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Research Article

Generalized Rough (m,n) Bi-Γ-ideals in Ordered LA-Γ-Semigroups

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Abstract. In this paper, generalized rough (m,n)-(bi-ideals, quasi-ideals and interior ideals) have been introduced in ordered LA-Γ-semigroups by means of pseudo order of relations. Some combined results have been proved. Properties of rough m-left Γ-ideals, n-right Γ-ideals rough interior (m,n)-Γ-ideals, rough (m,n)-quasi-Γ-ideals and rough (m,n) bi-Γ-ideals in ordered LA-Γ-Semigroups have been studied and discussed by using pseudo order of relations.

Keywords. Ordered LA-Γ-semigroups; Rough sets; Rough (m,n)-(left, right, quasi, bi and interior)-Γ-ideals

Mathematics Subject Classification (2020). 20M10; 20M12; 20N99

1. Introduction

Pawlak [15] introduced rough sets. In recent time concept of rough set theory has become an important and major mathematical tool for managing uncertainty that arises from inexact, noisy or incomplete information. Many authors enlighten ideas based on this theory. In connection with algebraic structures, Biswas and Nanda [9] introduced the notion of rough subgroups, whereas Kuroki [14] introduced it for semigroups. Generalized (m,n) bi-ideals in case of semigroups with involution was introduced by Moin *et al*. [7] whereas (m,n) quasi-ideals in semigroups was defined by Moin *et al*. [8]. Notes on (m,n) bi-Γ-ideals in Γ-semigroups was

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introduced by Moin and Rais in [5] where authors studied properties of (m-left, n-right, quasi and bi)-Γ-ideals in case of Γ-semigroups. Furthermore, Moin and Rais [6] defined rough (m,n) quasi-Γ-ideals in Γ-semigroups.

The concept of an AG-groupoid was first given by Kazim and Naseeruddin [2] in 1972 and they called it left almost semigroups (LA-semigroups). Holgate [12] called LA-semigroup to left invertive groupoid. The concept of a Γ-semigroup (generalization of semigroups) was introduced by Sen [17] in 1981. Further in 1986, Sen and Saha [18] defined Γ-semigroups in different way as compared to defined in [17]. They also defined Γ-groupoid in the same paper. In some direction of fuzziness ordered AG-groupoids has been studied by Faisal *et al*. [10]. It should be noted that an ordered Γ-AG-groupoid is the generalization of an ordered Γ-semigroups. In this paper we have studied roughness in generalized (m,n) bi-Γ-ideals in ordered LA-Γ semigroups. Generalized roughness in $(\in, \in \vee qk)$ of ordered semigroups have been studied by Muhammad *et al*. [13]. Further, generalized roughness in LA-Semigroups was studied by Noor *et al*. [16].

Fuzzy $(2,2)$-regular ordered G-AG**-Groupoids is investigates and studied by Faisal *et al*. [11]. Generalized roughness in ordered semigroups is studied by Moin [3] recently whereas T-roughness and its ideals in ternary semigroups were introduced by Moin and Naveed in [4]. On generalized fuzzy ideals of ordered LA-semigroups is discussed by Amjad *et al*. in [1].

Let S and Γ be two nonempty sets. Then a triplet $(S,\Gamma,\cdot)$ is called an LA-Γ-semigroup, where $\cdot$ is a ternary operation $S\times\Gamma\times S \Rightarrow S$ such that $(x\cdot\alpha\cdot y)\cdot\beta\cdot z = (z\cdot\alpha\cdot y)\cdot\beta\cdot x$ for all $x,y,z\in S$ and all $\alpha,\beta\in\Gamma$ [19]. An ordered LA-Γ-semigroup is an ordered set $(S,\leq)$ at the same time an LA-Γ-semigroup $(S,\Gamma,\cdot)$ such that $a\leq b \Rightarrow a\cdot\alpha\cdot x\leq b\cdot\alpha\cdot x$ and $x\cdot\beta\cdot a\leq x\cdot\beta\cdot b$ $\forall$ $a,b,x\in S$ and $\alpha,\beta\in\Gamma$. For the sake of convenience we write $a\cdot\alpha\cdot x\leq b\cdot\alpha\cdot x$ as $a\alpha x\leq b\alpha x$. In what follows we denote the ordered LA-Γ-semigroup $(S,\Gamma,\cdot,\leq)$ by S unless otherwise specified.

