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Research Article

Some Best Proximity Point Results for $\mathcal{MT}$-Rational Cyclic Contractions in *S*-Metric Space

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Abstract. In this paper, we use the concept of $\mathcal{MT}$-function to establish the best proximity point results for a certain class of proximal cyclic contractive mappings in S-metric spaces. Our results extend and improve some known results in the literature. We give an example to analyze and support our main results.

Keywords. Cyclic mapping; Best proximity point; $\mathcal{MT}$-function ($\mathcal{R}$-function); S-metric space

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1. Introduction

The remarkable Banach contraction principle states that every self mapping on a complete metric space has a unique fixed point. This principle has been generalized and extended in variety of settings. Let P and Q be nonempty subsets of metric space (X,d) and let $f : P \to Q$ be a non-selfmapping. the equation $fx = x$ does not necessarily have a solution, in which case best approximation theorems explore the existence of an approximate solution, whereas best proximity point theorems analyze the existence of an approximate solution that is optimal. Despite the fact that best approximation theorems produce an approximate solution to the equation $fx = x$, they may not render an approximate solution that is optimal. On the contrary, best proximity point theorems are intended to furnish an approximate solution x that is optimal in the sense that the error $d(x, fx)$ is minimum. Indeed, if P and Q are nonempty subsets of X, in light of the fact that $d(x,fx)$ is at least $d(P,Q)$ where $d(P,Q) = \inf\{d(x,y) : x \in P, y \in Q\}$, a best proximity point theorem guarantees the global minimization of $d(x,fx)$ when $d(x,fx) = d(P,Q)$. Such optimal approximate solutions are called best proximity points of the mapping f.

Let P and Q be two nonempty subsets of a metric space (X,d). An element $x \in P$ is said to be a fixed point of a given map $f : P \to Q$ if $fx = x$. Clearly, $f(P) \cap P \neq \varnothing$ is a necessary(but not sufficient) condition for the existence of a fixed point of f. If $f(P) \cap P = \varnothing$, then $d(x,fx) > 0$ for all $x \in P$ that is, the set of fixed points of f is empty. In a such situation, one often attempts to find an element x which is in some sense closest to fx. Best proximity point analysis has been developed in this direction.

An element $a \in P$ is called a *best proximity point* of f if

$$d(a,fa) = \mathrm{d}(P,Q),$$

where

$$\mathrm{d}(P,Q) = \inf\{d(x,y) : x \in P, y \in Q\}.$$

Because of the fact that $d(x,fx) \to \mathrm{d}(P,Q)$ for all $x \in P$, the global minimum of the mapping $x \to d(x,fx)$ is attained at a best proximity point. Clearly, if the underlying mapping is self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The goal of the best proximity point theorem is to provide sufficient conditions to ascertain the existence of an optimal solution to the problem of globally minimizing the error $d(x,fx)$. For more details on this approach, we refer the reader to ([1,2,5,10]) and references therein.

In the case of cyclic contractive mapping $f : P \cup Q \to P \cup Q$, a point $x \in P \cup Q$ is called the best proximity point if $d(x,fx) = \mathrm{d}(P,Q)$. Notice that a best proximity point x is a fixed point of f whenever $P \cap Q \neq \varnothing$. Thus it generalizes the notion of fixed point in case when $P \cap Q = \varnothing$. Further [2,3,11,12,14] examine several variants of contractions for the existence of a best proximity point.

On the other hand, Sedghi *et al.* [19] have defined the concept of an S-metric space. This notion is a generalization of a G-metric space [16] and a D^*-metric space [20], respectively. We refer the reader to ([6,13,17,18,18]) and references therein.

In 2003, the concepts of cyclic mapping and best proximity point were innovated by Kirk *et al.* [15]. Let P and Q be nonempty subsets of a metric space (X,d). A mapping $f : P \cup Q \to P \cup Q$ is called a cyclic mapping if $f(P) \subset Q$ and $f(Q) \subset P$. In 2006, Eldered and Veeramani [9] demonstrated some existence results about best proximity points of cyclic contraction mappings.

A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-function (or $\mathcal{R}$-function). If $\limsup_{s \to t^+} \varphi(s) < 1$ for all $t \in [0,\infty)$.

In this paper, we establish some new existence and convergence theorems of iterates of best proximity points for $\mathcal{MT}$-cyclic contractions in S-metric space.

2. Preliminaries

First we recall some necessary definitions and results in this direction.

The notion of S-metric spaces is defined as follows.

Definition 2.1 ([19]). Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x,y,z,a \in X$.

(S1) $S(x,y,z) \geq 0$;

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$;

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

Then the pair (X,S) is called an S-metric space.

Remark 2.2. Note that every S-metric on X induces a metric d_S on X defined by

$$d_S(x,y) = S(x,x,y) + S(y,y,x)$$

for all $x,y \in X$.

The following is an intuitive geometric example for S-metric spaces.

Example 2.3 ([19]). Let $X = \mathbb{R}$ and d be an ordinary metric on X. Put

$$S(x,y,z) = d(x,y) + d(x,z)$$

for all $x,y,z \in \mathbb{R}$. Then S is an S-metric on X.

