistic fuzzy congruence relation on S.*

Proof. For any $f_\alpha \in F$ and $(x_1,\dots,x_{n(\alpha)}),(y_1,\dots,y_{n(\alpha)}) \in X^{n(\alpha)}$, we have to prove the substitution property. From definition,

$$\begin{aligned} &A(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}))\\ &\qquad=\widetilde{f}\circ f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}))\\ &\qquad=\sup_{z\in Y}(f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),z)\wedge\widetilde{f}(z,f_\alpha(y_1,y_2,\dots,y_{n(\alpha)})))\\ &\qquad=\sup_{z\in Y}(f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),z)\wedge f(f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}),z)) \end{aligned}$$

and

$$\begin{aligned} A_i(x_i,y_i)&=\widetilde{f}\circ f(x_i,y_i)\\ &=\sup_{z\in Y}(f(x_i,z)\wedge f(y_i,z)). \end{aligned}$$

Let presume that

$$A(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}))\not\geq\min_{1\le i\le n(\alpha)}A(x_i,y_i).$$

There exist (β_1,β_2) such that $\beta_1,\beta_2\in[0,1]$ with $\beta_1+\beta_2\le 1$,

$$\begin{aligned} A(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}))&<(\beta_1,\beta_2)\\ &<A(x_i,y_i) \end{aligned}$$

that is

$$\sup_{z\in Y}(f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),z)\wedge f(f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}),z))<(\beta_1,\beta_2)$$

and for each $i=1,2,\dots,n(\alpha)$,

$$\sup_{z\in Y}(f(x_i,z)\wedge f(y_i,z))>(\beta_1,\beta_2).$$

It is clear that, there exist a z_i for each $i=1,2,\dots,n(\alpha)$ such that $f(x_i,z_i)>(\beta_1,\beta_2)$ and $f(y_i,z_i)>(\beta_1,\beta_2)$.

Let $z=f_\alpha(z_1,z_2,\dots,z_{n(\alpha)})$.

$$\begin{aligned} f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),f_\alpha(z_1,z_2,\dots,z_{n(\alpha)}))&\ge\min_{1\le i\le n(\alpha)}f(x_i,z_i)\\ &>(\beta_1,\beta_2) \end{aligned}$$

and

$$f(f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}),f_\alpha(z_1,z_2,\dots,z_{n(\alpha)}))\ge\min_{1\le i\le n(\alpha)}f(y_i,z_i).$$

Hence,

$$(f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),z)\wedge f(f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}),z))>(\beta_1,\beta_2)$$

and

$$\sup_{z\in Y}(f(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}),z)\wedge f(f_\alpha(y_1,y_2,\dots,y_{n(\alpha)}),z))>(\beta_1,\beta_2).$$

This is a contradiction with our acceptance. □

Theorem 3.19 (First Isomorphism Theorem). *Let $S=[X,F]$, $T=[Y,F]$ be two similar algebras and f is an intuitionistic fuzzy homomorphism from S to T. If A is the kernel of f then there exist a decomposition of f defined by following diagram. In this diagram all symbols have same meaning as Theorem* 3.10.

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "\varepsilon"'] & T \arrow[d, "i"] \\ S/A \arrow[r, "f'"'] & f(S) \end{tikzcd}$$

Figure 2

Proof. Thanks to previous proposition and Theorem 3.10 it is enough to prove f' is an isomorphism. $f'(A_{\overline{x}})=y$, where $\overline{x}$ is the class given by x such that $f(x,y)=(1,0)$. From definition f' is bijective. Also,

$$f'(f_\alpha(A_{\overline{x_1}},A_{\overline{x_2}},\dots,A_{\overline{x_{n(\alpha)}}}))=f_\alpha(f'(A_{\overline{x_1}}),f'(A_{\overline{x_2}}),\dots,f'(A_{\overline{x_{n(\alpha)}}}))$$

that is f' is homomorphism.

Murali [14] proved that second isomorphism theorem could not extend to fuzzy algebras. Likewise, we need a restriction of an intuitionistic fuzzy congruence relation to intuitionistic fuzzy subalgebra to introduced second isomorphism theorem on intuitionistic fuzzy algebra. But, we could not define a natural way for this restriction. So, we will deal on third isomorphism theorem.

Let $S=[X,F]$ be an algebras and A,B be two intuitionistic fuzzy congruence relations on S such that $B\le A$. Let $C(A)_{(1,0)}$ and $C(B)_{(1,0)}$ denotes the crisp $(1,0)$-equivalence class of A and B, respectively. It is clear that $C(B)_{(1,0)}$ is a refinement of $C(A)_{(1,0)}$. We define a congruence relation on following set:

$$S\diagup B=\{B_{\overline{x}}:\overline{x}\in C(B)_{(1,0)},x\in X\}.$$ □

Definition 3.20. Let $S=[X,F]$ be an algebras and A,B be two congruence relations on S such that $B\le A$.

$$A\diagup B:S\diagup B\times S\diagup B\to I\times I$$

$$A\diagup B(B_{\overline{x}},B_{\overline{y}})=A(x,y).\quad \text{for } x\in\overline{x},\ y\in\overline{y}$$

is called the quotient intuitionistic fuzzy congruence relation.