We prove that m-left, n-right, (m,n)-(quasi-, bi-, interior)-Γ-ideals of ordered LA-Γ-semigroups S is also rough m-left, n-right, (m,n)-(quasi-, bi-, interior)-Γ-ideals. By using pseudo order of relations, it is proved that m-left, n-right ordered Γ-ideals and (m,n) (resp., quasi-Γ, bi-Γ and interior Γ)-ideals in ordered LA-Γ-semigroups S becomes lower and upper rough m-left, n-right ordered Γ-ideals and (m,n) (resp., quasi-Γ, bi-Γ and interior Γ)-ideals of S.

2. Preliminaries and Basic Definitions

Definition 2.1. An ordered LA-Γ-semigroup (po-LA-Γ-semigroup) is a structure $(S,\Gamma,\cdot,\leq)$ such that $(S,\Gamma,\cdot)$ is an LA-Γ-semigroup, $(S,\leq)$ is a poset (i.e. reflexive, anti-symmetric and transitive) and $\forall$ $a,b,x\in S$ we have $a\leq b$ implies $a\alpha x\leq b\alpha x$ and $x\beta a\leq x\beta b$, where $\alpha,\beta\in\Gamma$.

Example 2.2. Consider an open interval $\mathbb{R}_{\mathbb{O}}=(0,1)$ of real numbers under the binary operation of multiplication. Define $a*b=a\xi b=ba^{-1}r^{-1}$, for all $a,b,r\in\mathbb{R}_{\mathbb{O}}$ and $\xi\in\Gamma$, then for $\Gamma=\{\xi\}$ it is easy to see that $(\mathbb{R}_{\mathbb{O}},*,\leq)$ is an ordered LA-Γ-semigroup or real ordered LA-Γ-semigroup under the usual order "≤". This shows that every ordered LA-semigroup is an ordered LA-Γ-semigroup i.e., an ordered LA-Γ-semigroup is the generalization of an ordered LA-semigroup.

Example 2.3. Consider $S = \{a,b,c\}$ and $\Gamma = \{\gamma_1,\gamma_2,\gamma_3\}$ be defined by as per the table given below

$\cdot$	a	b	c
a	a	c	c
b	a	b	b
c	a	b	b

γ_1	a	b	c
a	c	c	c
b	c	c	c
c	c	c	c

γ_2	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

γ_3	a	b	c
a	b	b	b
b	b	b	b
c	b	b	a

Observing the table above we find that $(a\gamma_1 b)\gamma_2 c = (c\gamma_1 b)\gamma_2 a \ \forall\ a,b,c \in S$ and $\forall\ \gamma_1,\gamma_2,\gamma_3 \in \Gamma$ whereas $(a\gamma_1 b)\gamma_2 c \neq a\gamma_1(b\gamma_2 c)$ i.e. associativity is not satified by S but left invertive law is satisfied. Hence S is an ordered LA-Γ-semiggroup but not an ordered Γ-semiggroups. For the above tables we define left invertive law as per the following order:

$$\leq := \{(a,a),(b,b),(c,c),(a,c),(a,b)\}.$$

Let A be a non-empty subset of an ordered LA-Γ-semigroup S, then we define

$$(A] = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we shall write $(a]$.

Definition 2.4. A non-empty subset A of an ordered LA-Γ-semigroup S, is called an LA-Γ-subsemigroup of S if $A\Gamma A \subseteq A$.

Definition 2.5. A non-empty subset A of an ordered LA-Γ-semigroup S, is called generalized ordered-m-left Γ-ideals of S (resp. generalized ordered-n-right Γ-ideals of S) if

(i) $A^m\Gamma S \subseteq A$ (resp. $S\Gamma A^n \subseteq A$);

(ii) $a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$.