Example 2.4 ([19]). Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put

$$S(x,y,z) = h\max\{d(x,y), d(y,z), d(x,z)\}$$

for some $h \in [0,1)$ and all $x,y,z \in \mathbb{R}$. Then S is an S-metric on X.

Lemma 2.5 ([19]). *Let (X,S) be an S-metric space. Then $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Remark 2.6. Let (X,S) be an S-metric space. From Definition 2.1 and Lemma 2.5, we have

$$S(x,x,z) \leq S(x,x,y) + 2S(y,y,z)$$

for all $x,y,z \in X$.

Definition 2.7 ([19]). Let (X,S) be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ is said to *converge to* $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n,x_n,x) < \varepsilon$. We write $x_n \to x$ for brevity.

(ii) A sequence $\{x_n\} \subset X$ is called a *Cauchy sequence* if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$ we have $S(x_n,x_n,x_m) < \varepsilon$.

(iii) The S-metric space (X,S) is said to be *complete* if every Cauchy sequence is a convergent sequence

Now we recall the notion of $\mathcal{MT}$-functions introduced in as follows.

Definition 2.8 ([7]). A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$-function (or $\mathcal{R}$-function). If

$$\limsup_{s \to t^+} \varphi(s) < 1, \quad \text{for all } t \in [0,\infty).$$

It is obvious that if $\varphi:[0,\infty)\to[0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

Example 2.9 ([8]). $\varphi:[0,\infty)\to[0,1)$ be defined by

$$\varphi(t)=\begin{cases}\frac{\sin(t)}{t}, & t\in(0,\pi/2],\\ 0, & \text{otherwise.}\end{cases}$$

Since $\lim_{s\to t^+}\varphi(s)=1$, φ is not an $\mathcal{MT}$-function.

Very recently, Du [8] first proved some characterizations of $\mathcal{MT}$-functions.

Theorem 2.10 ([8]). *Let $\varphi:[0,\infty)\to[0,1)$ be a function. Then the following statements are equivalent.*

(i) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0\le\sup_{n\in\mathbb{N}}\varphi(x_n)<1$.*

(ii) *φ is a function of contractive factor, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0\le\sup_{n\in\mathbb{N}}\varphi(x_n)<1$.*

In [4], the authors present some definitions about type of proximal contractions.

Definition 2.11 ([4]). A mapping $f:P\to Q$ is called *proximal contraction of the first kind* if there exists $k\in[0,1)$ such that

$$\left.\begin{aligned}d(u,fx)=d(P,Q)\\ d(v,fy)=d(P,Q)\end{aligned}\right\}\implies d(u,v)\le kd(x,y) \tag{2.1}$$

for all $u,v,x,y\in P$. It is easy to see that a self-mapping is a contraction of the first kind is precisely a contraction. However, a non self proximal contraction is not necessarily a contraction.

Definition 2.12 ([4]). A mapping $f:P\to Q$ is called *proximal contraction of the second kind* if there exists $k\in[0,1)$ such that

$$\left.\begin{aligned}d(u,fx)=d(P,Q)\\ d(v,fy)=d(P,Q)\end{aligned}\right\}\implies d(fu,fv)\le kd(fx,fy) \tag{2.2}$$

for all $u,v,x,y\in P$.

Definition 2.13 ([4]). Consider the non-self-mappings $g:P\to Q$ and $f:Q\to P$, the pair (g,f) is said to form a *proximal cyclic contraction* if there exists a non-negative number $\alpha<1$ such that

$$\left.\begin{aligned}d(u,gx)=d(P,Q)\\ d(v,fy)=d(P,Q)\end{aligned}\right\}\implies d(u,v)\le\alpha d(x,y)+(1-\alpha)d(P,Q) \tag{2.3}$$

for all $u,v,x,y\in P$.

Definition 2.14 ([7]). Let P and Q be nonempty subsets of a metric space (X,d). If a map $f:P\cup Q\to P\cup Q$ satisfies

(M1) $f(P)\subset Q$ and $f(Q)\subset P$.

(M2) there exists an $\mathbb{R}$-function $\varphi:[0,\infty)\to[0,1)$ such that

$$d(fx,fy)\le\varphi\big(d(x,y)\big)d(x,y)+\Big(1-\varphi\big(d(x,y)\big)\Big)\mathrm{d}(P,Q)$$

for any $x\in P$ and $y\in Q$,

then f is called an $\mathcal{MT}$-cyclic contraction with respect to φ on $P\cup Q$.

Remark 2.15. It is obvious that (M2) implies that f satisfies $d(fx,fy)\le d(x,y)$ for any $x\in P$ and $y\in Q$.

Recall that every S-metric on X induces a metric d_S on X defined by

$$d_S(x,y)=S(x,x,y)+S(y,y,x)\,;\quad\forall\ x,y\in X.$$

Let (X,S) be an S-metric space. Suppose that P and Q are nonempty subsets of an S-metric space (X,S). We define the following sets:

$$P_0=\{x\in P: d_S(x,y)=d_S(P,Q)\text{ for some } y\in Q\}\quad\text{and}$$

$$Q_0=\{y\in Q: d_S(x,y)=d_S(P,Q)\text{ for some } x\in P\},$$

where $d_S(P,Q)=\inf\{d_S(x,y): x\in P, y\in Q\}$.