Theorem 3.21. *Let $S=[X,F]$ be an algebras and A,B be two intuitionistic fuzzy congruence relations on S such that $B\le A$. The intuitionistic fuzzy quotient congruence relation $A\diagup B$ is an intuitionistic fuzzy congruence relation on $S\diagup B$.*

Proof. (i): If $x,x'\in\overline{x}$ and $y,y'\in\overline{y}$ then x,x' belong to the same $(1,0)$-equivalence class of A and also y,y' belong to the same class. Since $A(x,y)=A(x',y')$ then

$$A\diagup B(B_{\overline{x}},B_{\overline{y}})=A(x,y)=A(x',y').$$

We obtain that $A\diagup B$ is well-defined.

(ii): It is clear that A/B is reflexive and symmetric.

(iii): Shortly $A/B = \rho$. Let $x \in \overline{x}$, $y \in \overline{y}$ and $z \in \overline{z}$,

$$\begin{aligned}
\rho \circ \rho(B_{\overline{x}}, B_{\overline{y}}) &= \sup_{\overline{z} \in C(B)_{(1,0)}} (\rho(B_{\overline{x}}, B_{\overline{z}}) \wedge \rho(B_{\overline{z}}, B_{\overline{y}})) \\
&= \sup_{\overline{z} \in C(B)_{(1,0)}} \{A(x,z) \wedge A(z,y) : x \in \overline{x}, y \in \overline{y}, z \in \overline{z}\} \\
&\le \sup_{z \in X} (A(x,z) \wedge A(z,y)) \\
&= A \circ A(x,y) \le A(x,y) \\
&= \rho(B_{\overline{x}}, B_{\overline{y}}).
\end{aligned}$$

Thus, A/B transitive.

(iv): Let $f_\alpha \in F$ and $(B_{\overline{x_1}}, B_{\overline{x_2}}, \ldots, B_{\overline{x_{n(\alpha)}}}), (B_{\overline{y_1}}, B_{\overline{y_2}}, \ldots, B_{\overline{y_{n(\alpha)}}}) \in (S/B)^{n(\alpha)}$. Suppose that there exists a $(\beta_1, \beta_2) > \Theta$ with $\beta_1, \beta_2 \in [0,1]$ with $\beta_1 + \beta_2 \le 1$ such that

$$\rho(f_\alpha(B_{\overline{x_1}}, B_{\overline{x_2}}, \ldots, B_{\overline{x_{n(\alpha)}}}), f_\alpha(B_{\overline{y_1}}, B_{\overline{y_2}}, \ldots, B_{\overline{y_{n(\alpha)}}})) < (\beta_1, \beta_2)$$

and

$$(\beta_1, \beta_2) < \rho(B_{\overline{x_i}}, B_{\overline{y_i}}), \quad \text{for each } i = 1, 2, \ldots, n(\alpha).$$

From Theorem 3.17, $f_\alpha(B_{\overline{x_1}}, B_{\overline{x_2}}, \ldots, B_{\overline{x_{n(\alpha)}}}) = B_{\overline{x}}$ and $f_\alpha(B_{\overline{y_1}}, B_{\overline{y_2}}, \ldots, B_{\overline{y_{n(\alpha)}}}) = B_{\overline{y}}$ such that $\overline{x} =_1^0 [x]_B$ and $\overline{y} =_1^0 [y]_B$ with $x = f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}), x_i \in \overline{x_i}$ and $y = f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)})$, $y_i \in \overline{y_i}$ for each $i = 1, 2, \ldots, n(\alpha)$. Now,

$$\begin{aligned}
& A(x,y) = A(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}), f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)})) \\
& \qquad\quad \ge \min_{1 \le i \le n(\alpha)} A(x_i, y_i) \\
\Rightarrow \quad & A(x,y) > (\beta_1, \beta_2).
\end{aligned}$$

Hence,

$$\rho(f_\alpha(B_{\overline{x_1}}, B_{\overline{x_2}}, \ldots, B_{\overline{x_{n(\alpha)}}}), f_\alpha(B_{\overline{y_1}}, B_{\overline{y_2}}, \ldots, B_{\overline{y_{n(\alpha)}}})) > (\beta_1, \beta_2).$$

This contradicted our acceptance. □

Corollary 3.22 (Third Isomorphism Theorem)**.** *Let $S = [X, F]$ be an algebras and A, B be two intuitionistic fuzzy congruence relations on S such that $B \le A$. Then*

$$(S/B)/(A/B) \approxeq S/A.$$

Proof. Let take into account that ${}_1^0[B_{\overline{x}}]_{A/B} = \{B_{\overline{y}} \in S/B : A/B(B_{\overline{x}}, B_{\overline{y}}) = (1,0)\}$. By the definition of A/B, ${}_1^0[B_{\overline{x}}]_{A/B} \approxeq \overline{x}$ as crisp set. Therefore,

$$(A/B)_{{}_1^0[B_{\overline{x}}]_{A/B}} \approxeq A_{\overline{x}}$$

as an intuitionistic fuzzy subsets. We obtain that $(S/B)/(A/B) \approxeq S/A$. □

4. Conclusion

In this study, we studied intuitionistic fuzzy algebraic properties on intuitionistic fuzzy algebra. Thus, the common properties of intuitionistic fuzzy algebraic structures like groups, rings were extended.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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