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Research Article

Stability Results of Solution of Non-Homogeneous Impulsive Retarded Equation Using the Generalized Ordinary Differential Equation

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Abstract. This work is devoted to the study of a non-homogeneous impulsive retarded equation with bounded delays and variable impulse time using the generalized ordinary differential equations (GODEs). The integral solution of the system satisfying the Caratheodory and Lipschitz conditions obtained using the fundamental matrix theorem is embedded in the space of the generalized ordinary differential equations and investigate the problem of stability of the system in the Lyapunov sense. In particular, results on the necessary and sufficient conditions for stability and asymptotic stability of the impulsive retarded system via the generalized ordinary differential equation are obtained. An example is used to illustration the derived theory.

Keywords. Generalized ordinary differential equation; Regulated function; Fundamental matrix solution; Kurzweil integral; Bounded variation; Stability; asymptotic stability

Mathematics Subject Classification (2020). 45A05; 34A30; 34N05

1. Introduction

The analysis of the qualitative properties of systems of the dynamic model equation is of great importance in mathematical sciences. This is because of the significant roles of such model equations (especially differential equations) in the study of real-life problems. The impulsive retarded differential equation is one such model equations. An impulsive retarded differential

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equation is a delay equation couple with a difference equation known as the impulsive term. The early research work on impulsive retarded differential equations are found in ([10, 17]). Research interest in this area is on the increase as seen in ([4–7, 9, 12–14, 20]).

The establishment of the correspondence between the non-absolutely convergent integral (Kurzweil integral) and the ordinary differential equations in [16] was the first breakthrough in the study of a generalized ordinary differential equation. It is an extension of the Riemann theory of integration. The increased research interest in this field is largely base on the fact that investigation into the qualitative properties of other dynamic equations like the delay equations, impulsive equations and difference equations via the generalized ordinary differential equation is done in a unified viewpoint of dynamical flow theory. That is if an autonomous function is considered for $f(t,x) = f(x)$, then $\varphi(t,y) = \varphi(t,y,f)$ is a local flow, and the technique of topological dynamic can be applied. Otherwise, if the function is non-autonomous, then the local flow translate is not guaranteed. In this case, the topological dynamic of the Kurzweil equation will then consider the limit point of the translate $f_t = f(x,t+s)$ under the assumption that: — the limiting equation satisfying the Lipschitz and Caratheodory conditions is not an ordinary differential equation, and the space of the ordinary equation is not complete. But that if the ordinary differential equation is embedded in the Kurzweil equations we obtained a complete and compact space, such that the techniques of the topological dynamics can be applied (Igobi and Abasiekwere [12]).

Kurzwiel [16] defined the space W where the translate f_t of f is embedded to be complete uniform space (complete metric space) such that $(t,f) \to f_t$ is continuous and $\varphi(f,x,t)$ must also be continuous on $W \times R^n \times R$, and W is compact. Then the function f and the ordinary differential equation

$$\dot{x}(t) = f(x,t) \tag{1.0}$$

are members of the space W. That is, W is a function space and a space of equation satisfying

(i) $V \subset W$ be a compact set, then there exists a number M_V such that

$$|f(x,s)| \le M_V, \quad x \in V.$$

(ii) $V \subset W$ be a compact set, then there exists a number K_V such that

$$|f(x,t) - f(y,t)| \le K_V |x-y|, \quad x,y \in V.$$

Thus, the topology of the space containing all the functions $f(x,t)$ is a metric topology characterized by the convergence

$$f_k \to f_0 \text{ if } \int_0^t f_k(x,s)ds \to \int_0^t f_0(x,s)ds. \tag{1.1}$$

Artstein [3] presented more relaxed Kurzweil conditions as presented:

(i$'$) $V \subset W$ be a compact set, then there exists a locally Lebesgue integrable function $M_V(s)$ such that

$$|f(x,s)| \le M_V(s),\ x \in V \text{ and } \int_0^{t+h} M_V(s)ds \text{ is uniformly continuous in } s.$$

That is, there exists $\mu_V(\varepsilon) > 0$ such that $\int_0^{t+h} M_V(s)ds < \varepsilon$, $h \le \mu_V(\varepsilon)$.

(ii′) $V \subset W$ be a compact set, then there exists a locally Lebesgue integrable function $K_V(t)$ such that

$$|f(x,t)-f(y,t)| \le K_V(t)|x-y|, \quad x,y \in V \quad \text{and} \quad \int_0^t K_V(s)ds \le N_V$$

for all s and N_V fixed.

The precompactness of the function space and the continuity of $(t,f) \to f_t(t,x)$ is guarantee by equation (1.1) fulfilling (i$'$) and (ii$'$) for some N_V and μ_V. That is, the Cauchy sequence f_i implies $\int_0^t f(x,s)ds$ converges for all (x,t). However,

$$A(t)x = \lim_{\tau \to t} \int_0^\tau f_i(x,s)ds$$

does not necessarily imply the integral representation

$$A(t)x = \int_0^t f(x,s)ds.$$

Consider the integral equivalent of the ordinary equation (1.0)

$$x(t)-x(t_0) = \int_{t_0}^t f(x,\tau)d\tau. \tag{1.2}$$

Suppose the integral is a Riemann integral, we can define a δ-fine partition $t_0 = s_1 \le s_2 \le \ldots \le s_{n+1} = t$ on $[t_0,t]$ and an $\eta_i \in [s_{i-1}, s_i]$, $i = 1,2,3,\ldots$. The differential approximation of equation (1.2) in the Riemann sense is

$$\sum_{i=1}^n \int_{s_i}^{s_{i+1}} f(x(\delta_i)\delta_i)d\delta. \tag{1.3}$$

If we defined

$$A(s)x(t) = \int_{t_0}^t f(x,\delta)d\delta,$$

so that by equation (1.3) we have

$$\sum_{i=1}^t [A(s_{i+1})-A(s_i)]x(\eta_i) = S(dA,x,P). \tag{1.4}$$

Equation (1.4) defined the Riemann-Kurzweil sum approximation of $x(t)-x(t_0)$ if and only if $x(t)$ is a solution of (1.2) and $[s_i, s_{i+1}]$ is a fine partition. Thus for any $I \in R^n$,

$$I = \int_{t_0}^t dA(s)x(\eta) \tag{1.5}$$

defines the Kurzweil integral if there exists an $\varepsilon > 0$ such that

$$|I - S(dA,x,P)| \le \varepsilon.$$

The differential equation resulting from equation (1.5) is known as the generalized ordinary differential equation.

The establishment of the correspondence between the generalized ordinary differential equation and other types of differential systems are found in [12, 19].

In this work, we established results on stability and asymptotic stability of a non-homogeneous impulsive retarded equation with bounded delays and variable impulse time in the Lyapunov sense via the generalized ordinary differential equation. It is an extension of the works: Igobi and Abasiekwere [12] where results on uniqueness were established, Federson and

Schwabik [8], Afonso *et al.* [1] where stability results of homogeneous systems were considered.

We consider a non-homogeneous impulsive retarded functional system of the form

$$\left.\begin{aligned} \dot{y}(t) &= B_0 y(t) + B_1 y_t + u(t), && h(t, y_t(t)) \neq 0 \\ \Delta y(t) &= I_k(y(t)), && h(t, y_t(t)) = 0, \end{aligned}\right\} \tag{1.6}$$

where $h(t, y_t(t))$ defines a spatial-temporal relation. If $h(t, y_t(t)) \neq 0$ then system (1.6) is governed by the retarded equation only, otherwise at any point $h(t, y_t(t)) = 0$ the system undergoes impulse and instantly changes by some amount $I_k(y(t_k))$.

Assumption 1.0.

(a) We assume that the set of points $(t, y) \in G([t_0, t_0 + \eta], X)$ for which $h(t, y_t(t)) = 0$ consist of a sequence of hypersurfaces satisfying:

(i) $t = \tau_k(y(t))$, $\tau \in G([t_0, t_0 + \eta], X)$, $k = 0, 1, 2, \dots$

(ii) $\tau_0(y) < \tau_1(y) < \tau_2(y) < \dots$

(iii) $\lim_{k \to \infty} \tau_k(y) = \infty$

(b) The solution $y(t) \in G([t_0, t_0 + \eta], X)$ is right continuous, and so the initial data $y_{t_0} = \varphi$ is generalized to a regulated function that incorporates the left-hand and right-hand limit.

By Assumption 1.0 we re-defined (1.6) as initial-value problem

$$\left.\begin{aligned} \dot{y}(t) &= B_0 y(t) + B_1 y_t + u(t), && t \neq \tau_k(y(t^-)) \\ \Delta y(t_k) &= I_k(y(t^-), && t = \tau_k(y(t^-)) \\ y_{t_0} &= \varphi, \end{aligned}\right\} \tag{1.7}$$

where $B_{0_{n\times n}}$, $B_{1_{n\times 1}}$ are constant matrices, $y(t) \in G([t_0, t_0 + \eta], X)$, $u(t) : [t_0, t_0 + \eta] \to X$, are regulated right continuous Lebesgue integrable functions on $[t_0, t_0 + \eta]$, $y_t \in G([-r, 0], X)$ for $t \in [t_0, t_0 + \eta]$ expresses the history of $y(t)$ such that $y_t(\phi) = y(t + \phi)$, $\phi \in [-r, 0]$. $\varphi \in G([-r, 0], X)$ and $\Delta y(t) = y(t+) - y(t-) = y(t+) - y(t)$.

