Communications in Mathematics and Applications
Vol. 11, No. 4, pp. 549–558, 2020
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v11i4.1441

Research Article

Cycle Neighbor Polynomial of Graphs

Annie Sabitha Paul*[1] and Raji Pilakkat[2]

[1] *Department of Mathematics, Government College of Engineering, Kannur, Kerala, India*
[2] *Department of Mathematics, University of Calicut, Kerala, India*
***Corresponding author:** anniesabithapaul@gmail.com

Abstract. In this paper, a new univariate graph polynomial called *Cycle Neighbor Polynomial* $CN[G;x]$ of a graph G is introduced. We obtain some interesting properties of this polynomial and compute cycle neighbor polynomial of some specific graphs.

Keywords. Cycle neighbor free vertex; Cycle neighbor polynomial of a graph

MSC. 05C31

Received: July 15, 2020 **Accepted:** November 6, 2020

1. Introduction

All graphs $G=(V(G),E(G))$ discussed in this paper are finite, simple and undirected. Any undefined term in this paper may be found in [1]. There are plenty of graph polynomials in the literature of graph theory. These polynomials are studied because some of them are generating functions of some graph properties, some count the number of occurrences of certain graph features and some others make an attempt to find complete graph invariants.

Cycle Neighbor Polynomial is essentially a generating function for the number of cycles of various lengths in a graph G. The motivation for the authors to define this polynomial is many graphical properties like girth, circumference [5], number of cycles of different lengths, whether the graph is Hamiltonian, whether it is pancyclic [2], whether it is unicyclic [5], whether it is acyclic or not, whether it is a bipartite graph etc., can be directly obtained from the polynomial expression.

2. Cycle Neighbor Polynomial of Graphs

In this section we introduce a new univariate graph polynomial called cycle neighbor polynomial of a graph. Also, some properties of cycle neighbor polynomials are observed.

Definition 2.1. Let $G(V,E)$ be any graph. A vertex of the graph G is said to be a cycle neighbor free vertex if it does not belong to any cycle of length greater than or equal to three in the graph G.

Definition 2.2. Let G be any simple graph of order n. The *Cycle Neighbor Polynomial* (CYNP) of G denoted by $CN[G,x]$ is defined as

$$CN[G,x] = \sum_{k=0}^{c(G)} c_k(G)x^k, \tag{2.1}$$

where $c_0(G)$ is the number of cycle neighbor free vertices in G, $c(G)$ is the circumference of G and $c_k(G)$ is the number of cycles of length k in the graph G, where $3 \le g(G) \le k \le c(G) \le n$ and $g(G)$ is the girth of G.

We use the abbreviation CYNP for cycle neighbor polynomial of a graph.

Remark 2.3. (i) For any simple graph G, $c_1(G) = c_2(G) = 0$ in $CN[G,x]$.

(ii) If G_1 and G_2 are isomorphic graphs, then $CN[G_1,x] = CN[G_2,x]$.

(iii) If H is an induced subgraph of G, then $\deg(CN[G,x]) \ge \deg(CN[H,x])$.

(iv) $c_0(G)$, the constant term in CYNP of G is the number of cycle neighbor free vertices in G.

(v) If a graph G contains no cycle neighbor free vertices, then zero is a root of its CYNP.

Proposition 2.4 follows directly from the definition of CYNP.

Proposition 2.4. *Let $G(V,E)$ be any graph of order n. If $CN[G,x]$ is a nonconstant polynomial then,*

(i) *The lowest exponent of x of the nonconstant term in CYNP is the girth of G and the highest exponent is the circumference of G.*

(ii) *If $c_k(G) \ne 0$ for all k, where $3 \le k \le n$ then G is pancyclic.*

(iii) *The degree of the CYNP of G is n if and only if G is Hamiltonian.*

Theorem 2.5. *Let G be a nontrivial graph of order n. Then $CN[G,x]$ is a constant polynomial if and only if G is a forest.*

Proof. Suppose if possible, $CN[G,x]$ be a nonconstant polynomial of degree m. Then $m \ge 3$ and $c_m(G) \ne 0$. That is, G has at least one cycle of length m. Hence it is not a forest.