Equivalently, $(A^m\Gamma S] \subseteq A$ (resp. $S\Gamma A^n \subseteq A$) for non-negative integers m and n. An m-left ordered Γ-ideal in S is $(m,0)$-ordered Γ-ideal and an n-right ordered Γ is $(0,n)$-ordered Γ-ideal in S.

Definition 2.6. A non-empty subset A of S is called (m,n) quasi-Γ-ideal of S if

(i) $A^m\Gamma S \cap S\Gamma A^n \subseteq A$;

(ii) $a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$.

Definition 2.7. Let A be non-empty subset of S then A is called oredred (m,n) bi-Γ-ideal of A if

(i) $A^m\Gamma S\Gamma A^n \subseteq A$.

(ii) $a \in A$ and $b \in S, b \leq a \Rightarrow b \in A$.

Every m-left oredered Γ-ideal and n-right oredered Γ-ideal in orederd Γ-semigroup S is an (m,n) bi-Γ-ideal of S where A^0 is defined as $A^0\Gamma S\Gamma A^n = S\Gamma A^n = S$ when $m = 0$ and $A^m S\Gamma A^0 = A^m\Gamma S = S$ when $n = 0$.

Definition 2.8. An ordered (m,n) interior Γ-ideal of S a non-empty subset A such that

(i) $S^m\Gamma A\Gamma S^n \subseteq A$.

(ii) If $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 2.9. A non-empty subset A of S is called an ordered prime if $x\gamma y \in A$ implies $x \in A$ or $y \in A$ for all $x, y \in S$, $\gamma \in \Gamma$, $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$. If A is a Γ-ideal and prime subset of S, then A is called prime Γ-ideal of S.

Definition 2.10. A relation Θ on an ordered LA-Γ-semigroup S is called a pseudo order if $\leq \subseteq \Theta$, Θ is transitive, that is $(a,b),(b,c) \in \Theta$ implies $(a,c) \in \Theta$ for all $a,b,c \in S$ and Θ is compatible, that is if $(a,b) \in \Theta$ then $(a\gamma x, b\gamma x) \in \Theta$ and $(x\gamma a, x\gamma b) \in \Theta$ for all $a,b,x \in S$ and $\gamma \in \Gamma$.

An equivalence relation Θ on an ordered LA-Γ-semigroup S is called a congruence relation if $(a,b) \in \Theta$, then $(a\gamma x, b\gamma x) \in \Theta$ and $(x\gamma a, x\gamma b) \in \Theta$, for all $a,b,x \in S$ and $\gamma \in \Gamma$. A congruence Θ on S is called complete if $[a]_\Theta\gamma[b]_\Theta = [a\gamma b]_\Theta$ for all $a,b \in S$ and $\gamma \in \Gamma$ and $[a]_\Theta$ is the congruence class containing the element $a \in S$.

3. Generalized Rough Subsets in Ordered LA-Γ-semigroups

In this section we define generalized rough subsets in ordered LA-Γ-semigroup and shall show results based on them.

Definition 3.1. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S and A be a non-empty subset of S. Then the sets

$$\Theta_-(A) = \{x \in S : \forall\ y,\ x\Theta y \Rightarrow y \in A\} = \{x \in S : \Theta N(x) \subseteq A\}$$

and

$$\Theta_+(A) = \{x \in S : \exists\ y \in A, \text{ such that } x\Theta y\} = \{x \in S : \Theta N(x) \cap A \neq \emptyset\}$$

are called the Θ-lower approximation and the Θ-upper approximation of A.

For a non-empty subset A of S, $\Theta(A) = (\Theta_-(A), \Theta_+(A))$ is called a rough set with respect to Θ if $\Theta_-(A)$ and $\Theta_+(A)$ are not same.