In [12] the fundamental matrix theorem was used to present the solution of equation (1.7) as

$$y(t) = \varphi + \int_{t_0}^{t} L(s, y_s) ds + \int_{t_0}^{t} f(s) ds + \sum_{k=1}^{n} I_k(y(t_k)), \tag{1.8}$$

for $L : PC([t_0 - r, t_0 + \eta], L(X)) \times [t_0, t_0 + \eta] \to X$ and $f(t) : [t_0, t_0 + \delta] \to X$ such that $t \to L(t, y_t)$ and $t \to f(t)$ are Lebesgue integrable functions satisfying the following Caratheodory and Lipschitz assumptions (1.1).

Assumption 1.1. (A_1): there exists a Kurzweil integrable function $M_0 : [t_0, t_0 + \eta] \to R$, such that

$$\left\| \int_{s_0}^{s_1} L(s, y_s) ds \right\| \leq \int_{s_0}^{s_1} M_0(s) ds, \quad \text{for } s_1, s_2 \in [t_0, t_0 + \eta],\ y \in G([t_0, t_0 + \eta], R^n),$$

(A_2): there exists a Kurzweil integrable function $M_1 : [t_0, t_0 + \eta] \to R$, such that

$$\left\| \int_{s_0}^{s_1} L(s, x_s) - L(s, y_s) ds \right\| \leq \int_{s_0}^{s_1} M_1(s) \|x_s - y_s\| ds, \ \text{for } s_1, s_2 \in [t_0, t_0+\eta],\ x, y \in G([t_0, t_0+\eta], R^n),$$

(A_3): there exists a real constant $M > 0$ such that

$$\left\| \int_{s_0}^{s_1} f(s)ds \right\| \le \int_{s_0}^{s_1} M ds, \quad s_1, s_2 \in [t_0, t_0+\eta],$$

(A_4): there exist positives constants K_1, K_2 such that for $k = 1,2,\dots,n$ and all $x, y \in R^n$

$$|I_k(y)| \le K_1 |y| \text{ and } |I_k(x) - I_k(y)| \le K_2 |x-y|.$$

2. Generalized Ordinary Differential Equation

Let X be a Banach space and $L(X)$ a Banach space of bounded linear operators on X, with $\|\cdot\|_X$ and $\|\cdot\|_{L(X)}$ defining the topological norms in X and $L(X)$, respectively. A partition is any finite set $U = \{s_0, s_1, \dots, s_{i+1}\}$ such that $a = s_0 < s_1 < \dots < s_{i+1} = b$. Given any finite step function $A(t) : [a,b] \to L(X)$, for $A(t)$ being a constant on (s_{i-1}, s_i), then $\operatorname{var}_a^b A(t) = \sup \left\{ \sum_{i=1}^{n(P)} \|A(s_i) - A(s_{i-1})\|_X \right\}$ is the variation of $A(t)$ on $[a,b]$. The function $A(t)$ is of bounded variation on $[a,b]$ if $\operatorname{var}_a^b A(t) < \infty$.

The function $A(t) : [a,b] \to L(X)$ is regulated on $[a,b]$ if the one sided-limits $A(t-) = \lim_{s\to t-} A(s)$ and $A(t+) = \lim_{s\to t+} A(s)$ exist at every point of $t \in [a,b]$. That is $A(a-) = A(a)$ and $A(b+) = A(b)$ such that $\Delta^- A(t) = A(t) - A(t-)$ and $\Delta^+ A(t) = A(t+) - A(t)$ for all $t \in [a,b]$. By $G([a,b], L(X))$ we denote the set of all regulated functions $A(t) : [a,b] \to L(X)$, which is a Banach space when endowed with the usual supremum norm $\|A\|_\infty = \sup\{\|A(t)\|_X,\ t \in [a,b]\}$.

A tagged division of a compact interval $U[a,b] \subset R$ is a finite collection of point-interval pairs $P = (v, U)$, where $U = \{s_0, s_1, \dots, s_{i+1}\}$ and $v_i \in [s_{i-1}, s_i]$ (that is $(v_i, [s_{i-1}, s_i])$) . A gauge on $[a,b]$ is any positive function $\delta : [a,b] \to (0,\infty)$. A tagged division $P(v_i[s_{i-1}, s_i])$ is δ-*fine* if for every $i = 1,2,\dots,[s_{i-1}, s_i] \subset (v_i - \delta(v_i), v_i + \delta(v_i))$.

Consequent of Definitions 1, 6 and Lemmas 2, 4 in [12], the initial value problem of the linear non-homogeneous generalized ordinary differential equation is presented as

$$\left.\begin{array}{l} \frac{dx}{dt} = d[A(t)x(t) + g(t)] \\ x(t_0) = x_0, \end{array}\right\} \tag{2.0}$$

for $A : [t_0, t_0+\eta] \to L(X)$, $g : [t_0, t_0+\eta] \to X$, $x_0 \in X$ and $\operatorname{var}_{t_1}^{t_2} A < \infty$, $t_1, t_2 \in [t_0, t_0+\eta]$.

2.1 Solution of System (2.0) using A Fundamental Matrix Equation

Proposition 2.0 (Schwabik [19]). *Assume that* $A : [t_0, t_0+\eta] \to X$, $\operatorname{var}_{t_0}^{t_0+\eta} A < \infty$ *and the matrix* $I - \Delta^- A(t)$ *is regular for all* $t \in [t_0, t_0+\eta]$. *Then the matrix equation*

$$K(t) = K_0 + \int_{t_0}^{t} d[A(s)K(s)], \tag{2.1}$$

has for every $K_0 \in X$ *a unique solution* $K(t)$ *on* $[t_0, t_0+\eta]$.

Definition 2.0. A matrix $K(t)$ satisfying the matrix equation (2.1) isa fundamental matrix of the system (2.0).

Theorem 2.0 (Schwabik [19]). *If the assumption of Proposition* 2.0 *is satisfied, then there exists a unique* $n \times n$ *matrix value function* $U(t,s)$ *defined on* $t_0 \le s \le t_0 + \eta$ *such that*

$$U(t,s) = I + \int_s^t d[A(r)]U(r,s). \tag{2.2}$$

Lemma 2.0 (Schwabik [19]). *Suppose that the assumption of Proposition* 2.0 *is fulfilled, then there exists a constant* $M > 0$ *such that* $\|U(t,s)\| \le M$ *for all* (t,s)*, and* $t,s \in [t_0, t_0+\eta]$. *Moreover, we have*

$$\|U(t_2,s) - U(t_1,s)\| \le M \operatorname{var}_{t_1}^{t_2} A,$$

for all $t_0 \le t_1 \le t_2 \le s \le t_0 + \eta$*, and consequently* $\operatorname{var}_{t_0}^{s} U(\cdot,s) \le M \operatorname{var}_{t_0}^{s} A$.

Lemma 2.1 (Schwabik [19]). *Suppose that the assumption of Proposition* 2.0 *is satisfied. If we define*

$$\bar{U}(t,s) = U(t,s), \qquad \text{for } t_0 \le s \le t \le t_0 + \eta$$
$$\bar{U}(t,s) = U(t,t) = I, \quad \text{for } t_0 \le t \le s \le t_0 + \eta$$

where $U(t,s) \in X$ *is given by equation* (2.2)*, then for the two-dimensional variations of* $\bar{U}$ *on the squares* $[t_0, t_0+\eta] \times [t_0, t_0+\eta]$ *on which* $\bar{U}$ *is defined we have* $\operatorname{var}_{t_0}^{t_0+\eta} \bar{U} < \infty$.

Let us assume that $U(t,s) \in X$ satisfied Theorem 2.0, Lemma 2.0 and 2.1, then by variation of constant formula equation (2.0) implies that

$$x(t) = U(t,t_0)x_0 + g(t) - g(t_0) - \int_{t_0}^t d[U(t,s)](g(s) - g(t_0)). \tag{2.3}$$

Integrating the last term by parts,

$$\int_{t_0}^t d[U(t,s)](g(s) - u(t_0))$$
$$= U(t,t)[g(t) - g(t_0)] - U(t,t_0)[g(t) - g(t_0)] - \int_{t_0}^t U(t,s)d[(g(s) - g(t_0)].$$

Substituting the term back into equation (2.3) we obtained

$$x(t) = U(t,t_0)x_0 - \int_{t_0}^t U(t,s)d[(g(s) - g(t_0)] + \sum_{t_0 \le r \le t} \Delta^- U(t,\sigma)\Delta^- g(\sigma), \quad t_0 \le \sigma \le t. \tag{2.4}$$

Equation (2.4) is the integral solution of the generalized ordinary differential equation (2.0) using the fundamental matrix equation (2.2).