Conversely, if G is a forest, then $CN[G,x] = c_0(G)$, the number of cycle neighbor free vertices, a constant polynomial. □

As the maximum number of edges in any acyclic graph of order n is $n-1$, we have the following corollary.

Corollary 2.6. *Let $G(V,E)$ be a graph of order n and size m. If $m \geq n$, then $CN[G,x]$ is a nonconstant polynomial.*

Proposition 2.7. *Let G be any graph. Then $\deg(CN[G,x]) \leq n-1$ if and only if G is non hamiltonian.*

Proposition 2.8. *The CYNP $CN[G,x]$ of any graph G contains at most $n-2$ terms.*

Proof. The general expression for CYNP of G is $CN[G,x] = c_0(G) + c_3(G)x^3 + c_4(G)x^4 + \ldots + c_{c(G)}(G)x^{c(G)}$. If G is non hamiltonian, then $c(G) < n$ and when G is Hamiltonian, $c(G) = n$ and $c_0(G) = 0$. □

According to Whiteny's Theorem [1], a graph G of order $n \geq 3$ is two connected if and only if any two vertices of G are connected by at least two internally disjoint paths. Therefore, if G is two connected, every vertex of G belongs to a cycle and hence we have Proposition 2.9.

Proposition 2.9. *Let G be any graph of order $n \geq 3$. If G is two connected, then $CN[G,x]$ is a polynomial of degree greater than or equal to 3*

Remark 2.10. The converse of Proposition 2.9 need not be true. That is there are graphs for which $\deg(CN[G,x]) \geq 3$ but G is not two connected. The CYNP of the graph in Figure 1 is $x^5 + x^6$, whose degree is four, even though it is not two connected.

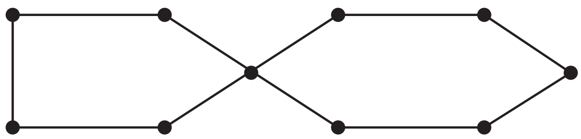

Figure 1

A graph G is bipartite if and only if it has no odd Cycles [1]. Hence the CYNP of bipartite graphs contain no odd powers of x and we have the result.

Theorem 2.11. *A graph is bipartite if and only if $CN[G,x]$ of G is an even polynomial.*

A graph G is 2-colorable if and only if it is bipartite [1]. Hence it follows that:

Corollary 2.12. *Let G be any graph. Then $CN[G,x]$ is an even polynomial if and only if $\chi(G) = 2$, where $\chi(G)$ is the chromatic number of G.*

Let G and H be two disjoint graphs with circumferences $c(G)$ and $c(H)$, respectively. Since there are no edges between G and H, a cycle of length k, $0 \leq k \leq p$ where $p = \max\{c(G), c(H)\}$ in $G \cup H$ is either a k-cycle in G or a k-cycle in H. So that we have

Proposition 2.13. *Let G and H be any two graphs and let $G \cup H$ be the disjoint union of G and H. Then $CN[G \cup H, x] = CN[G, x] + CN[H, x]$.*

Corollary 2.14. *If a graph G has n components $G_1, G_2, \ldots, G_n$, then $CN[G, x] = CN[G_1, x] + CN[G_2, x] + \ldots + CN[G_n, x]$.*

3. CYNP of Some Graphs

Proposition 3.1. *Let G be a unicyclic graph of order n. If the length of the cycle is m, $3 \le m \le n$ then the CYNP of G is $CN[G, x] = x^m + (n - m)$.*

Proof. Since G has only one cycle say C_m of length m and the remaining vertices $V(G) - V(C_m)$ are cycle neighbor free, the result follows. □

Corollary 3.2. *Let $G \cong C_n$, a cycle on n vertices. Then $CN[G, x] = x^n$.*

Definition 3.3 ([4])**.** A Tadpole graph (or dragon graph) $C_{n,m}$, $n \ge 3$, $m \ge 1$ is obtained by joining a cycle C_n, $n \ge 3$ to a path P_m on m vertices with a bridge.