3. Main Result

Definition 3.1. Let P and Q be two non-empty subsets of S-metric space (X,S). Let $f:P\to Q$ is called

(i) *S-$\mathcal{MT}$-K-proximal cyclic contraction of the first kind with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x,u,a,b,y,v\in P$

$$\left.\begin{array}{l} d_S(u,fx)=d_S(P,Q)\\ d_S(b,fa)=d_S(P,Q)\\ d_S(v,fy)=d_S(P,Q)\end{array}\right\}\implies S(u,b,v)\le\varphi(S(x,a,y))\frac{S(x,a,y)S(y,u,b)}{S(x,a,y)+2S(x,a,b)}\tag{3.1}$$

where $S(x,a,y)+2S(x,a,b)\neq0$.

(ii) *S-$\mathcal{MT}$-C-proximal cyclic contraction of the first kind with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x,u,a,b,y,v\in P$

$$\left.\begin{array}{l} d_S(u,fx)=d_S(P,Q)\\ d_S(b,fa)=d_S(P,Q)\\ d_S(v,fy)=d_S(P,Q)\end{array}\right\}\implies S(u,b,v)\le\varphi(S(x,a,y))\frac{S(x,a,y)S(x,u,b)}{S(y,u,b)+2S(x,a,y)}\tag{3.2}$$

where $S(x,a,y)+2S(x,a,b)\neq0$.

Definition 3.2. Let P and Q be two non-empty subsets of S-metric space (X,S). Let $f:P\to Q$ is called

(i) *S-$\mathcal{MT}$-K-proximal cyclic contraction of the second kind with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x,u,a,b,y,v\in P$,

$$\left.\begin{array}{l} d_S(u,fx)=d_S(P,Q)\\ d_S(b,fa)=d_S(P,Q)\\ d_S(v,fy)=d_S(P,Q)\end{array}\right\}\implies S(fu,fb,fv)\le\varphi(S(fx,fa,fy))\frac{S(fx,fa,fy)S(fy,fu,fb)}{S(fx,fa,fy)+2S(fx,fa,fb)}\tag{3.3}$$

where $S(fx,fa,fy)+2S(fx,fa,fb)\neq0$.

(ii) *S-$\mathcal{MT}$-C-proximal cyclic contraction of the second kind with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x,u,a,b,y,v \in P$,

$$\left.\begin{aligned} d_S(u,fx) &= d_S(P,Q) \\ d_S(b,fa) &= d_S(P,Q) \\ d_S(v,fy) &= d_S(P,Q) \end{aligned}\right\} \Longrightarrow S(fu,fb,fv) \le \varphi(S(fx,fa,fy))\frac{S(fx,fa,fy)S(fy,fu,fb)}{S(fx,fa,fy)+2S(fx,fa,fb)} \quad (3.4)$$

where $S(fx,fa,fy)+2S(fx,fa,fb) \neq 0$.

Definition 3.3. Let P and Q be two non-empty subsets of S-metric space (X,S). Suppose that $f: P \to Q$ and $\hat{f}: Q \to P$ are mappings. The pair $(f,\hat{f})$ is called *S-$\mathcal{MT}$-proximal cyclic contraction with respect to φ* if there exist an $\mathcal{MT}$-function φ such that for any $x,u \in P$ and $y,v \in Q$

$$\left.\begin{aligned} d_S(u,fx) &= d_S(P,Q) \\ d_S(v,\hat{f}y) &= d_S(P,Q) \end{aligned}\right\} \Longrightarrow d_S(u,v) \le \varphi(d_S(x,y))d_S(x,y) + \big(1-\varphi(d_S(x,y))\big)d_S(P,Q). \quad (3.5)$$

Theorem 3.4. *Let P,Q be two nonempty subsets of a S-metric space (X,S) such that $P_0,Q_0 \neq \emptyset$. Let $f: P \to Q$, $\hat{f}: Q \to P$ satisfy the following conditions:*

(i) *f and $\hat{f}$ are S-$\mathcal{MT}$-K-proximal cyclic contractions of the first kind;*

(ii) *the pair $(f,\hat{f})$ is S-$\mathcal{MT}$-proximal cyclic contraction;*

(iii) *$f(P_0) \subseteq Q_0$ and $\hat{f}(Q_0) \subseteq P_0$;*

Then there exists a point $x \in P$ and there exists a point $y \in Q$ such that

$$d_S(x,fx) = d_S(y,\hat{f}y) = d_S(x,y) = d_S(P,Q).$$

Proof. Let $x_0 \in P_0$, since $f(P_0) \subseteq Q_0$, there exists $x_1 \in P_0$ such that $d_S(x_1,fx_0) = d_S(P,Q)$. Also, since $fx_1 \in Q_0$, there exists $x_2 \in P_0$ such that $d_S(x_2,fx_1) = d_S(P,Q)$. Recursively, we obtain a sequence $\{x_n\}$ in P_0 satisfying

$$d_S(x_{n+1},fx_n) = d_S(P,Q), \quad \forall\, n \in \mathbb{N} \cup \{0\}. \quad (3.6)$$