Lemma 2.2. *If* $g(t) \in BV([a,b])$*, then* $\sum_{a<t<b} \|g(t)\| = \operatorname{var}_a^b g(t)$.

Proof. By the assertion in [11], the set of discontinuities of $g(t)$ in the interval $[a,b]$ can be written as $\{t_k \in X; k \in N\}$, we can assume without loss of generality that $t_k < t_{k+k}$, $k \in N$. So that for each $n \in N$,

$$S_n = \|\Delta^- g(a)\| + \sum_{a<t<b} \|\Delta^- g(t_k)\|.$$

Let $\varepsilon > 0$ and $n \in N$ be given. We can choose a $\delta_k > 0$, in such a way that

$$\|g(t_k - \eta_k) - g(t_k -)\| < \frac{\varepsilon}{2(n+1)} \quad \text{and} \quad [t_k -_k, t_k] \bigcap \{t_1, t_2, \dots, t_n\} = \{t_k\}.$$

We choose $\eta_0 > 0$ such that

$$\|g(a - \eta_0) - g(a-)\| < \frac{\varepsilon}{2}, \quad a - \eta > t_n.$$

Then,

$$\begin{aligned} S_n &\le \|g(a) - g(a - \eta_0)\| + \|g(a - \eta_0) - g(a-)\| + \sum_{k=1}^{n} \|g(t_k) - g(t_k - \eta_k)\| \\ &\quad + \sum_{k=1}^{n} \|g(t_k - \eta_k) - g(t_k -)\| \\ &< \|g(a) - g(a - \eta_0)\| + \frac{\varepsilon}{2} + \sum_{k=1}^{n} \|g(t_k) - g(t_k - \eta_k)\| + \frac{\varepsilon}{2(n-1)} \\ &< \varepsilon + \|g(a) - g(a - \eta_0)\| + \sum_{k=1}^{n} \|g(t_k) - g(t_k - \eta_k)\|, \end{aligned}$$

and

$$\lim_{n \to \infty} S_n \le \sup_{a \le s_k \le b} \{\|g(s_k) - g(s_{k-1})\|\} = \mathrm{var}_a^b g.$$

Given that $y \in PC([t_0 - r, t_0 + \eta], X)$, and $t \to L(t, y_t)$ are Kurzweil integrable on $[t_0, t_0 + \delta]$, and using the claim of Lemma 14 in [5] and the results of [15], [18] we define the functions $F(t, y) : PC([t_0 - r, t_0 + \eta], X) \to C([t_0 - r, t_0 + \eta], X)$ and $g(t) \in C([t_0, t_0 + \eta], X)$ on $[t_0, t_0 + \delta]$ as follow:

$$\text{(i)} \quad F(t, y)(v) = \begin{cases} 0, & t_0 - r \le v \le t_0 \\ \displaystyle\int_{t_0}^{v} L(s, y_s) ds, & t_0 \le v \le t \le t_0 + \eta \\ \displaystyle\int_{t_0}^{t} L(s, y_s) ds, & t_0 \le t \le v \le t_0 + \eta \end{cases} \tag{2.5}$$

$$\text{(ii)} \quad g(t)(v) = \begin{cases} \displaystyle\int_{t_0}^{v} f(s) ds, & t_0 \le v \le t \le t_0 + \delta \\ \displaystyle\int_{t_0}^{t} f(s) ds, & t_0 \le t \le v \le t_0 + \delta \end{cases} \tag{2.6}$$

Also, we define a unit step function

$$\int_k^c (t) = \begin{cases} 0, & t_0 < t < t_k^c \\ 1, & t \ge t_k^c, \end{cases}$$

concentrated at $t_k^c \in [t_0, \infty)$ so that given $v \in [t_0, t_0 + \eta]$ and $x \in G([t_0 - r, t_0 + \eta], X)$, the impulsive term in equation (1.8) is defined as

$$\text{(iii)} \quad J(y, t)(v) = \sum_{k=1}^{n} \int_k^c (t) \int_k^c (v) I_k(y(t_k)). \tag{2.7}$$

Consider the regulated function $A(t, y)(v) : G([t_0, t_0 + \eta], L(X)) \to G([t_0 - r, t_0 + \eta], L(X))$ defined by

$$A(t, y)(v) = F(t, y)(v) + J(t, y)(v), \tag{2.8}$$

for $v \in [t_0, t_0+\eta]$, such that given $s_1, s_2 \in [t_0, t_0+\eta]$ and $x, y \in G([t_0, t_0+\eta], L(x))$ we have

$$\|F(x,s_2) - F(x,s_1)\| \le |h(s_2) - h(s_1)|$$

and

$$\|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| \le \|x-y\|\,|h_1(s_2) - h_1(s_1)|$$

where $h_1(t) = \int_{t_0}^{t} [M_0(s) + M_1(s)]ds$, $t \in [t_0, \infty)$.

Similarly, for $J(y,t)(v) = \sum_{k=1}^{n} \int_k^c(t) \int_k^c(v) I_k(y(t_k))$, we have

$$\|J(x,s_2) - J(x,s_1)\| \le \sum_{k=1}^{n} \left[\int_{t_k}(s_2) - \int_{t_k}(s_1) \right] K_1$$

and

$$\|J(x,s_2) - J(x,s_1) - J(y,s_2) + J(y,s_1)\| \le \sum_{k=1}^{n} \left[\int_{t_k}(s_2) - \int_{t_k}(s_1) \right] K_2 \|x-y\|$$

where $h_2(t) = \max(K_1, K_2) \sum_{k=1}^{+\infty} \int_k^c(t_k)(t)$.

Now, for $A(t,y)(v) : G([t_0, t_0+\eta], L(X)) \to G([t_0 - r, t_0+\eta], L(X))$ defined in equation (2.8), we have

$$\|A(x,s_2) - A(x,s_1)\| \le h_1(s_2) - h_1(s_1) + h_2(s_2) - h_2(s_1) = (h(s_2) - h(s_1)),$$

$$\|A(x,s_2) - A(x,s_1) - A(y,s_2) + A(y,s_1)\| \le \|x-y\| |h(s_2) - h(s_1)|,$$

where $h(t) = h_1(t) + h_2(t)$ is a non-decreasing real function which is continuous from the left (Federson and Shwabik [9]). Then the generalized non-homogenous linear ordinary differential equation (2.0) is properly defined and has the integral equation form as

$$\begin{aligned} x &= x_0(t_0) + \int_{t_0}^{v} dF(t,x) + \int_{t_0}^{v} dJ(t,x) + g(t) - g(t_0) \\ &= x_0(t_0) + \int_{t_0}^{v} d[A(t)]x + g(t) - g(t_0) \end{aligned} \tag{2.9}$$

satisfying the initial condition

$$x(t_0)(v) = \begin{cases} \varphi(v - t_0), & v \in [t_0 - r, t_0] \\ \varphi(t_0)(t_0), & v \in [t_0, t_0+\delta]. \end{cases} \tag{2.10}$$

□

The equivalence result given one-to-one correspondence between solutions (2.4) and (1.8), and the existence and uniqueness theorems of the solution of system (1.7) using system (2.0) are well established in ([3, 7, 9, 12]).

3. Main Results

3.1 Stability of the Non-homogeneous Generalized Differential Equation

Definition 3.0. Let $A : G([t_0, t_0+\eta], L(X) \to G([t_0 - r, t_0+\eta], L(X)$ be a regulated function, and $g \in BV[t_0, t_0+\eta]$ is of locally bounded variation such that $\mathrm{var}_{t_0}^{t_0+\delta} g < \infty$. Then the solution function $T(t,x) \in G([t_0, t_0+\eta], X)$ is of bounded variation if there exists a non-decreasing function

$h:[0,\infty)\to R$ such that $h(0)=0$ and

$$\text{(a)}\quad \|T(x,t_2)-T(x,t_1)\| = \left\|\int_{t_1}^{t_2} dA(x(s),t)+g(t)\right\| \le \|A(x,t_2)+g(t_2)-A(x,t_1)-g(t_1)\| \le |h(t_2)-h(t_1)|+\operatorname{var}_{t_1}^{t_2} g \le \infty$$

$$\text{(b)}\quad \|T(x,t_2)-T(x,t_1)-T(y,t_2)+T(y,t_1)\| = \left\|\int_{t_1}^{t_2} d[A(x(s),t)+A(y(s),t)]\right\| \le \|A(x,t_2)-A(x,t_1)+A(y,t_2)-A(y,t_1)\| \le \|x-y)\|\ |h(t_2)-h(t_1)|.$$

Definition 3.1. Let $T\in F(\Omega,h)$, for $\Omega\subset B_q\times[t_0,t_0+\eta]$, for $B_q=\{x\in X,\|x\|<q\}$, $0<q<c$ and $h:[0,\infty)\to R$ is left continuous non-decreasing function, such that $T(0,t)-T(0,\eta)=0$ for $t,\eta>0$. Then for every $[v_1,v_2]\subset[t_0,t_0+\eta]$, we have

$$x(t)=\int_{v_1}^{v_2} d[T(0,t)] = T(0,v_2)-T(0,v_1)=0, \tag{3.0}$$

and $x\equiv 0$ is the trivial solution of the generalized differential equation (2.0), and $x:[v_1,v_2]\to X$ is of bounded variation on $[v_1,v_2]$.

