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Research Article

Some Special Families of Holomorphic and Sălăgean Type Bi-univalent Functions Associated with (*m*,*n*)-Lucas Polynomials

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Abstract. The aim of the present paper is to introduce some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m,n)-Lucas polynomials in the open unit disc $\mathfrak{D}$. We investigate the upper bounds on initial coefficients for functions in these newly introduced special families and also discuss the Fekete-Szegö problem. Some interesting consequences of the results established here are indicated.

Keywords. Holomorphic function; Bi-univalent function; Fekete-Szegö inequality; Lucas polynomial; Sălăgean operator

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1. Introduction

Let $\mathcal{N}$ be the set of natural numbers, $\mathcal{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. Let $\mathcal{A}$ be the family of normalized functions that have the form

$$g(z) = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

which are holomorphic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the collection of all members of $\mathcal{A}$ that are univalent in $\mathfrak{D}$. It is well-known (see [5]) that every function $g \in \mathcal{S}$ has an inverse g^{-1} satisfying $z = g^{-1}(g(z))$, $z \in \mathfrak{D}$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \dots . \tag{1.2}$$

A member g of $\mathcal{A}$ is said to be bi-univalent in $\mathfrak{D}$ if both g and g^{-1} are univalent in $\mathfrak{D}$. We denote the family of bi-univalent functions that have the form (1.1), by $\sum$. For detailed study and various subfamilies of the family $\sum$, one can refer the works of [2], [4], [8], [11] and [14].

We recall the principle of subordination between two holomorphic functions $g(z)$ and $f(z)$ in $\mathfrak{D}$. It is known that $g(z)$ is subordinate to $f(z)$, written as $g(z) \prec f(z)$, $z \in \mathfrak{D}$, if there is a $\psi(z)$ holomorphic in $\mathfrak{D}$, with $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathfrak{D}$, such that $g(z) = f(\psi(z))$. Moreover, $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$, if f is univalent in $\mathfrak{D}$.

Let $m(x)$ and $n(x)$ be polynomials with real coefficients. The (m,n)-Lucas polynomials $L_j(m(x), n(x), x)$ or briefly $L_j(x)$ are given by the following recurrence relation (see [10]):

$$L_j(x) = m(x)L_{j-1}(x) + n(x)L_{j-2}(x), \quad L_0(x) = 2,\ L_1(x) = m(x), \tag{1.3}$$

where $j \in \mathbb{N} - \{1\}$. It is clear from (1.3) that $L_2(x) = m^2(x) + 2n(x)$, $L_3(x) = m^3(x) + 3m(x)n(x)$. The generating function of the (m,n)-Lucas polynomial sequence $L_j(x)$ is given by

$$\mathcal{G}(x,z) := \sum_{j=0}^{\infty} L_j(x)z^j = \frac{2 - m(x)z}{1 - m(x)z - n(x)z^2}. \tag{1.4}$$

Note that for particular chioces of $m(x)$ and $n(x)$, the (m,n)-Lucas polynomial $L_j(x)$ leads to various polynomials, among those we list following few here (see, for more details [3]): (i) $L_j(x,1,x) = \mathcal{L}_j(x)$, the Lucas polynomials, (ii) $L_j(2x,1,x) = P_j(x)$, the Pell-Lucas polynomials, (iii) $L_j(1,2x,x) = J_j(x)$, the Jacobsthal polynomials, (iv) $L_j(3x,-2,x) = F_j(x)$, the Fermat-Lucas polynomials, (v) $L_j(2x,-1,x) = T_j(x)$, the first kind Chebyshev polynomials.

In literature, the coefficient estimates and celebrated Fekete-Szegö inequality are found for bi-univalent functions associated with certain polynomials like the Chebyshev polynomials, the (m,n)-Lucas polynomials. We also note that the above polynomials and other special polynomials are potentially important in the mathematical, physical, statistical and engineering sciences. More details associated with these polynomials can be found in [1], [7], [9], [12] and [16].

For $g \in \mathcal{A}$, $k \in \mathbb{N} \cup \{0\}$, Sălăgean differential operator [13] $D^k : \mathcal{A} \to \mathcal{A}$, is defined by

$$D^0 g(z) = g(z), D^1 g(z) = zg'(z), \dots, D^k g(z) = D(D^{k-1}g(z)), \quad z \in \mathfrak{D}.$$

It is easy to see that if $g \in \mathcal{A}$ and $g(z) = z + \sum\limits_{j=2}^{\infty} d_j z^j$, then $D^k g(z) = z + \sum\limits_{j=2}^{\infty} j^k d_j z^j$, $z \in \mathfrak{D}$.