This shows that

$$\begin{aligned} d_S(u,fx) &= d_S(P,Q), \\ d_S(b,fa) &= d_S(P,Q), \\ d_S(v,fy) &= d_S(P,Q), \end{aligned} \quad (3.7)$$

where $u = x_{n+1} = b$, $x = x_n = a = v$ and $y = x_{n-1}$. From (3.1), we have

$$\begin{aligned} S(x_{n+1},x_{n+1},x_n) &\le \varphi\big(S(x_n,x_n,x_{n-1})\big)\frac{S(x_n,x_n,x_{n-1})S(x_{n-1},x_{n+1},x_{n+1})}{S(x_n,x_n,x_{n-1})+2S(x_n,x_n,x_{n+1})} \\ &\le \varphi\big(S(x_{n-1},x_{n-1},x_n)\big)\frac{S(x_n,x_n,x_{n-1})S(x_{n-1},x_{n+1},x_{n+1})}{S(x_{n-1},x_{n-1},x_{n+1})} \\ &\le \varphi\big(S(x_{n-1},x_{n-1},x_n)\big)S(x_n,x_n,x_{n-1}). \end{aligned} \quad (3.8)$$

From (3.8), we have

$$S(x_n,x_n,x_{n+1}) \le \varphi\big(S(x_{n-1},x_{n-1},x_n)\big)S(x_{n-1},x_{n-1},x_n). \quad (3.9)$$

Since φ is an $\mathcal{MT}$-function, then from Theorem 2.10,

$$0 \le \sup_{n\in\mathbb{N}} \varphi\big(S(x_{n-1},x_{n-1},x_n)\big) < 1. \quad (3.10)$$

Let $\lambda := \sup_{n\in\mathbb{N}} \varphi\big(S(x_{n-1},x_{n-1},x_n)\big)$ then

$$0 \le \varphi\big(S(x_{n-1},x_{n-1},x_n)\big) \le \lambda < 1 \tag{3.11}$$

for all $n \in \mathbb{N}$. From (3.9) we have

$$S(x_n,x_n,x_{n+1}) \le \lambda S(x_{n-1},x_{n-1},x_n) \le S(x_{n-1},x_{n-1},x_n). \tag{3.12}$$

So the sequence $\{S(x_n,x_n,x_{n+1})\}$ is non-increasing sequence in $[0,\infty)$ and thus $\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = \inf_{n\in\mathbb{N}} S(x_n,x_n,x_{n+1})$ exists.

From the first inequality of (3.12), we obtain

$$S(x_n,x_n,x_{n+1}) \le \lambda^n S(x_0,x_0,x_1)\ ; \quad \forall n \in \mathbb{N}. \tag{3.13}$$

Due to $\lambda \in [0,1)$, $\lim_{n\to\infty} \lambda^n = 0$. By taking limit in (3.13) as $n \to \infty$, we deduce

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \tag{3.14}$$

Suppose $n,m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} S(x_n,x_n,x_m) &\le 2S(x_{m-1},x_{m-1},x_m) + S(x_n,x_n,x_{m-1}) \\ &\le 2S(x_{m-1},x_{m-1},x_m) + 2S(x_{m-2},x_{m-2},x_{m-1}) + S(x_n,x_n,x_{m-2}) \\ &\le 2S(x_{m-1},x_{m-1},x_m) + 2S(x_{m-2},x_{m-2},x_{m-1}) \cdots + S(x_n,x_n,x_{n+1}). \end{aligned}$$

Now, for $m = n + r$; $r \ge 1$ and (3.13), we obtain

$$S(x_n,x_n,x_{n+r}) \le 2\lambda^{n+r-1} S(x_0,x_0,x_1) + 2\lambda^{n+r-2} S(x_0,x_0,x_1) + \cdots + \lambda^n S(x_0,x_0,x_1).$$

By taking limit as $n \to \infty$, we deduce

$$\lim_{n\to\infty} S(x_n,x_n,x_m) = 0. \tag{3.15}$$

That is, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since (P,S) is a complete S-metric space, so there exists $x \in P$ such that $x_n \to x$ as $n \to \infty$. Similarly, since $\hat{f}(Q_0) \subseteq P_0$ and $Q_0 \subseteq g(Q_0)$, there exists a sequence $\{y_n\}$ such that it converges to some element $y \in Q$. Since the pair $(f,\hat{f})$ is S-$\mathcal{MT}$-proximal cyclic contraction, we have for $x_{n+1} \in P$, $y_{n+1} \in Q$,

$$d_S(x_{n+1}, f x_n) = d_S(P,Q) \quad \text{and} \quad d_S(y_{n+1}, \hat{f} y_n) = d_S(P,Q).$$

Then

$$d_S(x_{n+1},y_{n+1}) \le \varphi(d_S(x_n,y_n)) d_S(x_n,y_n) + \big(1 - \varphi(d_S(x_n,y_n))\big) d_S(P,Q).$$