Corollary 3.4. *Let $G \cong C_{n,m}$, then $CN[G, x] = x^n + m$.*

Proposition 3.5. *For a wheel graph $W_n \cong C_{n-1} + K_1$, $n \ge 4$*

$$CN[W_n, x] = (n-1) \sum_{k=3}^{n} x^k + x^{n-1}.$$

Proof. In $W_n = C_{n-1} + K_1$, let $v \in V(K_1)$ be the central vertex of W_n. Then v is adjacent to every vertex of C_n and vertices of C_n has only two neighbors other than v. It can be easily verified that the number of cycles of length k, in W_n is $n-1$ for $3 \le k \le n$, $k \ne n-1$ and there are n cycles of length $n-1$. Therefore, $CN[W_n, x] = (n-1) \sum\limits_{k=3}^{n} x^k + x^{n-1}$. □

Definition 3.6 ([4])**.** A Helm graph H_n, $n > 3$ is obtained from a wheel graph W_n by attaching a pendant edge at each vertex on the rim of the wheel W_n.

Corollary 3.7. $CN[H_n, x] = CN[W_n, x] + (n-1)$*, $n \ge 4$.*

Proposition 3.8. *For any complete graph K_n, $n \ge 3$*

$$CN[K_n, x] = \frac{n!}{2}\left[\frac{x^3}{3(n-3)!} + \frac{x^4}{4(n-4)!} + \ldots + \frac{x^{n-2}}{(n-2)2!} + \frac{x^{n-1}}{(n-1)} + \frac{x^n}{n}\right].$$

Proof. In K_n every vertex is adjacent to every other vertex. Hence to get the number of k-cycles of length k, $3 \le k \le n$ in K_n, choose k vertices out of n in $\binom{n}{k}$ ways and multiply it with the number of permutations ($k!$) of these k vertices and divide it by $2k$, in order to avoid the repetition of the cycle count with which each cycle is represented by the permutation. That is in $CN[K_n, x]$, $c_k(G) = \binom{n}{k}\frac{k!}{2k} = \frac{n!}{2k(n-k)!}$.

Therefore, $CN[K_n,x] = \frac{n!}{2}\left[\frac{x^3}{3(n-3)!} + \frac{x^4}{4(n-4)!} + \ldots + \frac{x^{n-2}}{(n-2)2!} + \frac{x^{n-1}}{(n-1)} + \frac{x^n}{n}\right]$. □

Definition 3.9 ([4]). A Lollipop graph $L_{n,m}$, $n \geq 3$, $m \geq 1$ is obtained by joining a complete graph K_n, $n \geq 3$ to a path P_m on m vertices with a bridge.

Corollary 3.10. $CN[L_{n,m},x] = CN[K_n,x] + m$.

Definition 3.11 ([4]). A Windmill graph $W_n^{(m)}$, is the graph obtained by taking m copies of the complete graph K_n, $n \geq 3$ with a common vertex. $W_3^{(m)}$ is also called the friendship graph and it is denoted by F_m.

Corollary 3.12. $CN[W_n^{(m))},x] = mCN[K_n,x]$.

Corollary 3.13. $CN[F_m,x] = mx^3$.

Definition 3.14 ([3]). A Shell graph $S_n \cong P_{n-1} + K_1$, which can also be defined as the graph obtained from the cycle C_n by adding the edges corresponding to the $n-3$ concurrent chords of the cycle. The vertex at which all chords are concurrent is called the apex of the shell.

Proposition 3.15. $CN[S_n,x] = (n-2)x^3 + (n-3)x^4 + (n-4)x^5 + \ldots + 2x^{n-1} + x^n$.

Proof. Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of P_{n-1} and let v be the vertex of K_1. Every cycle of length k, $3 \leq k \leq n$ contains the consecutive vertices $v_i, v_{i+1}, \ldots, v_{i+(k-2)}$, $1 \leq i \leq n-k$ of P_{n-1}. Hence the number of such cycles of length k is $n-(k-1)$, $3 \leq k \leq n$. Therefore, it follows that,

$$CN[S_n,x] = (n-2)x^3 + (n-3)x^4 + (n-4)x^5 + \ldots + 2x^{n-1} + x^n\,.$$ □

Definition 3.16 ([6]). A bow graph is a double shell with same apex in which each shell has any order.

Corollary 3.17. *Let B_N be a bow graph of order $N \geq 5$, which includes shells S_n and S_m such that $N = m+n-1$, then*

$$CN[B_N,x] = CN[S_n,x] + CN[S_m,x].$$

Proof. B_N includes the shells S_n and S_m with the same apex v, so that v is a cut vertex of B_N. Hence there are no cycles which have edges on both sides of v in B_N. Hence the result. □

Definition 3.18 ([8]). A butterfly graph BF is a bow graph with exactly two pendant edges at the apex.