Inspired by recent trends on bi-univalent functions and motivated by the paper [15], we define the following special families of $\sum$ by making use of the (m,n)-Lucas polynomials, which are given by the recurrence relation (1.3) and the generating function (1.4).

Definition 1.1. A function $g(z)$ in $\sum$ of the form (1.1) is said to be in the family $\mathfrak{S}_{\sum}(x,\gamma,\mu,k)$, $0 \leq \gamma \leq 1$, $\mu \geq 0$, $\mu \geq \gamma$ and $k \in \mathbb{N} \cup \{0\}$, if

$$\frac{z(D^k g(z))' + \mu z^2 (D^k g(z))''}{(1-\gamma)D^k g(z) + \gamma z (D^k g(z))'} \prec \mathcal{G}(x,z) - 1, \quad z \in \mathfrak{D}$$

and

$$\frac{\omega(D^k f(\omega))' + \mu\omega^2(D^k f(\omega))''}{(1-\gamma)D^k f(\omega) + \gamma\omega(D^k f(\omega))'} \prec \mathcal{G}(x,\omega) - 1, \quad \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4).

Definition 1.2. A function $g(z)$ in $\sum$ of the form (1.1) is said to be in the family $\mathfrak{M}_{\sum}(x,\gamma,\mu,k)$, $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$ and $k \in \mathbb{N} \cup \{0\}$, if

$$\frac{z(D^k g(z))' + \mu z^2(D^k g(z))''}{(1-\gamma)z + \gamma z(D^k g(z))'} \prec \mathcal{G}(x,z) - 1, \quad z \in \mathfrak{D}$$

and

$$\frac{\omega(D^k f(\omega))' + \mu\omega^2(D^k f(\omega))''}{(1-\gamma)\omega + \gamma\omega(D^k f(\omega))'} \prec \mathcal{G}(x,\omega) - 1, \quad \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4).

Definition 1.3. A function $g(z)$ in $\sum$ of the form (1.1) is said to be in the family $\mathfrak{B}_{\sum}(x,\xi,\tau,k)$, $\xi \ge 1$, $\tau \ge 1$ and $k \in \mathbb{N} \cup \{0\}$, if

$$\frac{(1-\xi) + \xi[(z(D^k g(z))')']^{\tau}}{(D^k g(z))'} \prec \mathcal{G}(x,z) - 1, \quad z \in \mathfrak{D}$$

and

$$\frac{(1-\xi) + \xi[(\omega(D^k f(\omega))')']^{\tau}}{(D^k f(\omega))'} \prec \mathcal{G}(x,\omega) - 1, \quad \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4).

For functions belonging to these newly defined families $\mathfrak{S}_{\sum}(x,\gamma,\mu,k)$, $\mathfrak{M}_{\sum}(x,\gamma,\mu,k)$ and $\mathfrak{B}_{\sum}(x,\xi,\tau,k)$, we derive the estimates for the coefficients $|d_2|$ and $|d_3|$ and also, we consider the celebrated Fekete-Szegö problem [6] in Section 2.

2. Coefficient Estimates and Fekete-Szegö Inequality

Theorem 2.1. *Let $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$, $k \in \mathbb{N} \cup \{0\}$ and $g(z) = z + \sum_{j=2}^{\infty} d_j z^j$ be in the family $\mathfrak{S}_{\sum}(x,\gamma,\mu,k)$. Then*