Taking limit as $n \to \infty$, we have

$$d_S(x,y) \le \varphi(d_S(x,y)) d_S(x,y) + \big(1 - \varphi(d_S(x,y))\big) d_S(P,Q),$$

$$\big(1 - \varphi(d_S(x,y))\big) d_S(x,y) \le \big(1 - \varphi(d_S(x,y))\big) d_S(P,Q),$$

yields

$$d_S(x,y) = d_S(P,Q). \tag{3.16}$$

Thus, $x \in P_0$ and $y \in Q_0$. Since $f(P_0) \subseteq Q_0$ and $\hat{f}(Q_0) \subseteq P_0$, there exist $u \in P$ and $v \in Q$ such that

$$d_S(u, f x) = d_S(P,Q) \quad \text{and} \quad d_S(v, \hat{f} y) = d_S(P,Q). \tag{3.17}$$

Since f is S-$\mathcal{MT}$-K-proximal cyclic contraction of the first kind, we get from $d_S(u,fx) = d_S(P,Q)$ and $d_S(x_{n+1},fx_n) = d_S(P,Q)$ as

$$S(u,u,x_{n+1}) \le \varphi(S(x,x,x_n))\frac{S(x,x,x_n)S(x_n,u,u)}{S(x,x,x_n)+2S(x,x,u)} \le S(x,x,x_n).$$

Taking limit as $n \to \infty$, we have $S(u,u,x) = 0$ and so $u = x$. Therefore

$$d_S(x,fx) = d_S(P,Q). \tag{3.18}$$

Similarly, we have $v = y$ and so

$$d_S(y,\hat{f}y) = d_S(P,Q). \tag{3.19}$$

Thus, from (3.16), (3.18) and (3.19), we get

$$d_S(x,y) = d_S(x,fx) = d_S(y,\hat{f}y) = d_S(P,Q).$$

This completes the proof. □

Example 3.5. Consider the space $X = \mathbb{R}^2$ with S-metric given in example then $S(x,y,z) = \frac{1}{2}\max\{|x-y|,|x-z|,|y-z|\}$ for all $x,y,z \in X$, where $|a-b| = |a_1-b_1|+|a_2-b_2|$ for $(a_1,a_2),(b_1,b_2) \in X$. Then $S(x,x,y) = \frac{1}{2}|x-y|$ and $d_S(x,y) = |x-y|$. Let

$$P = \{(-1,x) : x \in [-\tfrac{1}{2},\tfrac{1}{2}]\} \quad \text{and} \quad Q = \{(1,y) : y \in [-\tfrac{1}{2},\tfrac{1}{2}]\}.$$

Define the mappings $f : P \to Q$ and $\hat{f} : Q \to P$ as follows:

$$f\big((-1,x)\big) = \begin{cases} \big(1,\frac{x}{4}\big); & \text{if } x \in [-\frac{1}{2},0], \\ (1,0); & \text{otherwise}, \end{cases}$$

$$\hat{f}\big((1,y)\big) = \begin{cases} \big(-1,\frac{y}{4}\big); & \text{if } y \in [-\frac{1}{2},0], \\ (-1,0); & \text{otherwise}. \end{cases}$$

Then $d_S(P,Q) = 2$, $P_0 = P$, $Q_0 = Q$,

$$f(P_0) = \{(1,x) : x \in [-\tfrac{1}{8},0]\} \subseteq \{(1,x) : x \in [-\tfrac{1}{2},\tfrac{1}{2}]\} = Q_0,$$

$$\hat{f}(Q_0) = \{(-1,y) : y \in [-\tfrac{1}{8},0]\} \subseteq \{(-1,y) : y \in [-\tfrac{1}{2},\tfrac{1}{2}]\} = P_0.$$

Define $\varphi(t) = \frac{4}{5}$; $\forall\ t \in [0,\infty)$, we will show that f and $\hat{f}$ are S-$\mathcal{MT}$-K-proximal cyclic contractions.

Let $(-1,x_1),(-1,x_2),(-1,a_1),(-1,a_2) \in P$ satisfying

$$d_S\big((-1,a_1),f(-1,x_1)\big) = 2 \quad \text{and} \quad d_S\big((-1,a_2),f(-1,x_2)\big) = 2.$$

***Case* I: $x_1,x_2 \in [-\frac{1}{2},0]$ and $x_1 > x_2$.** We have

$$S\big((-1,a_1),(-1,a_1),(-1,a_2)\big) = \tfrac{1}{2}\left|\tfrac{x_1}{4}-\tfrac{x_2}{4}\right| = \tfrac{1}{8}|x_1-x_2|,$$

$$k := \varphi\Big(S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)\Big) = \tfrac{4}{5},$$

$$S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) = \tfrac{1}{2}|x_1-x_2|,$$

$$S\big((-1,x_2),(-1,a_1),(-1,a_1)\big) = \tfrac{1}{2}\left|\tfrac{x_1}{4}-x_2\right| \ \big(\ge \tfrac{1}{2}(|x_1-x_2|\big),$$

$$S\big((-1,x_1),(-1,a_2),(-1,a_1)\big) = \tfrac{1}{2}\left|x_1-\tfrac{x_2}{4}\right| \ \big(\le \tfrac{1}{2}(|x_1-x_2|\big).$$