Example 3.2. Consider $S = \{1,2,3,4,5\}$ and $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ with the following operation "$\cdot$" and the order "$\le$" :

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	5	3	4
5	1	2	3	4	5

γ_1	1	2	3	4	5
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2
5	2	2	2	2	2

γ_2	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	4

γ_3	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	5

$$\le := \{(1,1),(2,2),(2,3),(2,4),(2,5),(3,3),(4,4),(5,5)\}.$$

Hence S is an ordered LA-Γ-semigroup because the elements of S satisfies left invertive law. Now, let

$$\Theta = \{(1,1),(1,4),(2,2),(2,3),(2,4),(2,5),(3,3),(4,4),(5,3),(5,4),(5,5)\}$$

be a complete pseudo order on S, such that

$$\Theta N(1) = \{1,4\},\ \Theta N(2) = \{2,3,4,5\} \text{ and } \Theta N(3) = \{3\},\ \Theta N(4) = \{4\},\ \Theta N(5) = \{3,4,5\}.$$

Now for $A = \{1,2,4\} \subseteq S$,

$$\Theta_-(\{1,2,4\}) = \{1,4\} \quad \text{and} \quad \Theta_+(\{1,2,4\}) = \{1,2,3,4,5\}.$$

So, $\Theta_-(\{1,2,4\})$ is Θ-lower approximation of A and $\Theta_+(\{1,2,4\})$ is Θ-upper approximation of A.

For a non-empty subset A of S, $\Theta(A) = (\Theta_-(A), \Theta_+(A))$ is called a rough set with respect to Θ if $\Theta_-(A) \neq \Theta_+(A)$.

Theorem 3.3. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. If A and B are non-empty subsets of S. Then*

$$\Theta_+(A)\Gamma\Theta_+(B) \subseteq \Theta_+(A\Gamma B).$$

Proof. Let z be any element of $\Theta_+(A)\Gamma\Theta_+(B)$. Then $z = x\gamma y$ where $x \in \Theta_+(A)$ and $y \in \Theta_+(B)$ and $\gamma \in \Gamma$. Thus there exist elements l, $m \in S$ such that

$$l \in A \text{ and } x\Theta l; \quad m \in B \text{ and } y\Theta m.$$

Since Θ is a pseudo order on S, so $x\gamma y\Theta l\gamma m$. As $a\gamma b \in A\Gamma B$, so we have

$$z = x\gamma y \in \Theta_+(A\Gamma B).$$

Thus $\Theta_+(A)\Gamma\Theta_+(B) \subseteq \Theta_+(A\Gamma B)$. □

Definition 3.4. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S, then for each $x, y \in S$ and $\gamma \in \Gamma$, $\Theta N(x)\Gamma\Theta N(y) \subseteq \Theta N(x\gamma y)$. If

$$\Theta N(x)\Gamma\Theta N(y) = \Theta N(x\gamma y),$$

then Θ is called complete pseudo order relation.

Theorem 3.5. *Let Θ be a complete pseudo order on an ordered LA-Γ-semigroup S. If A and B are non-empty subsets of S. Then*

$$\Theta_-(A)\Gamma\Theta_-(B) \subseteq \Theta_-(A\Gamma B).$$

Proof. Let z be any element of $\Theta_-(A)\Gamma\Theta_-(B)$. Then $z = x\gamma y$ where $x \in \Theta_-(A)$ and $y \in \Theta_-(B)$. Thus, we have $\Theta N(x) \subseteq A$ and $\Theta N(y) \subseteq B$. Since Θ is complete pseudo order on S, so we have

$$\Theta N(x\gamma y) = \Theta N(x)\Theta N(y) \subseteq A\Gamma B,$$

which implies that $x\gamma y \in \Theta_-(A\Gamma B)$. Thus $\Theta_-(A)\Gamma\Theta_-(B) \subseteq \Theta_-(A\Gamma B)$. □

4. Generalized Ordered Rough (m,n)-(quasi-Γ, bi-Γ, interier-Γ)-ideals in Ordered LA-Γ-semigroups

Definition 4.1. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a Θ-upper (resp., Θ-lower) rough LA-Γ-subsemigroup of S if $\Theta_+(A)$ (resp., $\Theta_-(A)$) is an LA-Γ-subsemigroup of S.