Definition 3.2. The trivial solution $(x\equiv 0)$ of equation (2.0) is variationally stable if and only if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that if $\|x_0\|<\delta_0$ and $g\in BV[t_0,t_0+\eta]$ continuous from the left, with $\alpha(t)>0$, for all $t\in[t_0,t_0+\eta]$

$$\operatorname{var}_{t_0}^{t_0+\eta}\left(x(t)-M\alpha(t)\int_{t_0}^{t} D[A(x,s)]\right)=\operatorname{var}_{t_0}^{t_0+\eta} g<\delta(\varepsilon),$$

then $\|x(t,t_0,x_0)\|<\varepsilon$.

Definition 3.3. The trivial solution $(x\equiv 0)$ of (2.0) is variationally attracting if there exists a $\delta_0>0$ and for every $\varepsilon>0$ there exists a $T=T(\varepsilon)\ge 0$ and $\delta=\delta(\varepsilon)>0$ such that if $x:[t_0,t_0+\eta]\to X$ is a function of bounded variation on $[t_0,t_0+\eta]$, and $g\in BV[t_0,t_0+\eta]$ continuous from the left, with $\alpha(t)>0$ for all $t\in[t_0,t_0+\eta]$ such that

$$\|x(t_0)\|<\delta_0 \ \text{ and } \ \operatorname{var}_{t_0}^{t_0+\eta}\left(x(t)-M\alpha(t)\int_{t_0}^{t} D[A(x,s)]\right)=\operatorname{var}_{t_0}^{t_0+\eta} g<\delta,$$

then

$$\|x(t,t_0,x_0)\|<\varepsilon.$$

Definition 3.4. The trivial solution $(x\equiv 0)$ of (2.0) is variationally asymptotically stable if it is variationally stable and variationally attracting.

Lemma 3.0. *Let $A(x,t):G([t_0,t_0+\eta],X)\to G([t_0-r,t_0+\eta],X)$ be a regulated function for all $t>0$, $g\in BV([t_0,t_0+\eta])$ is a locally bounded variation on $[t_0,t_0+\eta]$. Then the trivial solution $(x\equiv 0)$ of equation (2.0) is variationally stable if $\|U(t,t_0)\|\le M$.*

Proof. Assume $\|U(t,t_0)\| \le M$, and also using the integral solution (2.4), for $g \in BV[t_0,t_0+\eta]$ satisfying Lemma 2.1, then

$$\begin{aligned}\|x(t,t_0,x_0)\| &\le \|U(t,t_0)\|x_0 - \int_{t_0}^{t} \|U(t,s)\|D_s[g(s)-g(t_0)] + \sum_{t_0<\sigma<t} \|\Delta^- U(t,\sigma)\|\Delta^- g(\sigma)\\ &\le M\|x_0\| - \int_{t_0}^{t} MD_s[g(s)-g(t_0)] + \sum_{t_0<\delta<t} \|U(t,\sigma)-U(t,\sigma-)\|\Delta^- g(\sigma)\\ &\le M\|x_0\| - M\operatorname{var}_{t_0}^{t} g + 2M\operatorname{var}_{t_0}^{t} g\\ &\le 2M\delta < \varepsilon,\end{aligned}$$

for $\|x_0\| < \delta$, $\operatorname{var}_{t_0}^{t_0+\delta} g < \delta$, and $\delta = \frac{\varepsilon}{2M}$. By Definition 3.2, the trivial solution of equation (2.4) is variationally stable.

Conversely, assume the fundamental matrix is bounded then,

$$\begin{aligned}&\|U(t,t_0+\eta) - U(t,t_0)\|\alpha(t)\\ &\quad\le M\|A(x,t_0+\eta) - A(x,t_0)\|\alpha(t)\\ &\quad\le M\alpha(t)\operatorname{var}_{t_0}^{t_0+\eta}\left[\int_{t_0}^{t_0+\eta} D[A(x,s)]\right] + \|x(t_0+\eta) - x(t_0)\|\\ &\quad\le \left[\left(x(t_0+\eta) - M\alpha(t)\int_0^{t_0+\eta} D[A(x,s)]\right) - \left(x(t_0) - M\alpha(t)\int_0^{t_0} D[A(x,s)]\right)\right]\\ &\quad\le \operatorname{var}_0^{t_0+\eta} g - \operatorname{var}_0^{t_0}\\ &\quad= \operatorname{var}_{t_0}^{t_0+\eta} \le \delta.\end{aligned}$$

Hence,

$$\|x(t)\| = \|x(t,t_0,x_0)\| < \varepsilon. \qquad \square$$

The Lyapunov functional theorem as proved in [19] for the generalized ordinary differential equation is presented in the next lemma, but in this case, the solution of equation (2.0) is via the fundamental matrix equation.

Lemma 3.1. *Let* $V : [t_0,\infty) \times X \to \Re$ *be such that* $V(\bullet,x)$ *is continuous from the left and* $V(\bullet,x) \in BV([t_0,\infty))$ *satisfies*

$$\|V(t,x) - V(t,y)\| \le K\|x-y\|, \quad x,y \in X,\ t \in [0,\infty), \tag{3.1}$$

where $K > 0$. *Also assume that for any solution* $x : (a,b) \to X$, *for* $(a,b) \subset [t_0,\infty)$, *there exists a function* $\phi(x(t)) : X \to \Re$ *such that*

$$\lim_{\eta\to 0} \frac{V(t+\eta, x(t+\eta)) - V(t,x(t))}{\eta} \le -\phi(x(t)), \tag{3.2}$$

then

$$V(t,y(t)) \le V(t_0,y(t_0)) - K\operatorname{var}_{t_0}^{t_0+\eta}\left(y(t) + M\alpha(t)\int_{t_0}^{t} d[A(t,x(s))]\right) + \eta\phi(y(t)) \tag{3.3}$$

holds.

Proof. Let x be a solution of the generalized differential equation $\frac{dx}{dt} = DA(t,x)$, such that

$$y(t_0) \le \|U(t_0,t_0)\|x(t_0) \le x(t_0), \quad \text{for } y:[t_0,t_0+\eta] \to X,$$

and by equation (3.1)

$$\begin{aligned}
&V(t_0+\eta, y(t_0+\eta)) - V(t_0+\eta, x(t_0+\eta)) \\
&\quad \le K\|y(t_0+\eta) - x(t_0+\eta)\| \\
&\quad = K\left\| y(t_0+\eta) + U(t,t_0)x_0 - \int_{t_0}^{t_0+\eta} U(s,t_0)d[g(t_0+\eta)+g(t_0)]\right\| \\
&\quad \le K\left\| y(t_0+\eta) + U(t,t_0)x_0 - \alpha(t)\int_{t_0}^{t_0+\eta} d[U(s,t_0)]\right\| \\
&\quad \le K\left\| y(t_0+\eta) + y(t_0) - M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,x(s)]\right\|. \qquad (3.4)
\end{aligned}$$

By the presentation in [19],

$$\begin{aligned}
V(t_0+\eta, y(t_0+\eta)) - V(t_0, y(t_0)) &= V(t_0+\eta, y(t_0+\eta)) - V(t_0+\eta, x(t_0+\eta)) \\
&\quad + V(t_0+\eta, x(t_0+\eta)) - V(t_0, x(t_0)).
\end{aligned}$$

Using hypothesis (3.2) and equation (3.4), we obtained

$$\begin{aligned}
&V(t_0+\eta, y(t_0+\eta)) - V(t_0, y(t_0)) \\
&\quad \le K\left\| y(t_0+\eta) - y(t_0) - M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,x(s)])\right\| + \eta\phi(y(t)) \\
&\quad \le K\left\| y(t_0+\eta) - y(t_0) - M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,y(s)])\right\| \\
&\qquad + K\left\| M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,y(s)]) - M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,x(s)])\right\| + \eta\phi(y(t)) \\
&\quad \le K\left\| y(t_0+\eta) - M\alpha(t)\int_{0}^{t_0+\eta} d[A(t,y(s)])\right\| + K\left\| y(t_0) - M\alpha(t)\int_{0}^{t_0} d[A(t,y(s)])\right\| \\
&\qquad + K\left\| M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,y(s)]) - M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,x(s)])\right\| + \eta\phi(y(t)) \\
&\quad \le K[\mathrm{var}_0^{t_0+\eta} g + \mathrm{var}_0^{t_0} g] \\
&\qquad + K\left\| M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,y(s)]) - M\alpha(t)\int_{t_0}^{t_0+\eta} d[A(t,x(s)])\right\| + \eta\phi(y(t)) \\
&\quad = K[\mathrm{var}_0^{t_0+\eta} g + \mathrm{var}_0^{t_0} g] + KM\|\alpha(t)\|\left\| \int_{t_0}^{t_0+\eta} d[A(t,y(s) - A(t,x(s)])\right\| + \eta\phi(y(t)).
\end{aligned}$$