Consider

$$k \cdot \frac{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)S\big((-1,x_2),(-1,a_1),(-1,a_1)\big)}{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)+2S\big((-1,x_1),(-1,a_2),(-1,a_1)\big)}$$

$$= k \cdot \frac{\frac{1}{2}|x_1 - x_2| \cdot \frac{1}{2}\left|\frac{x_1}{4} - x_2\right|}{\frac{1}{2}|x_1 - x_2| + \left|x_1 - \frac{x_2}{4}\right|}$$

$$\geq k \cdot \frac{\frac{1}{4}|x_1-x_2|^2}{\frac{3}{2}|x_1-x_2|} = \frac{2}{15}|x_1 - x_2|$$

$$\geq \frac{1}{8}|x_1 - x_2| = S\big((-1,a_1),(-1,a_1),(-1,a_2)\big).$$

***Case* II: $x_1, x_2 \notin \left[-\frac{1}{2}, 0\right]$ and $x_1 > x_2$.** We have

$$S\big((-1,a_1),(-1,a_1),(-1,a_2)\big) = 0,$$

$$k := \varphi\Big(S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)\Big) = \frac{4}{5},$$

$$S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) = \frac{1}{2}|x_1 - x_2|,$$

$$S\big((-1,x_2),(-1,a_1),(-1,a_1)\big) = \frac{1}{2}|x_2|,$$

$$S\big((-1,x_1),(-1,a_2),(-1,a_1)\big) = \frac{1}{2}|x_1|.$$

Consider

$$k \cdot \frac{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)S\big((-1,x_2),(-1,a_1),(-1,a_1)\big)}{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) + 2S\big((-1,x_1),(-1,a_2),(-1,a_1)\big)}$$

$$= k \cdot \frac{\frac{1}{2}|x_1-x_2| \cdot \frac{1}{2}|x_2|}{\frac{1}{2}|x_1-x_2| + |x_1|} \geq 0 = S\big((-1,a_1),(-1,a_1),(-1,a_2)\big)$$

***Case* III: $x_2 \in \left[-\frac{1}{2}, 0\right]$ and $x_1 \notin \left[-\frac{1}{2}, 0\right]$.** We have

$$S\big((-1,a_1),(-1,a_1),(-1,a_2)\big) = \frac{1}{2}\left|\frac{x_2}{4}\right| = \frac{1}{8}|x_2|,$$

$$k := \varphi\Big(S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)\Big) = \frac{4}{5},$$

$$S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) = \frac{1}{2}|x_1 - x_2|,$$

$$S\big((-1,x_2),(-1,a_1),(-1,a_1)\big) = \frac{1}{2}|x_2| \left(\leq \frac{1}{2}(|x_1 - x_2|\right),$$

$$S\big((-1,x_1),(-1,a_2),(-1,a_1)\big) = \frac{1}{2}\left|\frac{x_2}{4}\right| = \frac{1}{8}|x_2|.$$

Consider

$$k \cdot \frac{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big)S\big((-1,x_2),(-1,a_1),(-1,a_1)\big)}{S\big((-1,x_1),(-1,x_1),(-1,x_2)\big) + 2S\big((-1,x_1),(-1,a_1),(-1,a_1)\big)}$$

$$= k \cdot \frac{\frac{1}{2}|x_1-x_2| \cdot \frac{1}{2}|x_2|}{\frac{1}{2}|x_1-x_2| + \frac{1}{4}|x_2|}$$

$$\geq k \cdot \frac{\frac{1}{4}|x_2|}{\frac{3}{2}|x_1-x_2|}$$

$$\geq \left(\frac{4}{5}|x_1 - x_2|\right) \frac{\frac{1}{4}|x_2|}{\frac{3}{2}|x_1-x_2|} = \frac{2}{15}|x_2|$$

$$\geq \frac{1}{8}|x_2| = S\big((-1,a_1),(-1,a_1),(-1,a_2)\big).$$

From all the above cases, we conclude that f is a S-$\mathcal{MT}$-K-proximal cyclic contractions. Similarly, we can show that $\hat{f}$ is a S-$\mathcal{MT}$-K-proximal cyclic contractions too. Next, we show that the pair $(f, \hat{f})$ is a S-$\mathcal{MT}$-proximal cyclic contraction. Let $(-1,u),(-1,x) \in P$ and $(1,v),(1,y) \in Q$ be such that

$$d_S\big((-1,u),f(-1,x)\big)=d_S(P,Q)=2,\quad d_S\big((1,v),\hat{f}(1,y)\big)=d_S(P,Q)=2.$$

Then, we get

$$u=\begin{cases}\frac{x}{4}; & \text{if } x\in\left[-\frac{1}{2},0\right],\\ 0; & \text{otherwise,}\end{cases}\qquad v=\begin{cases}\frac{y}{4}; & \text{if } y\in\left[-\frac{1}{2},0\right],\\ 0; & \text{otherwise.}\end{cases}$$

***Case* I: $x,y\in\left[-\frac{1}{2},0\right]$ and $x>y$.** We have $d_S\big((-1,u),(1,v)\big)=\frac{1}{4}|x-y|$.

***Case* II: $x,y\notin\left[-\frac{1}{2},0\right]$ and $x>y$.** We have, $d_S\big((-1,u),(1,v)\big)=0$.

***Case* III: $y\in\left[-\frac{1}{2},0\right]$ and $x\notin\left[-\frac{1}{2},0\right]$.** We have, $d_S\big((-1,u),(1,v)\big)=\frac{1}{4}|y|$.