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{2|\mu(2\mu-\gamma)m^2(x) + (1-\gamma+2\mu)^2 n(x)|}}, \tag{2.1}$$

$$|d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{(1-\gamma+2\mu)^2} + \frac{|m(x)|}{2(1-\gamma+3\mu)}\right] \tag{2.2}$$

and for $\delta \in \mathcal{R}$

$$|d_3 - \delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{2(3^k)(1-\gamma+3\mu)}\,; & \left|1 - \dfrac{3^k\delta}{2^{2k}}\right| \le J \\[2ex] \dfrac{|m(x)|^3\left|1 - \frac{3^k\delta}{2^{2k}}\right|}{2(3^k)|\mu(2\mu-\gamma)m^2(x) + (1-\gamma+2\mu)^2 n(x)|}\,; & \left|1 - \dfrac{3^k\delta}{2^{2k}}\right| \ge J, \end{cases} \tag{2.3}$$

where

$$J=\frac{1}{(1-\gamma+3\mu)}\left|\mu(2\mu-\gamma)+(1-\gamma+2\mu)^2\left(\frac{n(x)}{m^2(x)}\right)\right|. \tag{2.4}$$

Proof. Let $g(z)\in\mathfrak{S}_{\Sigma}(x,\gamma,\mu,k)$. Then, for two holomorphic functions r and s such that $r(0)=s(0)=0$, $|r(z)|<1$ and $|s(\omega)|<1$, $z,\omega\in\mathfrak{D}$, and using Definition 1.1, we can write

$$\frac{z(D^kg(z))'+\mu z^2(D^kg(z))''}{(1-\gamma)D^kg(z)+\gamma z(D^kg(z))'}=\mathcal{G}(x,r(z))-1$$

and

$$\frac{\omega(D^kf(\omega))'+\mu\omega^2(D^kf(\omega))''}{(1-\gamma)D^kf(\omega)+\gamma\omega(D^kf(\omega))'}=\mathcal{G}(x,s(\omega))-1$$

or, equivalently

$$\frac{z(D^kg(z))'+\mu z^2(D^kg(z))''}{(1-\gamma)D^kg(z)+\gamma z(D^kg(z))'}=-1+L_0(x)+L_1(x)r(z)+L_2(x)r^2(z)+\ldots \tag{2.5}$$

and

$$\frac{\omega(D^kf(\omega))'+\mu\omega^2(D^kf(\omega))''}{(1-\gamma)D^kf(\omega)+\gamma\omega(D^kf(\omega))'}=-1+L_0(x)+L_1(x)s(\omega)+L_2(x)s^2(\omega)+\ldots. \tag{2.6}$$

From (2.5) and (2.6), in view of (1.3), we obtain

$$\frac{z(D^kg(z))'+\mu z^2(D^kg(z))''}{(1-\gamma)D^kg(z)+\gamma z(D^k(z))'}=1+L_1(x)r_1z+[L_1(x)r_2+L_2(x)r_1^2]z^2+\ldots \tag{2.7}$$

and

$$\frac{\omega(D^kf(\omega))'+\mu\omega^2(D^kf(\omega))''}{(1-\gamma)D^kf(\omega)+\gamma\omega(D^kf(\omega))'}=1+L_1(x)s_1\omega+[L_1(x)s_2+L_2(x)s_1^2]\omega^2+\ldots. \tag{2.8}$$

It is well known that if $|r(z)|=|r_1z+r_2z^2+r_3z^3+\ldots|<1$, $z\in\mathfrak{D}$ and $|s(\omega)|=|s_1\omega+s_2\omega^2+s_3\omega^3+\ldots|<1$, $\omega\in\mathfrak{D}$, then

$$|r_i|\le 1 \text{ and } |s_i|\le 1 \quad (i\in\mathbb{N}). \tag{2.9}$$

Comparing the corresponding coefficients in (2.7) and (2.8), we have

$$2^k(1-\gamma+2\mu)d_2=L_1(x)r_1, \tag{2.10}$$

$$2(3^k)(1-\gamma+3\mu)d_3-2^{2k}(1+\gamma)(1-\gamma+2\mu)d_2^2=L_1(x)r_2+L_2(x)r_1^2, \tag{2.11}$$

$$-2^k(1-\gamma+2\mu)d_2=L_1(x)s_1, \tag{2.12}$$

$$-2(3^k)(1-\gamma+3\mu)d_3+2^{2k}(\gamma^2-(4+2\mu)\gamma+3+10\mu)d_2^2=L_1(x)s_2+L_2(x)s_1^2. \tag{2.13}$$

From (2.10) and (2.12), we can easily see that

$$r_1=-s_1 \tag{2.14}$$

and also

$$2^{2k+1}(1-\gamma+2\mu)^2d_2^2=(r_1^2+s_1^2)(L_1(x))^2. \tag{2.15}$$

If we add (2.11) and (2.13), then we obtain

$$2^{2k+1}((1-\gamma)(1-\gamma+2\mu)+2\mu)d_2^2=L_1(x)(r_2+s_2)+L_2(x)(r_1^2+s_1^2). \tag{2.16}$$