Corollary 3.19. *If BF is a butterfly graph with $N \geq 7$ vertices, then*

$$CN[BF,x] = CN[B_{N-2},x] + 2\,.$$

4. Graphs with the Maximum and Minimum Number of Terms in the CYNP

The CYNP of any graph of order n has at most $n-2$ terms and at least one term. In this section, we characterize connected graphs having maximum and minimum number of terms in its cycle neighbor polynomial.

Theorem 4.1. *Let G be a connected graph of order n, $n \geq 4$. Then $CN[G,x]$ has exactly $n-2$ terms if and only if G is pancyclic or $G \cong H_{n-1,1}$, where $H_{n-1,1}$ is a graph consisting of a pancyclic graph H on $n-1$ vertices and a vertex (K_1) connected to any one of the vertices of H by a bridge.*

Proof. If G is pancyclic or $G \cong H_{n-1,1}$, then it is clear that $CN[G,x]$ has exactly $(n-2)$ terms.

Now, assume that for the graph G, $CN[G,x]$ has exactly $(n-2)$ terms. Suppose if possible, G is neither pancyclic nor $G \cong H_{n-1,1}$ but $CN[G,x]$ contains $(n-2)$ terms. Since G is not pancyclic, G does not contain at least one cycle of length l for $3 \leq l \leq n$.

Claim: $l \neq n$.
Suppose if possible $l = n$. Then G must contain cycles of all lengths k, $3 \leq k \leq n-1$. Otherwise, the number of terms in $CN[G,x]$ will be less than $(n-2)$, contradicting our assumption. Therefore, G must be a connected graph of order n and contains cycles of all lengths k, $3 \leq k \leq n-1$. Hence $G \cong H_{n-1,1}$, another contradiction to the assumption that G not isomorphic to $H_{n-1,1}$. Therefore $l \neq n$.

So let $l = k$, $3 \leq k \leq n-1$. When G contain no cycles of length k, in order for $CN[G,x]$ to have $(n-2)$ terms, G must contain a Hamilton cycle and therefore, G contains no cycle neighbor free vertices. Hence the cycle neighbor polynomial of G has no constant term and hence $CN[G,x]$ has less than $(n-2)$ terms, a contradiction. □

Since a Lollipop graph $L_{n-1,1}$, $n \geq 4$ is obtained by attaching complete graphs on one and $n-1$ vertices (K_1 and K_{n-1}) by a bridge, where K_{n-1} is pancyclic, we have the following corollary:

Corollary 4.2. *The CYNP of Lollipop graph on $n \geq 4$ vertices contains $n-1$ terms.*

Definition 4.3 ([7]). A cactus graph is a connected graph in which no two cycles have an edge in common.

Definition 4.4. A k-cycle neighbor graph is a cactus graph G in which the length of every cycle in G is k and every vertex belongs at least one cycle of G.

Theorem 4.5. *Let G be a connected graph of order n. Then the CYNP of G has exactly one term if and only if one of the following conditions holds.*

(i) *G is a tree*

(ii) *G is k-cycle neighbor graph.*

Proof. If (i) holds, then trivially, $CN[G,x] = n$, and when (ii) holds, every vertex of G belongs to atleast one cycle of G, hence G contains no cycle neighbor free vertices and the lengths of all cycles in G are k, therefore $CN[G,x] = c_k(G)x^k$, where $c_k(G)$ is the number of k-cycles in G.

Conversely, suppose that $CN[G,x] = c_k(G)x^k$, $k \neq 1$ or 2. If $k = 0$, then $CN[G,x] = c_0(G)$, and since G is connected, it is a tree. If $k \neq 0$, then $3 \leq k \leq n$. Hence G has $c_k(G)$ cycles of length k. Also, since G is connected, each of these $c_k(G)$ cycles are connected to m other k-cycles either by a common vertex or by a bridge, where $1 \leq m \leq (c_k(G) - 1)$. But no pair of these cycles have an edge in common. Otherwise, these two cycles will then form a new cycle of length greater than k, which contradicts $CN[G,x] = c_k(G)x^k$. □

Corollary 4.6. *If G is not connected, the CYNP of G contains exactly one term if and only if*

(i) *G is a forest*

(ii) *Each component of G is k-cycle neighbor graph for the same value of k, $k = 3,4,5,\ldots$.*

5. Cycles and Trees Which Have the Same CYNP as Their Complements

In this section we prove that among all connected acyclic graphs, only paths on n vertices, $n = 2,3$ or 4 and among all cycles C_n, only C_5 have the same CYNP as their complements.