From all the above cases, we get

$$\begin{aligned}d_S\big((-1,u),(1,v)\big)&\le\frac{4}{5}|x-y|+2\left(\frac{1}{5}\right)\\&=kd_S\big((-1,x),(1,y)\big)+(1-k)d_S(P,Q),\end{aligned}$$

where $k=\varphi\big(d_S((-1,x),(1,y))\big)$. Hence the pair $(f,\hat{f})$ is a S-$\mathcal{MT}$-proximal cyclic contraction. Therefore, all the hypotheses of Theorem 3.4 are satisfied. Thus, $(-1,0)\in P$ and $(1,0)\in Q$ are elements such that

$$d_S\big((1,0),f(1,0)\big)=d_S((-1,0),\hat{f}(-1,0)\big)=d_S\big((-1,0),(-1,0)\big)=d_S(P,Q).$$

Theorem 3.6. *Let P,Q be two nonempty subsets of a S-metric space (X,S) such that $P_0,Q_0\neq\varnothing$. Let $f:P\to Q,\hat{f}:Q\to P$ satisfy the following conditions:*

(i) *f and $\hat{f}$ are S-$\mathcal{MT}$-C-proximal cyclic contractions of the first kind;*

(ii) *the pair $(f,\hat{f})$ is S-$\mathcal{MT}$-proximal cyclic contraction;*

(iii) *$f(P_0)\subseteq Q_0$ and $\hat{f}(Q_0)\subseteq P_0$;*

Then there exists a point $x\in P$ and there exists a point $y\in Q$ such that

$$d_S(x,fx)=d_S(y,\hat{f}y)=d_S(x,y)=d_S(P,Q).$$

Proof. Proceeding as in Theorem 3.4, we can construct a sequence $\{x_n\}$ in P_0 such that

$$d_S(x_{n+1},fx_n)=d_S(P,Q),\quad \forall\ n\in\mathbb{N}\cup\{0\}. \tag{3.20}$$

This shows that

$$\begin{aligned}&d_S(u,fx)=d_S(P,Q),\\&d_S(b,fa)=d_S(P,Q),\\&d_S(v,fy)=d_S(P,Q),\end{aligned}\tag{3.21}$$

where $u=x_{n+1}=b,x=x_n=a=v$ and $y=x_{n-1}$. Since f is S-$\mathcal{MT}$-C-proximal cyclic contraction of the first kind, from (3.2), we have

$$\begin{aligned}S(x_{n+1},x_{n+1},x_n)&\le\varphi\big(S(x_n,x_n,x_{n-1})\big)\frac{S(x_n,x_n,x_{n-1})S(x_n,x_{n+1},x_{n+1})}{S(x_{n-1},x_{n+1},x_{n+1})+2S(x_n,x_n,x_{n-1})}\\&\le\varphi\big(S(x_{n-1},x_{n-1},x_n)\big)\frac{S(x_n,x_n,x_{n-1})S(x_n,x_{n+1},x_{n+1})}{S(x_n,x_n,x_{n+1})}\\&\le\varphi\big(S(x_{n-1},x_{n-1},x_n)\big)S(x_n,x_n,x_{n-1}).\end{aligned}\tag{3.22}$$

From (3.22) we have

$$S(x_n,x_n,x_{n+1}) \le \varphi\big(S(x_{n-1},x_{n-1},x_n)\big)S(x_{n-1},x_{n-1},x_n). \tag{3.23}$$

With a similar idea to the proof of Theorem 3.4, we can deduce that there exists a point $x \in P$ and there exists a point $y \in Q$ such that

$$d_S(x,fx) = d_S(y,\hat{f}y) = d_S(x,y) = d_S(P,Q).$$

□

Corollary 3.7. *Let P,Q be two nonempty subsets of a S-metric space (X,S) such that $P_0,Q_0 \neq \varnothing$. Let $f : P \to Q, \hat{f} : Q \to P$ satisfy the following conditions:*

(i) *f is S-$\mathcal{MT}$-K-proximal cyclic contractions of the first kind;*

(ii) *$\hat{f}$ is S-$\mathcal{MT}$-C-proximal cyclic contractions of the first kind;*

(iii) *the pair $(f,\hat{f})$ is S-$\mathcal{MT}$-proximal cyclic contraction;*

(iv) *$f(P_0) \subseteq Q_0$ and $\hat{f}(Q_0) \subseteq P_0$;*

Then there exists a point $x \in P$ and there exists a point $y \in Q$ such that

$$d_S(x,fx) = d_S(y,\hat{f}y) = d_S(x,y) = d_S(P,Q).$$

Corollary 3.8. *Let P,Q be two nonempty subsets of a S-metric space (X,S) such that $P_0,Q_0 \neq \varnothing$. Let $f : P \to Q, \hat{f} : Q \to P$ satisfy the following conditions:*

(i) *f is S-$\mathcal{MT}$-C-proximal cyclic contractions of the first kind;*

(ii) *$\hat{f}$ is S-$\mathcal{MT}$-K-proximal cyclic contractions of the first kind;*