Substituting the value of $(r_1^2+s_1^2)$ from (2.15) in (2.16), we get

$$d_2^2=\frac{(L_1(x))^3(r_2+s_2)}{2^{2k+1}[((1-\gamma)(1-\gamma+2\mu)+2\mu)(L_1(x))^2-(1-\gamma+2\mu)^2L_2(x)]}, \tag{2.17}$$

which yields (2.1), on using (2.9).

Using (2.14) in the subtraction of (2.13) from (2.11), we obtain

$$d_3=\frac{2^{2k}}{3^k}d_2^2+\frac{L_1(x)(r_2-s_2)}{4(3^k)(1-\gamma+3\mu)}. \tag{2.18}$$

Then in view of (2.15), (2.18) becomes

$$d_3=\frac{(L_1(x))^2(r_1^2+s_1^2)}{2(3^k)(1-\gamma+2\mu)^2}+\frac{L_1(x)(r_2-s_2)}{4(3^k)(1-\gamma+3\mu)},$$

which yields (2.2), on using (2.9).

From (2.17) and (2.18), for $\delta\in\mathcal{R}$, we get

$$|d_3-\delta d_2^2|=|m(x)|\left|\left(T(\delta,x)+\frac{1}{4(3^k)(1-\gamma+3\mu)}\right)r_2+\left(T(\delta,x)-\frac{1}{4(3^k)(1-\gamma+3\mu)}\right)s_2\right|,$$

where

$$T(\delta,x)=\frac{\left(\frac{2^{2k}}{3^k}-\delta\right)m^2(x)}{2^{2k+2}[\mu(\gamma-2\mu)m^2(x)+(1-\gamma+2\mu)^2n(x)]}.$$

In view of (1.3), we conclude that

$$|d_3-\delta d_2^2|\le\begin{cases}\dfrac{|m(x)|}{2(3^k)(1-\gamma+3\mu)}\ ; & 0\le|T(\delta,x)|\le\dfrac{1}{4(3^k)(1-\gamma+3\mu)}\\ 2|m(x)|\,|T(\delta,x)|\ ; & |T(\delta,x)|\ge\dfrac{1}{4(3^k)(1-\gamma+3\mu)},\end{cases}$$

which yields (2.3) with J as in (2.4). This evidently completes the proof of Theorem 2.1. □

Remark 1. The results obtained in Theorem 2.1 coincide with Theorem 2 and Theorem 3 of [3], for $k=0$ and $\mu=\gamma$, $(0\le\gamma\le1)$.

Remark 2. The results of Theorem 2.1 reduce to Corollary 1 and Corollary 3 of [3], when $k=\mu=\gamma=0$.

Remark 3. Corollary 2 and Corollary 4 of [3] can be obtained from Theorem 2.1, by putting $k=0$ and $\mu=\gamma=1$.

Theorem 2.2. *Let $0\le\gamma\le1$, $\mu\ge0$, $\mu\ge\gamma$, $k\in\mathbb{N}\cup\{0\}$ and $g(z)=z+\sum_{j=2}^{\infty}d_jz^j$ be in the family $\mathfrak{M}_{\Sigma}(x,\gamma,\mu,k)$. Then*

$$|d_2|\le\frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|(4\mu(\mu-\gamma)+(1-\gamma+2\mu))m^2(x)+8(1-\gamma+\mu)^2n(x)|}}, \tag{2.19}$$

$$|d_3|\le\frac{1}{3^k}\left[\frac{m^2(x)}{4(1-\gamma+\mu)^2}+\frac{|m(x)|}{3(1-\gamma+2\mu)}\right] \tag{2.20}$$

and for $\delta \in \mathcal{R}$

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{3^{k+1}(1-\gamma+2\mu)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le M \\ \dfrac{|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{3^k|(4\mu(\mu-\gamma)+(1-\gamma+2\mu))m^2(x)+8(1-\gamma+\mu)^2n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge M, \end{cases} \tag{2.21}$$

where

$$M=\frac{1}{3(1-\gamma+2\mu)}\left|(4\mu(\mu-\gamma)+(1-\gamma+2\mu))+8(1-\gamma+\mu)^2\left(\frac{n(x)}{m^2(x)}\right)\right|.$$