Hence

$$\begin{aligned}
&V(t_0+\eta, y(t_0+\eta)) - V(t_0, y(t_0)) \\
&\quad \le K[\mathrm{var}_0^{t_0+\eta} g + \mathrm{var}_0^{t_0} g] + KM\|\alpha(t)\|\left\| \int_{t_0}^{t_0+\eta} d[A(t,y(s) - A(t,x(s)])\right\| + \eta\phi(y(t)). \qquad (3.5)
\end{aligned}$$

Using Definition 3.0, we estimate the second term on the left as

$$\left\| \int_{t_0}^{t_0+\eta} d[A(t,y(s)) - A(t,x(s),t)]\right\|$$

$$
\begin{aligned}
&\le \int_{t_0}^{t_0+\eta} d(w\,\|y(s)) - x(s)\|)\,h(t) \\
&\le \int_{t_0}^{t_0+\eta} w(\|y(s)) - x(s)\|)\,dh(t) \\
&= \lim_{\tau\to 0}\left(\int_{t_0}^{t_0+\tau} + \int_{t_0+\tau}^{t_0+\eta}\right) w(\|y(s)) + x(s)\|)\,dh(t) \\
&= \lim_{\tau\to 0}\int_{t_0}^{t_0+\tau} w(\|y(s)) - x(s)\|)\,dh(t) + \lim_{\tau\to 0}\int_{t_0+\tau}^{t_0+\eta} w(\|y(s)) - x(s)\|)\,dh(t) \\
&\le \sup_{\sigma\in[t_0,t_0+\eta]} w(0)\lim_{\tau\to 0}(h(t_0+\tau) - h(t_0)) \\
&\quad + \sup_{\sigma\in[t_0,t_0+\eta]} w(\|y(\sigma)) - x(\sigma)\|)\lim_{\tau\to 0}(h(t_0+\eta) - h(t_0+\tau)) \\
&= \sup_{\sigma\in[t_0,t_0+\eta]} w(\|y(\sigma)) - x(\sigma)\|)(h(t_0+\eta) - h(t_0+)).
\end{aligned} \tag{3.6}
$$

Also, by Theorem 1.3.5 (Kurzweil [1]), for $\rho \in (t_0, t_0+\eta(t_0))$ and

$$x(\rho) = y(t_0) + \int_{t_0}^{\rho} D[A(t,x(s))] \quad (\text{for } y(t_0) = x(t_0))$$

being a solution, then

$$y(\rho) - x(\rho) = y(\rho) - y(t_0) - \int_{t_0}^{\rho} D[A(t,x(s))].$$

Therefore,

$$
\begin{aligned}
\lim_{\rho\to t_0+}(y(\rho) - x(\rho)) &= \lim_{\rho\to t_0+}\left(y(\rho) - y(t_0) - \int_{t_0}^{\rho} D[A(t,x(s))]\right) \\
&= y(t_0+) - y(t_0) - \lim_{\rho\to t_0+}\left(\int_{t_0}^{\rho} D[A(t,x(s))]\right) \\
&= y(t_0+) - y(t_0) - \lim_{\rho\to t_0+}\left(\int_{0}^{t_0} D[A(t,x(s))] - \int_{0}^{\rho} D[A(t,x(s))]\right) \\
&= y(t_0+) - \lim_{\rho\to t_0+}\left(-\int_{0}^{\rho} D[A(t,x(s))]\right) - y(t_0) - \lim_{\rho\to t_0+}\left(\int_{0}^{t_0} D[A(t,x(s))]\right) \\
&= \lim_{\rho\to t_0+} g(\rho) - g(t_0) \\
&= g(t_0+) - g(t_0).
\end{aligned}
$$

Hence

$$\lim_{\rho\to t_0+}\|(y(\rho) - x(\rho))\| = \|g(t_0+) - g(t_0)\| = \operatorname{var}_{t_0}^{t_0+} g. \tag{3.7}$$

Given any arbitrary $\varepsilon > 0$ and setting

$$\gamma = \frac{\varepsilon}{\Lambda} > 0, \quad \text{for } KM\,\|\alpha(t)\| \le \Lambda, \tag{3.8}$$

such that $\widehat{\rho}(\gamma) > 0$ for $\rho \in (0, \widehat{\rho}(\gamma)]$, we have $w(\rho) < \gamma$ and $\gamma \in (0, \widehat{\rho}(\alpha)/2)$. Let there exists $\eta_1(t_0) > 0$, $\eta_1(t_0) < \eta(t_0)$ such that for $\rho \in (t_0, t_0+\eta_1(t_0))$

$$\|(y(\rho) - x(\rho))\| = \operatorname{var}_{t_0}^{t_0+} g + \lambda. \tag{3.9}$$

We defined an infinite set $M(\gamma)$

$$M(\gamma)=\left\{t_0\in[t_0-\tau,t_0+\tau];\operatorname{var}_{t_0}^{t_0+}g\geq\frac{\widehat{\rho}(\gamma)}{2}\right\},$$

such that for γ defined in equation (3.8), and $g\in BV[t_0-\tau,t_0+\tau]$ we have

$$\omega(\|y(\rho)-x(\rho)\|)\leq\omega(\operatorname{var}_{t_0}^{t_0+}g+\lambda)<\omega(\widehat{\rho}(\gamma))<\gamma.$$

Hence by equation (3.6), we obtained

$$\left\|\int_{t_0}^{t_0+\eta}d[A(t,y(s)-A(t,x(s))]\right\|\leq\gamma(h(t_0+\eta-h(t_0+))),\quad \eta\in(0,\eta_2(t_0)). \tag{3.10}$$

If the $\lim_{\eta\to0^+}h(t_0+\eta)=h(t_0+)$ exists, then for any $\eta_3(t_0)>0$ such that $\eta_3(t_0)\leq\eta_2(t_0)$, and $0<\eta<\eta_3(t_0)$ we have

$$h(t_0+\eta)-h(t_0+)<\frac{\gamma}{(l(\gamma)+1)\omega(\operatorname{var}_{t_0}^{t_0+}+\lambda)}.$$

Using equation (3.6) and (3.9), we obtained

$$\left\|\int_{t_0}^{t_0+\eta}d[A(t,y(s)-A(t,x(s))]\right\|\leq\frac{\gamma}{l(\gamma)+1}, \tag{3.11}$$

for $l(\gamma)$ defining the number of elements in set $M(\gamma)$.

Schwabik [19] defined

$$h_\gamma(t)=\frac{\gamma}{l(\gamma)+1}\sum_{t_0\in M(\gamma)}H_{t_0}(t),\quad t\in[t_1,t_2],$$

where $H(t)=0$, for $t\leq t_0$, and $H(t)=1$, for $t_0<t$. The function $h_\gamma:[t_1,t_2]\to R$ is a non-decreasing and continuous from the left, such that

$$\operatorname{var}_{t_1}^{t_2}h_\gamma=h_\gamma(t_2)-h_\gamma(t_1)=\frac{\gamma}{l(\gamma)+1}<\gamma. \tag{3.12}$$

The points of discontinuity of h_γ are only the points in $M(\gamma)$, such that for any

$$t\in M(\gamma),\quad h_\gamma(t+)-h_\gamma(t)=\frac{\gamma}{l(\gamma)+1}.$$

Defining

$$\widehat{h}_\gamma(t)=\gamma h_c(t)+h_\gamma(t),\quad t\in[t_1,t_2],$$

where $h_c(t)$ is the continuous part of the function h. Then $\widehat{h}_\gamma$ is non-decreasing and continuous from the left on $[t_1,t_2]$ and

$$\widehat{h}_\gamma(t_1)-\widehat{h}_\gamma(t_0)=\gamma[h_c(t_2)-h_c(t_1)]+h_\gamma(t_2)-h_\gamma(t_1)\leq\gamma(h(t_2)-h(t_1)+1.$$

By equation (3.10) and (3.11), and for $\eta=\eta_2(t_0)$ and $\eta=\eta_2(t_0)$, where $t_0\in[t_1,t_2]$ and $\widehat{h}_\gamma:[t_1,t_2]\to R$ we have

$$\left\|\int_{t_0}^{t_0+\eta}d[A(t,y(s)-A(t,x(s))]\right\|\leq\widehat{h}_\gamma(t_0+\eta)-\widehat{h}_\gamma(t_0).$$

Using equations (3.5), (3.12), (3.13) for $\eta\in[t_0,t_0+\eta]$ we have

$$\begin{aligned}V(t_0+\eta,y(t_0+\eta))-V(t_0,y(t_0))&\leq K[\operatorname{var}_0^{t_0+\eta}g+\operatorname{var}_0^{t_0}g]+\Lambda\left(\widehat{h}_\gamma(t_0+\eta)-\widehat{h}_\gamma(t_0)\right)+\eta\phi(y(t))\\&\leq K[\operatorname{var}_0^{t_0+\eta}g+\operatorname{var}_0^{t_0}g]+\Lambda\gamma+\eta\phi(y(t))\\&\leq K[\operatorname{var}_{t_0}^{t_0+\eta}g]+\varepsilon+\eta\phi(y(t)),\end{aligned}$$

and the hypothesis of Lemma 3.1 holds. That is

$$V(t,y(t)) \le V(t_0,y(t_0)) - Kvar_{t_0}^{t_0+\eta}\left(y(t)+M\alpha(t)\int_{t_0}^{t} ds[A(s)]\right)+\eta\phi(y(t)). \qquad \square$$

Consequent to Lemma 3.1, we state the sufficient conditions for the variational stability and variational asymptotic stability for the trivial solution ($x \equiv 0$) of the generalized ordinary differential equation (2.0) as presented in [19].