Theorem 4.2. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S and A be an LA-Γ-subsemigroup of S. Then*

(1) $\Theta_+(A)$ *is an LA-Γ-subsemigroup of S.*

(2) *If Θ is complete, then $\Theta_-(A)$ is, if it is non-empty, an LA-Γ-subsemigroup of S.*

Proof. (1) Let A be an LA-Γ-subsemigroup of S. Then,

$$\emptyset \neq A \subseteq \Theta_+(A).$$

By Theorem 3.3, we have

$$\Theta_+(A)\Gamma\Theta_+(A) \subseteq \Theta_+(A\Gamma A) \subseteq \Theta_+(A).$$

Thus $\Theta_+(A)$ is an LA-Γ-subsemigroup of S, that is, A is a Θ-upper rough LA-Γ-subsemigroup of S.

(2) Let A be an LA-Γ-subsemigroup of S. Then by Theorem 3.5, we have

$$\Theta_-(A)\Gamma\Theta_-(A) \subseteq \Theta_-(A\Gamma A) \subseteq \Theta_-(A).$$

Thus $\Theta_-(A)$ is, if it is non-empty, an LA-Γ-subsemigroup of S, that is, A is a Θ-lower rough LA-Γ-subsemigroup of S. □

The following example shows that the converse of above theorem does not hold.

Example 4.3. We consider a set $S = \{1,2,3,4,5\}$ and $\Gamma = \{\gamma_1,\gamma_2,\gamma_3\}$ with the following operation "$\cdot$" and the order "$\leq$" :

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

γ_1	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

γ_2	1	2	3	4	5
1	4	4	4	4	4
2	4	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	4	4	4	4	2

γ_3	1	2	3	4	5
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	3
5	3	3	3	3	5

Then choosing

$$\leq := \{(1,1),(1,2),(2,2),(2,4),(3,3),(4,4),(5,5)\}.$$

Here S is not an ordered semigroup because $3 = 3\cdot(4\cdot5) \neq (3\cdot4)\cdot5 = 4$. But the elements of S satisfies left invertive law. Hence S is an ordered LA-Γ-semigroup.

Now let

$$\Theta = \{1,1),(1,2),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$$

be a complete pseudo order on S, such that

$$\Theta N(1) = \{1,2\},\ \Theta N(2) = \{2\} \text{ and } \Theta N(3) = \Theta N(4) = \Theta N(5) = \{3,4,5\}.$$

Now for $\{1,2,3\} \subseteq S$,

$$\Theta_-(\{1,2,3\}) = \{1,2\} \text{ and } \Theta_+(\{1,2,3\}) = \{1,2,3,4,5\}.$$

It is clear that $\Theta_-(\{1,2,3\})$ and $\Theta_+(\{1,2,3\})$ are both LA-Γ-subsemigroups of S but $\{1,2,3\}$ is not an LA-Γ-subsemigroup of S.

Definition 4.4. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a Θ-upper (resp., Θ-lower) ordered rough m-left Γ-ideal of S if $\Theta_+(A)$ (resp., $\Theta_-(A)$) is an ordered m-left Γ-ideal of S.

Similarly we can define Θ-upper, Θ-lower ordered rough n-right Γ-ideal and Θ-upper, Θ-lower ordered rough (m,n) bi-Γ-ideals of S.

Theorem 4.5. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S and A be an ordered m-left* (n-right, (m,n)) *bi-Γ-ideal of S. Then*

(1) $\Theta_+(A)$ *is an ordered m-left* (n-right, bi-(m,n))-Γ-ideals of S.

(2) If Θ is complete, then $\Theta_-(A)$ is, if it is non-empty, an ordered m-left (n-right, (m,n)-bi)-Γ-ideal of S.

Proof. (1) Let A be a ordered m-left Γ-ideal of S that is $S^m\Gamma A \subseteq A$. Then by Theorem 3.3 we have (i)

$$S^m\Gamma\Theta_+(A) \subseteq (\Theta_+(S))^m\Gamma(\Theta_+(A) \subseteq \Theta_+(S^m)\Gamma\Theta_+(A) \subseteq)\Theta_+(S^m\Gamma A) \subseteq \Theta_+(A).$$

(ii) Let $a \in \Theta_+(A)$ and $b \in S$ such that $b \le a$. Then there exist $y \in A$, such that $a\Theta y$ and $b\Theta a$. Since Θ is transitive, so $b\Theta y$ implies $b \in \Theta_+(A)$.