Proof. Let $g(z) \in \mathfrak{M}_\Sigma(x,\gamma,\mu,k)$. Then, for two holomorphic functions r and s such that $r(0)=s(0)=0$, $|r(z)|=|r_1z+r_2z^2+r_3z^3+\cdots|<1$ and $|s(\omega)|=|s_1\omega+s_2\omega^2+s_3\omega^3+\cdots|<1$, $z,\omega\in\mathfrak{D}$, and using Definition 1.2, we can write

$$\frac{z(D^kg(z))'+\mu z^2(D^kg(z))''}{(1-\gamma)z+\gamma z(D^kg(z))'}=\mathcal{G}(x,r(z))-1 \tag{2.22}$$

and

$$\frac{\omega(D^kf(\omega))'+\mu\omega^2(D^kf(\omega))''}{(1-\gamma)\omega+\gamma\omega(D^kf(\omega))'}=\mathcal{G}(x,s(\omega))-1. \tag{2.23}$$

Following (2.5), (2.6), (2.7), and (2.8) in the proof of Theorem 2.1, one gets in view of (2.22) and (2.23)

$$2^{k+1}(1-\gamma+\mu)d_2=L_1(x)r_1, \tag{2.24}$$

$$3^{k+1}(1-\gamma+2\mu)d_3-2^{2k+2}(1-\gamma+\mu)\gamma d_2^2=L_1(x)r_2+L_2(x)r_1^2, \tag{2.25}$$

$$-2^{k+1}(1-\gamma+\mu)d_2=L_1(x)s_1, \tag{2.26}$$

$$-3^{k+1}(1-\gamma+2\mu)d_3+2^{2k+1}[2\gamma^2-(5+2\mu)\gamma+3(1+2\mu)]d_2^2=L_1(x)s_2+L_2(x)s_1^2. \tag{2.27}$$

The results (2.19)-(2.21) of this theorem now follow from (2.24)-(2.27) by applying the procedure as in Theorem 2.1 with respect to (2.10)-(2.13). □

Theorem 2.3. *Let* $\xi \ge 1$, $\tau \ge 1$ *and* $k \in \mathcal{N}\cup\{0\}$, $g(z)=z+\sum_{j=2}^{\infty}d_jz^j$ *be in the family* $\mathfrak{B}_\Sigma(x,\xi,\tau,k)$. *Then*

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|((8\xi\tau^2-7\xi\tau+1)-4(2\xi\tau-1)^2)m^2(x)-8(2\xi\tau-1)^2n(x)|}}, \tag{2.28}$$

$$|d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{4(2\xi\tau-1)^2}+\frac{|m(x)|}{3(3\xi\tau-1)}\right] \tag{2.29}$$

and for $\delta \in \mathcal{R}$

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{3^{k+1}(3\xi\tau-1)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le \Omega \\ \dfrac{\left|1-\frac{3^k\delta}{2^{2k}}\right||m(x)|^3}{3^k|((8\xi\tau^2-7\xi\tau+1)-4(2\xi\tau-1)^2)m^2(x)-8(2\xi\tau-1)^2n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \Omega, \end{cases} \tag{2.30}$$

where

$$\Omega = \frac{1}{3(3\xi\tau-1)}\left|((8\xi\tau^2-7\xi\tau+1)-4(2\xi\tau-1)^2)-8(2\xi\tau-1)^2\left(\frac{n(x)}{m^2(x)}\right)\right|.$$

Proof. Let $g(z)\in\mathfrak{B}_{\Sigma}(x,\xi,\tau,k)$. Then, for two holomorphic functions r and s such that $r(0)=s(0)=0$, $|r(z)|=|r_1z+r_2z^2+r_3z^3+\cdots|<1$ and $|s(\omega)|=|s_1\omega+s_2\omega^2+s_3\omega^3+\cdots|<1$, $z,\omega\in\mathfrak{D}$, and using Definition 1.3, we can write

$$\frac{(1-\xi)+\xi[(z(D^kg(z))')']^{\tau}}{(D^kg(z))'}=\mathcal{G}(x,(z))-1, \quad z\in\mathfrak{D} \tag{2.31}$$

and

$$\frac{(1-\xi)+\xi[(\omega(D^kf(\omega))')']^{\tau}}{(D^kf(\omega))'}=\mathcal{G}(x,(\omega))-1, \quad \omega\in\mathfrak{D}. \tag{2.32}$$