Theorem 3.0. *Assume* $V:[t_0,+\infty)\times B_q \to R_+$, *for* $B_q=\{x\in X, \|x\|<q\}$, $0<q<c$ *is positive definite, continuous from the left on* $[0,+\infty)$ *and satisfies the following:*

(i) $V(t,0)=0$, $t\in[t_0,\infty)$.

(ii) $\|V(t,x)-V(t,z)\|\le K|x-z|$, *for* $K>0$ *being a constant.*

(iii) $V(t,x)$ *is positive definite such that there exists a continuous increasing function* $b:[t_0,+\infty)\to R_+$, *for* $b(0)=0$ *and* $V(t,x)\ge b(\|x(t)\|_X)$, *for* $(t,x)\in[t_0,+\infty)\times B_q$.

(iv) *the right derivative along the integral path of equation* (2.0) *is negative definite. That is*

$$DV(t,x)^+ = \lim_{\eta\to 0^+}\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\le 0,$$

then the trivial solution ($x\equiv 0$) *of equation* (2.0) *is variationally stable.*

Theorem 3.1. *Assume* $V:[t_0,+\infty)\times B_q\to R_+$, *for* $B_q=\{x\in X,\|x\|<q\}$, $0<q<c$ *is positive definite, continuous from the left on* $[t_0,+\infty)$, *and satisfies* (i)-(iv) *of Theorem 3.0. Suppose also that there exists a continuous function* $\varphi: X\to R$, *such that* $\varphi(0)=0$ *and* $\varphi(x)>0$, *for* $x\ne 0$, *and if for every solution* $x:[t_0,t_0+\eta]\to B_q$, *we have*

$$\lim_{\eta\to 0^+}\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\le -\varphi(x(t)), \quad t\in[t_0,t_0+\eta].$$

Then the trivial solution $x\equiv 0$ *of equation* (2.0) *is variationally asymptotically stable.*

Consider the solution of equation (1.8) with the following assumptions:

$$L(0,t)=0,\ I_k(0)=0,\ k=1,2,\dots, \quad \text{for all } t\in[t_0,t_0+\delta], \text{ and } \lim_{t\to\infty} f(t)=0.$$

Definition 3.5. The function $y\equiv 0$ satisfying the above assumptions is the trivial solution of equation (1.8) and by implication system (1.7) on $[t_0,t_0+\eta]$.

The following classical definitions of stability of the trivial solution ($y\equiv 0$) of (1.7) are presented:

Definition 3.6. (i) The solution $y\equiv 0$ of the system (1.7) is stable if for every $\varepsilon>0$, $t_0\ge 0$ there exists $\delta=\delta(\varepsilon,t_0)>0$ such that if $\varphi\in G([t_0-r,t_{0+\eta}],X)$, and for any solution $\bar{y}=\bar{y}(t_0,\varphi):[t_0-r,v]\to X$ of (1.7) on $[t_0-r,v]$ such that $y_{t_0}=\varphi$ and

$$\|\varphi(\varepsilon,t_0)\|<\delta,$$

then

$$\|\bar{y}_t(t_0,\varphi)\|<\varepsilon, \quad \text{for } t\in[t_0,v].$$

(ii) The solution $y \equiv 0$ of the system (1.7) is uniformly stable if (i) hold and $\delta = \delta(\varepsilon) > 0$ (δ independent on t_0).

(iii) The solution $y \equiv 0$ of the system (1.7) is uniformly asymptotically stable if there exists $\delta_0 > 0$ for every $\varepsilon > 0$, $t_0 \ge 0$, there exists $T = T(\varepsilon, t_0) \ge 0$ such that if $\varphi \in \Gamma_c$, and for any solution $\bar{y} = \bar{y}(t_0, \varphi) : [t_0 - r, v] \to X$ of (1.7) on $[t_0 - r, v]$ such that $y_{t_0} = \varphi$ and

$$\|\varphi\| < \delta_0,$$

then

$$\|\bar{y}_t(t_0, \varphi)\| < \varepsilon, \quad \text{for } t \in [t_0, v] \bigcap [t_0 + T, \infty).$$

We employed results of Theorems 19 and 20 in [12], Theorems 3.0, 3.1 and Lemma 3.1 to present the stability results of the initial value problem (1.7).

Let $y \in G([t_0, t_0 + \eta], X)$ be the solution of the impulsive retarded differential equation (1.7) on $[t_0 - r, t_0 + \eta]$ and $x(t) \in G([t_0, t_0 + \eta], X)$ the solution of the generalized ordinary differential equation (2.0) satisfying the initial condition

$$x(t_0)(v) = \begin{cases} \varphi(v - t_0), & v \in [t_0 - r, t_0] \\ \varphi(t_0)(t_0), & v \in [t_0, t_0 + \eta], \end{cases}$$

then $x(t_0) = \varphi_{t_0}$, where $\varphi_{t_0} = \varphi(t_0 - r)$. Also, for $y_t \in G([t_0 - r, t_0 + \eta], X)$, we have

$$x(t)(t + \theta) = y(t + \theta) = y_t(\theta), \quad \text{for } \theta \in [t_0 - r, t_0],$$

such that $x(t)_t = y_t$, $t \in [t_0, t_0 + \eta]$, and at t_0, $x(t_0)_{t_0} = y_{t_0} = \varphi$.

Given that $L : G([t_0 - r, t_0 + \eta], L(X)) \times [t_0, t_0 + \eta] \to X$ depend on both t and y for $y_t \in G([t_0 - r, t_0 + \eta], X)$, then L is a functional such that we defined a set $\Gamma_a = \{\psi : [t_0, t_0 + \eta] \to X | \psi(t^-) = \psi(t)$ for all but finite number of points $t \in [t_0, t_0 + \eta]\}$ and $\|\psi\|_r = \sup_{t_0 - r \le s \le t_0} \|\psi(s)\| < a$, $0 < a < \kappa$. That is ψ has left-hand limit and is right continuous on $[t_0, t_0 + \eta]$ so that $L(y_t, t) \equiv L(\psi, t)$.

Remark 3.0. Consequent of the results of Theorems 19 and 20 in [12], for $y \in G([t_0 - r, t_0 + \eta], X)$ and $x \in G([t_0, t_0 + \eta], X)$ being solutions of system (1.7) and (2.0) respectively, then $(t, x(t)) \to (t, y_t)$ is one-to-one mapping and

$$\begin{aligned} \|x(t)_t\| &= \sup_{t_0 - r \le \theta \le t_0} |x(t + \theta)| \\ &= \sup_{t_0 - r \le v \le t} |x(v)| \\ &= \sup_{t_0 - r \le v \le t} |y(v)| \\ &= \sup_{t_0 - r \le \theta \le t_0} |y(t + \theta)| \\ &= \sup_{t_0 - r \le t \le t_0 + \eta} |y(t)| \\ &= \|y_t\| \\ &= \|\psi\|. \end{aligned}$$

Let $W : [t_0, \infty) \times \Gamma_a \to R_+$ be the associated Lyapunov functional of the impulsive retarded equation (1.7) which we relate to the functional $V : [t_0, +\infty) \times B_q \to R_+$ of the generalized ordinary differential equation (2.0) as

$$V(t,x) = W(t,y_t) = W(t,\psi). \tag{3.13}$$

Lemma 3.2. *Let $y \in G([t_0,t_0+\eta],X)$, $L : G([t_0-r,t_0+\eta],L(X)) \times [t_0,t_0+\eta] \to X$, $f(t):[t_0,t_0+\delta] \to X$ such that $t \to L(t,x_t)$ and $t \to f(t)$ are Lebesgue integrable and satisfies conditions A, B, A_1, A_2, A_3, and $I_k(t_k)$, $k = 1,2,\ldots$ satisfies A_4. Assume further that $W : [t_0,\infty) \times \Gamma_a \to R_+$ is positive definite, right continuous, locally Lipschitz in ψ and satisfies:*

(i) $W(t,0) = 0$, $t \in [t_0,\infty)$.

(ii) $\|W(t,\psi) - W(t,\bar{\psi})\| \le K|\psi - \bar{\psi}|$, *for* $K > 0$, *and* $\psi,\bar{\psi} \in \Gamma_a$

(iii) $D^+W(t,\psi) = \lim_{\eta\to0^+}\sup \dfrac{W(t+\eta,\psi(t+\eta)) - W(t,\psi(t))}{\eta}$.