Theorem 5.1. *Let T be any tree on $n \geq 2$ vertices and let $\overline{T}$ be the complement of T. Then $CN[T,x]$ and $CN[\overline{T},x]$ are the same if and only if $T \cong P_n$, path on n vertices, where $n = 2,3$ or 4.*

Proof. When $T \cong P_n$, $n = 2,3$ or 4, $\overline{T}$ is also acyclic with the same order. Hence $CN[T,x] = CN[\overline{T},x]$.

If T is a tree on $n > 4$ vertices, we have the following cases:

Case (i): T is a path.
Suppose that $T \cong P_n$, $n \geq 5$. Then T has exactly two pendant vertices. For $n \geq 5$, the support vertices of the pendant vertices are distinct and nonadjacent. These support vertices together with the pendant vertices will form a cycle of length 4 in $\overline{T}$.

Case (ii): T is not a path.
In this case, T has three or more pendant vertices. These pendant vertices will be adjacent in $\overline{T}$ and will form a cycle in $\overline{T}$. Hence in both cases $\overline{T}$ is not acyclic and $CN[T,x] \neq CN[\overline{T},x]$. □

Theorem 5.2. *Let C_n be any cycle, $n \geq 3$. Then $CN[C_n,x] = CN[\overline{C}_n,x]$ if and only if $n = 5$.*

Proof. For $n = 5$, C_n is self complementary. Since isomorphic graphs have the same cycle neighbor polynomial, $CN[C_5,x] = CN[\overline{C}_5,x]$. Now, let $CN[C_n,x] = CN[\overline{C}_n,x]$. Suppose if possible, $CN[C_n,x] = CN[\overline{C}_n,x]$ holds for $n \neq 5$. For $n = 3$ and 4, $\overline{C}_n$ is acyclic, a contradiction. When $n > 5$, $\overline{C}_n$ contains triangles whereas C_n does not, again a contradiction. Hence $CN[C_n,x] = CN[\overline{C}_n,x]$ if and only if C_n is of length five. □

6. Some Graph Modifications Which Do Not Affect The CYNP

In this section, we consider some graph modifications like edge removal, edge addition, edge contraction and a special case of vertex identification under which the cycle neighbor polynomial of a graph will be unaffected.

Theorem 6.1. *Let G be any graph. Then $CN[G,x] = CN[G \setminus e, x]$ if and only if e is a cut edge of G.*

Proof. Suppose that $CN[G,x] = CN[G \setminus e, x]$. If $e = uv$ is not a cut edge, then there are one or more internally disjoint paths joining u and v other than e. Hence it is clear that e belongs to a cycle of G and the removal of e from G will affect at least one of the coefficients $c_k(G)$, where $3 \le k \le c(G)$. Therefore, $CN[G,x] \neq CN[G \setminus e, x]$.

Conversely, if e is a cut edge of G, both G and $G \setminus e$ will have the same number of cycles of different lengths and the same number of cycle neighbor free vertices. Therefore, $CN[G,x] = CN[G \setminus e, x]$. □

Theorem 6.2. *For any edge e in $\overline{G}$, the complement of G, $CN[G,x] = CN[G+e,x]$ if and only if e is an edge joining different components of G.*

Proof. If e is an edge joining different components of a graph G, then e is a cut edge of $G+e$ and hence by Theorem 6.1, $CN[G+e,x] = CN[G,x]$.