This proves that $\Theta_+(A)$ is an ordered m-left Γ-ideal of S, that is, A is a generalized Θ-upper ordered rough m-left Γ-ideal of S. In the similar fashion we can show that generalized Θ-upper approximation of an n-right $((m,n)$-bi-)-Γ-ideal of S is an n-right $((m,n)$-bi-)-Γ-ideal of S.

(2) Let A is an ordered m-left Γ-ideal of S that is $A^m\Gamma S \subseteq A$. By Theorem 3.5(i)

$$S^m\Gamma\Theta_-(A) \subseteq (\Theta_-(S))^m\Gamma(\Theta_-(A) \subseteq \Theta_-(S^m)\Gamma\Theta_-(A) \subseteq)\Theta_-(S^m\Gamma A) \subseteq \Theta_-(A).$$

(ii) Let $a \in \Theta_-(A)$ and $b \in S$ such that $b \le a$. Then $[a]_\Theta \subseteq A$ and $b\Theta a$. This implies that $[a]_\Theta = [b]_\Theta$. Since $[a]_\Theta \subseteq A$, so $[b]_\Theta \subseteq A$. Thus $b \in \Theta_-(A)$.

This proves that $\Theta_-(A)$ is, if it is non-empty, an ordered m-left Γ-ideal of S, that is, A is a generalized Θ-lower ordered rough m-left, n-right $((m,n)$-bi)-Γ-ideal of S. In the similar fashion it can be proved that generalized Θ-lower approximation of an n-right $((m,n)$-bi-)-Γ-ideal of S is an n-right$((m,n)$-bi-)-Γ-ideal of S. □

Definition 4.6. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a Θ-upper (resp., Θ-lower) ordered rough (m,n)-bi-Γ-ideal of S if $\Theta_+(A)$ (resp., $\Theta_-(A)$) is an ordered (m,n)-bi-Γ-ideal of S.

Theorem 4.7. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered (m,n)-bi-Γ-ideal of S, then it is a generalized Θ-upper ordered rough (m,n)-bi-Γ-ideal of S.*

Proof. Let A is an ordered (m,n)-bi-Γ-ideal of S.

(i) By Theorem 3.5 we have

$$(\Theta_+(A))^m\Gamma S\Gamma(\Theta_+(A))^n \subseteq (\Theta_+(A^m)\Gamma\Theta_+(S))\Gamma\Theta_+(A^n) \subseteq \Theta_+((A^m\Gamma S)A^n) \subseteq \Theta_+(A^m\Gamma S\Gamma A^n) \subseteq \Theta_+(A).$$

(ii) Let $a \in \Theta_+(A)$ and $b \in S$ such that $b \le a$. Then there exist $y \in A$, such that $a\Theta y$ and $b\Theta a$. Since Θ is transitive, so $b\Theta y$ implies $b \in \Theta_+(A)$.

From above and Theorem 4.2(1), we get that $\Theta_+(A)$ is an ordered (m,n)-bi-Γ-ideal of S, that is, A is a generalized Θ-upper ordered rough (m,n)-bi-Γ-ideal of S. □

Theorem 4.8. *Let Θ be a complete pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered (m,n)-bi-Γ-ideal of S, then $\Theta_-(A)$ is, if it is non-empty, an ordered (m,n)-bi-Γ-ideal of S.*

Proof. Let A be an ordered (m,n)-bi-Γ-ideal of S.