Then $\|V(t,x) - V(t,\bar{x})\| \le K|x - \bar{x}|$, *and* $D^+W(t,\psi) = \lim_{\eta\to0^+}\sup \dfrac{V(t+\eta,x(t+\eta)) - V(t,x(t))}{\eta}$.

Proof. Let $y,\bar{y},\hat{y} \in G([t_0 - r,t_0 + \eta],X)$ be the solution of the retarded impulsive equation (1.7) such that for $\psi,\bar{\psi} \in \Gamma_a$, we have that $y_t = \psi$, $\bar{y}_t = \bar{\psi}$ and $\hat{y}_t = 0$. Also, let $x,\bar{x},\hat{x} \in G([t_0,t_0+\eta],X)$ be the solution of the generalized ordinary differential equation 23.0). Then by Remark 3.0 we have that $x(t)_t = \psi$ $\bar{x}(t)_t = \bar{\psi}$ and $\hat{x}(t)_t = 0$, and using Theorem (3.0) for $V : [t_0,+\infty) \times B_q \to R_+$, we have

$$V(t,0) = V(t,x_t(t,0)) = W(t,y_t(t,0)) = W(t,0) = 0, \quad t \in [t_0,\infty).$$

Considering the hypothesis (ii), for $K > 0$ we have

$$\begin{aligned}
\|W(t,\psi) - W(t,\bar{\psi})\| &= \|W(t,y_t) - W(t,\bar{y}_t)\| \\
&= \|V(t,x(t)_t) - V(t,\bar{x}(t)_t)\| \\
&\le K\|x(t)_t - \bar{x}(t)_t\| \\
&\le \sup_{t_0-r\le\theta\le t_0} K|x(t_0+\theta) - \bar{x}(t_0+\theta)| \\
&\le \sup_{t_0-r\le v\le t} K|x(v) - \bar{x}(v)| \\
&\le K|x(t) - \bar{x}(t)|.
\end{aligned}$$

By hypothesis (iii) we have

$$\begin{aligned}
D^+W(t,\psi) &= \lim_{\eta\to0^+}\sup \frac{W(t+\eta,\psi(t+\eta)) - W(t,\psi(t))}{\eta} \\
&= \lim_{\eta\to0^+}\sup\left(\frac{W(t+\eta,\psi(t+\eta)) - W(t,\psi(t))}{\eta}\right) + \lim_{\eta\to0^+}\left(\frac{W(t+\eta,y_t(t)) - W(t+\eta,\psi(t+\eta))}{\eta}\right) \\
&= \lim_{\eta\to0^+}\sup\left(\frac{W(t+\eta,\psi(t+\eta)) - W(t,\psi(t))}{\eta} + \frac{W(t+\eta,y_t(t)) - W(t+\eta,\psi(t+\eta))}{\eta}\right) \\
&= \lim_{\eta\to0^+}\sup\left(\frac{W(t+\eta,y_t(t)) - W(t,\psi(t))}{\eta}\right) \\
&= \lim_{\eta\to0^+}\sup\left(\frac{V(t+\eta,x(t)_t) - V(t,x(t))}{\eta}\right)
\end{aligned}$$

$$= \limsup_{\eta\to 0^+}\left(\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\right).$$

Hence the boundedness of the right derivative of $W[t_0,\infty)\times\Gamma_c\to R_+$ along $x\in G([t_0,t_0+\eta],X)$. □

Theorem 3.2. *Suppose $L(t,y_t)$ and $f(t)$ satisfies conditions A_1, A_2, A_3, and I_k, $k=1,2,\dots$ satisfies A_4. Assume further that $W:[t_0,\infty)\times\Gamma_a\to R_+$ satisfies the conditions*

(i) $W(t,0)=0$, $t\in[t_0,\infty)$,

(ii) $\|W(t,\psi)-W(t,\bar\psi)\|\le K|\psi-\bar\psi|$, *for* $K>0$, *and* $\psi,\bar\psi\in\Gamma_a$,

(iii) $b(\|\psi\|)\le W(t,\psi)$, *for* $t\in[t_0,\infty)$, $\psi\in\Gamma_a$,

(iv) $D^+W(t,\psi)\le 0$.

Then the trivial solution $(y\equiv 0)$ *of* (1.7) *is uniformly stable.*

Proof. Let $L(t,y_t)$, $f(t)$, satisfies conditions A_1, A_2, A_3, and I_k, $k=1,2,\dots$ satisfies A_4, also $A:[t_0,t_0+\delta]\to L(X)$, $g:[t_0,t_0+\delta]\to X$ be regulated functions. Using Lemma 3.2 and equation (3.13), for $V:[t_0,\infty)\times B_q\to R_+$ being right continuous and positive definite, then

$$V(t,0)=W(t,0),\quad \text{for } \hat{x}(t)_t=\hat{y}_t$$

and

$$\|V(t,x(t)_t)-V(t,\bar{x}(t)_t)\|=\|W(t,\psi)-W(t,\bar\psi)\|\le K\|\psi-\bar\psi\|=K\|x-\bar x\|,$$

then conditions (i) and (ii) implies.

Using Lemma 3.1, we have

$$\begin{aligned}V(\hat{t},x\hat{t}))&\le V(t_0,x(t_0))+K\operatorname{var}_{t_0}^{\hat{t}}\left(x(s)-M\alpha(t)\int_{t_0}^{t}dA(t,x(s))\right)\\&\le K\|x(t_0)\|+K\operatorname{var}_{t_0}^{\hat{t}}\left(x(s)-M\alpha(t)\int_{t_0}^{t}dA(t,x(s))\right).\end{aligned}$$

By Definition 3.2 and setting $2K\delta(\varepsilon)<a(\varepsilon)=\varepsilon$ we have

$$V(\hat{t},x\hat{t}))\le K\delta(\varepsilon)+K\delta(\varepsilon)=2K\delta(\varepsilon)<a(\varepsilon)=\varepsilon. \tag{3.14}$$

We define $[t_0-r,t_0+\eta]$ as the maximal interval of existence of the solution $x(t,\psi)_t$ and set

$$a(\varepsilon)=\inf\{b(t^*),\ t^*\in[t_0,t_0+\eta]|\ \|x(t*)\|>\varepsilon\}.$$

Then

$$a(\varepsilon)=\inf b(t^*)<b\|x(t^*)_t\|=b\|y_t(t^*)\|\le W(t^*,y_t(t^*))=V(t^*,x(t^*)_t), \tag{3.15}$$

which contradicts equation (3.14), hence $\|x(t)_t\|<\varepsilon$ and condition (iii) implies.

Proving condition (iv), we use equation (3.13) to imply

$$V(t+\eta,x(t+\eta))=W(t+\eta,x(t+\eta)_{t+\eta})$$

and

$$W(t+\eta,x(t+\eta)_{t+\eta})=W(t+\eta,y_{t+\eta})=W(t,\psi).$$

So that for $\eta\ge 0$,

$$\frac{1}{\eta}V(t+\eta,x(t+\eta))-V(t,x(t))=\frac{1}{\eta}W(t+\eta,y_{t+\eta})-W(t,y_t)$$

$$= \frac{1}{\eta} W(t+\eta, \psi(t+\eta)) - W(t, \psi(t)),$$

and

$$\lim_{\eta\to 0} \sup \frac{1}{\eta} V(t+\eta, x(t+\eta)) - V(t, x(t)) = \lim_{\eta\to 0} \sup \frac{1}{\eta} W(t+\eta, \psi(t+\eta)) - W(t, \psi(t))$$
$$= D^+(t, \psi) \le 0.$$

Hence, the trivial solution $x \equiv 0$ of system (2.0) is variationally stable, and so given any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $\bar{x} : [t_0, t_0+\eta) \times B_q \to R_+$ being a function of bounded variation on $[t_0, t_0+\eta]$, we have that

$$\|x_0\| < \delta(\varepsilon)$$

$$\operatorname{var}_{t_0}^{t_0+\eta}\left(\bar{x}(s) - M\alpha(t)\int_{t_0}^{t} D[A(x,s)]\right) = \operatorname{var}_{t_0}^{t_0+\eta} g < \delta(\varepsilon),$$

then

$$\|\bar{x}(t, t_0, y_0)\| < \varepsilon.$$

We prove the reverse $\|\bar{y}_t(t_0, \varphi)\| < \varepsilon$.

Let $\bar{y}_{t_0} = \varphi$ be the solution of system (1.7) on $[t_0, t_0+\eta]$, for $\varphi \in \Gamma_a$. Assume $\|\varphi\| < \delta$, then for

$$\bar{x}(t_0)(v) = \begin{cases} \varphi(v - t_0), & v \in [t_0 - r, t_0] \\ \varphi(t_0)(t_0), & v \in [t_0, t_0+\eta] \end{cases}$$

and by Remark 3.0, $\bar{x}(t_0)(v) = \sup_{t_0 - r < v < t_0} \bar{x}(v) = \|\varphi\| < \delta$.