Conversely, let $CN[G,x] = CN[G+e,x]$. Suppose if possible, e is not an edge joining different components of G. Let G_1 be any component of G. Since $e \in E(\overline{G})$ without loss of generality, let $e \in E(\overline{G}_1)$. Also, since G_1 is a connected component of G, e in $G_1 + e$ is either an edge of a cycle in G_1 or a chord of a cycle in G_1. In both cases, the number of cycles in G and $G+e$ are different which contradicts $CN[G,x] = CN[G+e,x]$. □

Definition 6.3 ([8])**.** Edge contraction is an operation which removes an edge e from G and simultaneously merging the two vertices that it previously joined. The resulting graph is denoted by G/e.

Theorem 6.4. *Let G be any triangle free graph, then $CN[G,x] = CN[G/e,x]$ if and only if e is a cut edge of G and both end points of e are not cycle neighbor free vertices.*

Proof. If e is a cut edge of G such that both end points of e are not cycle neighbor free vertices of G then both G and G/e have the same number of cycle neighbor free vetices and the same number of cycles of different lengths k for all possible values of k.

Conversely, let $CN[G,x] = CN[G/e,x]$. If e is not a cut edge, then e belongs to at least one cycle of G. Also, since G is triangle free, the length of one or more cycles in G will be diminished in G/e, a contradiction to the assumption that $CN[G,x] = CN[G/e,x]$. □

Theorem 6.5. *Suppose that G and H are graphs with disjoint vertex sets and let $G.H$ be a graph obtained by identifying a vertex of G with a vertex of H. Then $CN[G.H,x] = CN[G \cup H,x]$ if and only if both the vertices $v_1 \in V(G)$ and $v_2 \in V(H)$ which are being identified in $G.H$ belongs to some cycles of G and H, respectively.*

Proof. Let G and H be any two vertex disjoint graphs and let $v_1 \in V(G)$ and $v_2 \in V(H)$. Consider the following cases:

Case (i): Both v_1 and v_2 are cycle neighbor free vertices in G and H, respectively. Then clearly $CN[G.H,x] = CN[G \cup H,x] - 1$.

Case (ii): One of $v_1 \in V(G)$ or $v_2 \in V(H)$ is a cycle neighbor free vertex. Then also $CN[G.H,x] = CN[G \cup H,x] - 1$.

Case (iii): Let v_1 belongs to a cycle of G and v_2 belongs to a cycle of H. Then the identification of the vertices v_1 in G and v_2 in H will not affect the number of cycles in G and H and therefore $CN[G.H,x] = CN[G \cup H,x]$. This completes the proof. □

7. Conclusion

In this paper, a new univariate graph polynomial viz., cycle neighbor polynomial of a graph is introduced. This polynomial is a generating function of the number of cycles of various lengths in a graph. That is cycle neighbor polynomial of a graph directly encodes the number of cycles of different lengths together with the number of cycle neighbor free vertices in the graph. The concept of cycle neighbor polynomial of a graph is interesting and important because it reveals many graph properties of the underlying graph.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland/Elsevier, New York (1976), https://www.zib.de/groetschel/teaching/WS1314/BondyMurtyGTWA.pdf.

[2] J. A. Bondy, Pancyclic graphs I, *Journal of Combinatorial Theory, Series B* **11** (1971), 80 – 84, DOI: 10.1016/0095-8956(71)90016-5.

[3] P. K. Deb and N. B. Limaye, On harmonious labelling of some cycle related graphs, *Ars Combinatoria* **65** (2002), 177 – 197.

[4] J. A. Gallian, A dynamic survey of graph labelling, *The Electronic Journal of Combinatorics*, Article number DS6: December 21, 2018, https://www.combinatorics.org/files/Surveys/ds6/ds6v21-2018.pdf.

[5] F. Harary, *Graph Theory*, Addison Wesley, Reading MA (1969).

[6] J. J. Jesintha and K. E. Hilda, All uniform bowgraphs are graceful, *Mathematics in Computer Science* **9** (2015), 185 – 191, DOI: 10.1007/s11786-015-0224-2.

[7] B. B. Moshe, M. Segal, A. Dvir and A. Tamir, Centdian computation in cactus graphs, *Journal of Graph Algorithms and Applications* **16** (2012), 199 – 224, DOI: 10.7155/jgaa.00255.

[8] D. B. West, *Introduction to Graph Theory*, 2nd edition, Englewood Cliffs, Prentice Hall, NJ (2000), URL: `http://docshare01.docshare.tips/files/26167/261678089.pdf`.