(i) By Theorem 3.5 we have

$$\begin{aligned}(\Theta_-(A))^m\Gamma S\Gamma(\Theta_-(A))^n &\subseteq (\Theta_-(A^m))\Gamma(\Theta_-(S))\Gamma(\Theta_-(A^n))\\ &\subseteq \Theta_-((A^m\Gamma S)\Gamma\Theta_-(A^n)\\ &\subseteq \Theta_-(A^m\Gamma S\Gamma A^n)\\ &\subseteq \Theta_-(A).\end{aligned}$$

(ii) Let $a \in \Theta_-(A)$ and $b \in S$ such that $b \le a$. Then $[a]_\Theta \subseteq A$ and $b\Theta a$. This implies that $[a]_\Theta = [b]_\Theta$. Since $[a]_\Theta \subseteq A$, so $[b]_\Theta \subseteq A$. Thus $b \in \Theta_-(A)$.

From this and Theorem 4.2(2), we get that $\Theta_-(A)$ is, if it is non-empty, an ordered (m,n)-bi-Γ-ideal of S. □

Theorem 4.9. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. If A and B are an ordered n-right and an ordered m-left Γ-ordered ideals of S respectively, then*

$$\Theta_+(A\Gamma B) \subseteq \Theta_+(A) \cap \Theta_+(B).$$

Proof. The proof is straightforward. □

Theorem 4.10. *Let Θ be a pseudo order on an ordered LA-semigroup S. If A is an ordered n-right and B is an ordered m-left ideals of S, then*

$$\Theta_-(A\Gamma B) \subseteq \Theta_-(A) \cap \Theta_-(B).$$

Proof. The proof is straightforward. □

Definition 4.11. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a Θ-upper (resp., Θ-lower) ordered rough (m,n)-interior Γ-ideal of S if $\Theta_+(A)$ (resp., $\Theta_-(A)$) is an ordered (m,n)-interior Γ-ideal of S.

Theorem 4.12. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered interior (m,n)-Γ-ideal of S, then A is a Θ-upper ordered rough (m,n)-interior Γ-ideal of S.*

Proof. The proof of this theorem is similar to Theorem 4.7. □

Theorem 4.13. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered interior (m,n)-Γ-ideal of S, then $\Theta_-(A)$ is, if it is non-empty, an ordered interior (m,n)-Γ-ideal of S.*

Proof. The proof of this theorem is similar to Theorem 4.8. □

We call A an ordered rough (m,n)-interior Γ-ideal of S if it is both a Θ-lower and Θ-upper ordered rough (m,n)-interior Γ-ideal of S.

Theorem 4.14. *Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered quasi-Γ-ideal of S, then A is a generalized Θ-lower rough oredred quasi-Γ-ideal of S.*

Proof. Let A be a quasi-Γ-ideal of S. Then by Theorem 3.5 we get

$$\begin{aligned}\Theta_-(A)\Gamma S\cap S\Gamma\Theta_-(A) &= \Theta_-(A)\Gamma\Theta_-(S)\cap\Theta_-(S)\Gamma\Theta_-(A)\\ &\subseteq \Theta_-(A\Gamma S)\cap\Theta_-(S\Gamma A)\\ &\subseteq \Theta_-(A\Gamma S\cap S\Gamma A)\\ &\subseteq \Theta_-(A).\end{aligned}$$

(ii) Let $a\in\Theta_-(A)$ and $b\in S$ such that $b\le a$. Then $[a]_\Theta\subseteq A$ and $b\Theta a$. This implies that $[a]_\Theta=[b]_\Theta$. Since $[a]_\Theta\subseteq A$, so $[b]_\Theta\subseteq A$. Hence $b\in\Theta_-(A)$. Thus we get that $\Theta_-(A)$ is a quasi-Γ-ideal of S, that is A is a generalized Θ-lower rough ordered quasi-Γ-ideal of S. □

Definition 4.15. Let Θ be a pseudo order on an ordered LA-Γ-semigroup S. Then a non-empty subset A of S is called a Θ-upper (resp., Θ-lower) ordered rough (m,n)-quasi-Γ-ideal of S if $\Theta_+(A)$ (resp., $\Theta_-(A)$) is an ordered (m,n)-quasi-Γ-ideal of S.