(iii) *the pair $(f,\hat{f})$ is S-$\mathcal{MT}$-proximal cyclic contraction;*

(iv) *$f(P_0) \subseteq Q_0$ and $\hat{f}(Q_0) \subseteq P_0$;*

Then there exists a point $x \in P$ and there exists a point $y \in Q$ such that

$$d_S(x,fx) = d_S(y,\hat{f}y) = d_S(x,y) = d_S(P,Q).$$

The following is the best proximity point theorem for non self-mappings which are S-$\mathcal{MT}$-K-proximal cyclic contractions of the first kind and second kind:

Theorem 3.9. *Let P,Q be two nonempty subsets of a S-metric space (X,S) such that $P_0,Q_0 \neq \varnothing$. Let $f : P \to Q$ satisfy the following conditions:*

(i) *f is a generalized S-$\mathcal{MT}$-K-proximal cyclic contraction of the first kind and second kind;*

(ii) *$f(P_0) \subseteq Q_0$;*

Then there exists a point $x \in P$ such that

$$d_S(x,fx) = d_S(P,Q).$$

Proof. Proceeding as in Theorem 3.4, we can construct a sequence $\{x_n\}$ in P_0 such that

$$d_S(x_{n+1},fx_n) = d_S(P,Q), \quad \forall\ n \in \mathbb{N} \cup \{0\}. \tag{3.24}$$

Since f is an S-$\mathcal{MT}$-K-proximal cyclic contraction of the first kind, we have

$$S(x_n,x_n,x_{n+1}) \le \varphi\big(S(x_{n-1},x_{n-1},x_n)\big)S(x_{n-1},x_{n-1},x_n) \tag{3.25}$$

for all $n \geq 1$. Again, similarly, we can show that the sequence $\{x_n\}$ is a Cauchy sequence and so it converges to some $x \in P$.

Since f is S-$\mathcal{MT}$-K-proximal cyclic contraction of the second kind,

$$S(fx_n, fx_n, fx_{n+1}) \leq \varphi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big)S(fx_{n-1}, fx_{n-1}, fx_n)$$
$$\leq S(fx_{n-1}, fx_{n-1}, fx_n). \tag{3.26}$$

This shows that $\{S(fx_n, fx_n, fx_{n+1})\}$ is a decreasing sequence and bounded below. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} S(fx_n, fx_n, fx_{n+1}) = r$. Suppose that $r > 0$. From (3.26) observe that

$$\frac{S(fx_n, fx_n, fx_{n+1})}{S(fx_{n-1}, fx_{n-1}, fx_n)} \leq \varphi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big).$$

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} \varphi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big) = 1$$

which is a contradiction to φ is an $\mathcal{MT}$-function so $r = 0$, then we get,

$$\lim_{n\to\infty} S(fx_n, fx_n, fx_{n+1}) = 0.$$

similarly, in the proof of Theorem 3.4, we can show that $\{fx_n\}$ is a Cauchy sequence and converges to some element $y \in Q$. Therefore, we can conclude that $d_S(x, y) = \lim_{n\to\infty} d_S(x_{n+1}, fx_n) = d_S(P,Q)$, which implies that $x \in P_0$. Since $f(P_0) \subseteq Q_0$, there exists $u \in P$ such that

$$d_S(u, fx) = d_S(P,Q). \tag{3.27}$$

Since f is a generalized S-$\mathcal{MT}$-K-proximal cyclic contraction of the first kind, it follows from (3.24) and (3.27) that

$$S(u,u,x_{n+1}) \leq \varphi\big(S(x,x,x_n)\big)\frac{S(x,x,x_n)S(x_n,u,u)}{S(x,x,x_n)+2S(x,x,u)}$$
$$\leq \varphi\big(S(x,x,x_n)\big)\frac{S(x,x,x_n)S(x_n,u,u)}{S(x_n,x_n,u)}$$
$$\leq \varphi\big(S(x,x,x_n)\big)S(x,x,x_n).$$

Taking limit as $n \to \infty$, we have $u = x$. Therefore, $d_S(x, fx) = d_S(P,Q)$. This completes the proof. □

Corollary 3.10. *Let P,Q be two nonempty subsets of a S-metric space (X,S) such that $P_0, Q_0 \neq \varnothing$. Let $f : P \to Q$ satisfy the following conditions:*

(i) *f is a generalized S-$\mathcal{MT}$-C-proximal cyclic contraction of the first kind and second kind;*

(ii) *$f(P_0) \subseteq Q_0$;*

Then there exists a point $x \in P$ such that

$$d_S(x, fx) = d_S(P,Q).$$

4. Conclusion and Future Scope

In this attempt, we have proved some best proximity point results for $\mathcal{MT}$-rational cyclic contractions in S-metric space. These results generalizes and improves the recent results of $\mathcal{MT}$ contractive conditions. We are employing rational form for proximal cyclic contraction of the first kind and second kind.

Inspiring from the ideas presented in this paper, one can introduce the concept of generalized in b metric space, S_b-metric space and so on. An attempt can be made in the direction of best proximity point in these spaces.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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