Also for $\bar{x}(t) \in B_q$ a function of bounded variation such that

$$\operatorname{var}_{t_0}^{v}\left(\bar{x}(s) - M\alpha(t)\int_{t_0}^{t} d[A(s,x)]\right) = \operatorname{var}_{t_0}^{v} g < \delta,$$

then $\|\bar{x}(v)\| < \varepsilon$ holds.

Therefore, by Remark 3.0

$$\|\bar{y}_t(t_0, \varphi)\| = \sup_{t_0 - r \le \theta \le t_0} \|\bar{y}(t+\theta)\| = \sup_{t_0 - r \le v \le t_0} \|\bar{y}(v)\| = \sup_{t_0 - r \le v \le t_0} \|\bar{x}(v)_t\| = \|\bar{x}(v)\| < \varepsilon,$$

and hence, $\|\bar{y}_t(t_0, \varphi)\| < \varepsilon$ is proved. □

Theorem 3.3. *Suppose $L(t, y_t)$ and $f(t)$ satisfies conditions A_1, A_2, A_3, and I_k, $k = 1, 2, \dots$ satisfies A_4. Let $W : [t_0, \infty) \times \Gamma_a \to R_+$ satisfies the hypothesis* (i), (ii), (iii) *of Theorem* 3.2. *Assume further that there exists a continuous function $h : \Re_+ \to \Re_+$ with $h(0) = 0$, and $h(x) > 0$, for $x \ne 0$, such that for $\psi \in \Gamma_a$*

$$\lim_{\eta\to 0^+} \sup \frac{1}{\eta}(W(t+\eta, \psi(t+\eta)) - W(t, \psi(t))) \le -h(\|\psi\|), \quad t \in [t_0, t_0+\eta].$$

Then the trivial solution $y \equiv 0$ is uniformly asymptotically stable.

Proof. Given $V : [0, \infty) \times B_q \to R_+$ as defined in equation (3.13), then the hypothesis of Theorem 3.0 as proved in [19] is satisfies. Consequently, we defined a continuous function $\phi : B_q \to R$ such that $\phi(0) = 0$, $\phi(z) > 0$, for $z \ne 0$, and

$$\phi(z) = h(\|z\|), \quad z \in B_q. \tag{3.16}$$

By Remark 3.0 and Theorem 4.0 and Theorem 4.1 in [12], we have $(t,x(t)) \to (t,y_t)$, and so using equation (2.10)

$$x(t)(t+\theta) = y(t+\theta) = y_t(\theta) \quad \text{and}$$

$$x(t+\eta)_{t+\eta} = y_{t+\eta}, \quad t \in [t,t+\eta].$$

Hence

$$\lim_{\eta\to 0}\sup \frac{1}{\eta}(V(t+\eta,x(t+\eta)) - V(t,x(t))) = \lim_{\eta\to 0}\sup \frac{1}{\eta}(W(t+\eta,y_{t+\eta}) - W(t,y_t)) \le -h \sup_{v\le t}(|y(v)|).$$

Again, from Remark 3.0 and equation (3.2) we have

$$\sup_{v\le t}|y(v)| = \sup_{v\le t}|x(v)| = \|x(t)_t\| = \|x(t)\|$$

and so

$$\lim_{\eta\to 0}\sup \frac{1}{\eta}(V(t+\eta,x(t+\eta)) - V(t,x(t))) \le -h \sup_{v\le t}(|y(v)|) = -h(\|x(t)\|) = -\phi(x(t)),$$

which satisfies Lemma 3.1.

Therefore by Definition 3.4, $x \equiv 0$ is variationally asymptotically stable and so for $\varepsilon > 0$, there exists $\delta_0 > 0$, $T = T(\varepsilon) \ge 0$ and $\delta = \delta(\varepsilon) > 0$ such that for any solution $\bar{x} : [t_0, t_0+\eta] \to X$ and $g \in BV[t_0,t_0+\eta]$ continuous from the left with $\alpha(t) > 0, t \in [t_0,t_0+\eta]$ such that

$$\|\bar{x}(t_0)\| < \delta_0$$

and

$$\text{var}_{t_0}^{t_0+\eta}\left(\bar{x}(t) - M\alpha(t)\int_{t_0}^{t} d[A(x,s)]\right) = \text{var}_{t_0}^{t_0+\eta} g < \delta,$$

then $\|\bar{x}(t)\| < \varepsilon$.

Also for $\varepsilon > 0$, let $\delta_0 > 0$ and $T = T(\varepsilon) \ge 0$ be defined. Given $\phi \in \Gamma_a$ and any solution $\bar{y} : [t_0 - r, t_0 + \eta] \to X$ such that $\bar{y}_0 = \phi$ and $\|\phi\| < 0$, then by the proof of Theorem 3.2 and the assertion of Theorem 3.3, $x \equiv 0$ is variationally stable and variational asymptotically stable. Consequently, $y \equiv 0$ of (1.7) for

$$\|\bar{y}_t(t)\| < \varepsilon$$

is variationally stable and variational asymptotically stable. □

Illustration

We consider the general form of equation (1.7) with matrix B_0 satisfying Hurwitz inequality. Then there exists a positive definite matrix $P \in R^{n\times m}$ that solves the Lyapunov equation

$$B_0^T P + PB_0 = -I.$$

Also for $\lambda_{\min}(P)$ defining the smallest eigenvalue of matrix P, we assume that $\|PB_0\| \le \frac{1}{2}$ and $u(t)$ a non-negative function. Then we defined the Lyapunov function associated with system (1.7) for $W : [t_0,\infty) \times \Gamma_a \to R_+$ as

$$W(t,\psi) = \psi(t)^T P\psi(t) + \frac{1}{2}\int_{t_0-r}^{t} \psi(s)^T \psi(s)ds.$$

Since $(t,\psi) \in ([t_0,\infty) \times \Gamma_a, R_+)$, then conditions (i) and (iii) of theorem (3.2) are easily confirmed. That is $W(t,0) = 0$ and $W(t,\psi) \ge \lambda_{\min}(P)\|\psi\|^2$.

The time derivative of W along the integral path of system (1.7) is computed as

$$\begin{aligned}
D^+W(t,\psi) &= \psi(t)^T P(B_0\psi(t)+B_1\psi(t-r)+u(t))+(B_0\psi(t)+B_1\psi(t-r)+u(t))^T P\psi(t) \\
&\quad +\frac{1}{2}(\psi(t)^T\psi(t))-\frac{1}{2}(\psi(t-r)^T\psi(t-r)) \\
&= \psi(t)^T PB_0\psi(t)+\psi(t)^T PB_1\psi(t-r)+\psi(t)^T Pu(t)+\psi(t)^T PB_0^T\psi(t) \\
&\quad +\psi(t)PB_1^T\psi^T(t-r)+u(t)^T P\psi(t)+\frac{1}{2}(\psi(t)^T\psi(t))-\frac{1}{2}(\psi(t-r)^T\psi(t-r)) \\
&= \psi(t)^T(PB_0+PB_0^T)\psi(t)+2\psi(t)^T PB_1\psi(t-r)+2\psi(t)^T Pu(t) \\
&\quad +\frac{1}{2}(\psi(t)^T\psi(t))-\frac{1}{2}(\psi(t-r)^T\psi(t-r)) \\
&= -\frac{1}{2}(\psi(t)^T\psi(t))+2\psi(t)^T PB_1\psi(t-r)+2\psi(t)^T Pu(t)-\frac{1}{2}(\psi(t-r)^T\psi(t-r)) \\
&\le -\frac{1}{2}\|\psi(t)\|^2+2PB_1\|\psi(t)\|^2-\frac{1}{2}\|\psi(t-r)\|^2+2\psi(t)^T Pu(t) \\
&\le -\frac{1}{2}\|\psi(t)\|^2+PB_1\|\psi(t)\|^2+PB_1\|\psi(t-r)\|^2-\frac{1}{2}\|\psi(t-r)\|^2+2\psi(t)^T Pu(t) \\
&\le -\left[\left(\frac{1}{2}-PB_1\right)(\|\psi(t)\|^2+\|\psi(t-r)\|^2)-Pu(t)\|\psi(t)\|\right].
\end{aligned}$$

So for any $\eta>0$ sufficiently small, we have $W(\psi(t+\eta))\le W(\psi(t))$ by the continuity of W, so that at $t=t_k^i$ we have

$$D^+W(\psi(t))=\lim_{\eta\to 0^+}\frac{W(\psi(t+\eta))-W(\psi(t))}{\eta}\le 0.$$

Hence, we defined a functional $V:[t_0,\infty)\times B_q\to R_+$ satisfying equation (3.13) such that Theorem 3.2 holds and

$$D^+V(x(t))=\lim_{\eta\to 0^+}\frac{1}{\eta}V(x(t+\eta))-V(x(t))\le 0.$$

4. Conclusion

A non-homogeneous impulsive retarded equation with bounded delays and variable impulse time was considered. Results on the necessary and sufficient conditions for stability and asymptotic stability of the impulsive retarded system via the generalized ordinary differential equation were obtained. An illustration was used to demonstrate the suitability of the results. This work is an extension of the results of [1, 8, 12] to a non-homogeneous system.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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