Theorem 4.16. *Let Θ be a complete pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered (m,n)-quasi-Γ-ideal of S, then A is a Θ-lower ordered rough (m,n)-quasi-Γ-ideal of S.*

Proof. Let A be an ordered (m,n)-quasi-Γ-ideal of S.

(i) Now by Theorem 3.5 we get

$$\begin{aligned}S^m\Gamma\Theta_-(A)\cap\Theta_-(A)\Gamma S^n &= (\Theta_-(S))^m\Gamma\Theta_-(A)\cap\Theta_-(A)\Gamma(\Theta_-(S))^n\\ &\subseteq \Theta_-(S^m)\Gamma\Theta_-(A)\cap\Theta_-(A)\Gamma\Theta_-(S^n)\\ &\subseteq \Theta_-(S^m\Gamma A)\cap\Theta_-(A\Gamma S^n\end{aligned}$$

$$\subseteq \Theta_-(S^m\Gamma A \cap A\Gamma S^n)$$
$$\subseteq \Theta_-(A).$$

(ii) Let $a \in \Theta_-(A)$ and $b \in S$ such that $b \leq a$. Then $[a]_\Theta \subseteq A$ and $b\Theta a$. This implies that $[a]_\Theta = [b]_\Theta$. Since $[a]_\Theta \subseteq A$, so $[b]_\Theta \subseteq A$. Thus $b \in \Theta_-(A)$.

Thus we obtain that $\Theta_-(A)$ is an ordered (m,n)-quasi-Γ-ideal of S, that is, A is a Θ-lower ordered rough (m,n)-quasi-Γ-ideal of S. □

Theorem 4.17. *Let Θ be a complete pseudo order on an ordered LA-Γ-semigroup S. If A is an ordered (m,n)-quasi-Γ-ideal of S, then A is a Θ-upper ordered rough (m,n)-quasi-Γ-ideal of S.*

Proof. Let A be an ordered (m,n)-quasi-Γ-ideal of S.

(i) Now by Theorem 3.3, we get

$$\begin{aligned}\Theta_+(A^m)\Gamma S \cap S\Gamma\Theta_+(A^n) &= \Theta_+(A^m)\Gamma\Theta_+(S)\cap\Theta_+(S)\Gamma\Theta_+(A^n)\\ &\subseteq \Theta_+(A^m\Gamma S)\cap\Theta_+(S\Gamma A^n)\\ &= \Theta_+(A^m\Gamma S\cap S\Gamma A^n)\\ &\subseteq \Theta_+(A).\end{aligned}$$

(ii) Let $a \in \Theta_+(A)$ and $b \in S$ such that $b \leq a$. Then $[a]_\Theta \subseteq A$ and $b\Theta a$. This implies that $[a]_\Theta = [b]_\Theta$. Since $[a]_\Theta \subseteq A$, so $[b]_\Theta \subseteq A$. Thus $b \in \Theta_+(A)$.

Thus, we obtain that $\Theta_+(A)$ is an ordered (m,n)-quasi-Γ-ideal of S, that is, A is a Θ-upper ordered rough (m,n)-quasi-Γ-ideal of S. □

Theorem 4.18. *Let Θ be a complete pseudo order on an ordered LA-Γ-semigroup S. Let L and R be a Θ-lower ordered rough m-left Γ-ideal and a Θ-lower ordered rough n-right Γ-ideal of S, respectively. Then $L \cap R$ is a Θ-lower ordered rough (m,n)-quasi-Γ-ideal of S.*

Proof. The proof is straightforward. □

5. Conclusion

(m,n)-(quasi-Γ, bi-Γ, interior-Γ)-ideals of ordered LA-Γ-semigroups in terms of rough sets have been discussed. Through pseudo orders, it is proved that the two-sided ideals and (m,n) (resp., quasi-Γ-ideals, bi-Γ-ideals and interior-Γ-ideals) in ordered LA-Γ-semigroups becomes lower and upper rough two-sided ideals and (m,n) (resp., quasi-Γ-ideals, bi-Γ-ideals and interior-Γ-ideals) in ordered LA-Γ-semigroups.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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