Following (2.5), (2.6), (2.7), and (2.8) in the proof of Theorem 2.1, one gets in view of (2.31) and (2.32)

$$2^{k+1}(2\xi\tau-1)d_2=L_1(x)r_1, \tag{2.33}$$

$$2^{2k+2}(2\xi\tau^2-4\xi\tau+1)d_2^2+3^{k+1}(3\xi\tau-1)d_3=L_1(x)r_2+L_2(x)r_1^2, \tag{2.34}$$

$$-2^{k+1}(2\xi\tau-1)d_2=L_1(x)s_1, \tag{2.35}$$

$$2^{2k+1}(4\xi\tau^2+\xi\tau-1)d_2^2-3^{k+1}(3\xi\tau-1)d_3=L_1(x)s_2+L_2(x)s_1^2. \tag{2.36}$$

The results (2.28)-(2.30) of this theorem now follow from (2.33)-(2.36) by applying the procedure as in Theorem 2.1 with respect to (2.10)-(2.13). □

In next section, we present some interesting consequences of our main result.

3. Corollaries and Consequences

Theorem 2.1 would yield the following corollary for the family $\mathscr{K}_{\Sigma}(x,k)$, when $\gamma=1/2$ and $\mu=1/2$.

Corollary 3.1. *If $g(z)\in\mathscr{K}_{\Sigma}(x,k)$, a subfamily of Σ satisfying*

$$\frac{(z^2(D^kg(z))')'}{(zD^kg(z))'}\prec\mathcal{G}(x,z)-1,\ z\in\mathfrak{D} \quad \textit{and} \quad \frac{(\omega^2(D^kf(\omega))')'}{(\omega D^kf(\omega))'}\prec\mathcal{G}(x,\omega)-1,\ \omega\in\mathfrak{D},$$

where $f(\omega)=g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4), then

$$|d_2|\le\frac{\sqrt{2}|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|m^2(x)+9n(x)|}}, \qquad |d_3|\le\frac{1}{3^k}\left[\frac{4m^2(x)}{9}+\frac{|m(x)|}{4}\right]$$

and for some $\delta\in\mathcal{R}$,

$$|d_3-\delta d_2^2|\le\begin{cases}\dfrac{|m(x)|}{4(3^k)}; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right|\le\dfrac{1}{8}\left|1+9\left(\dfrac{n(x)}{m^2(x)}\right)\right| \\ \dfrac{2|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{3^k|m^2(x)+9n(x)|}; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right|\ge\dfrac{1}{8}\left|1+9\left(\dfrac{n(x)}{m^2(x)}\right)\right|.\end{cases}$$

Corollary 3.2 asserts immediate consequence of Theorem 2.1 for the family $\mathscr{J}_{\Sigma}(x,k)$, when $\gamma=0$ and $\mu=1/2$.

Corollary 3.2. *If $g(z) \in \mathscr{J}_{\Sigma}(x,k)$, a subfamily of Σ satisfying*

$$\frac{(z^2(D^k g(z))')'}{2D^k g(z)} \prec \mathcal{G}(x,z)-1,\ z \in \mathfrak{D} \ \text{ and } \ \frac{(\omega^2(D^k f(\omega))')'}{2D^k f(\omega)} \prec \mathcal{G}(x,\omega)-1, \quad \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4), then

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|m^2(x)+8n(x)|}}, \qquad |d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{4}+\frac{|m(x)|}{5}\right]$$

and for $\delta \in \mathcal{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{5(3^k)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le \dfrac{1}{5}\left|1+8\left(\dfrac{n(x)}{m^2(x)}\right)\right| \\[2ex] \dfrac{|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{3^k|m^2(x)+8n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \dfrac{1}{5}\left|1+8\left(\dfrac{n(x)}{m^2(x)}\right)\right|. \end{cases}$$

We conclude the below result for the family $\mathscr{L}_{\Sigma}(x,k)$ by putting $\gamma = 1/2$ and $\mu = 1$ in Theorem 2.1.

Corollary 3.3. *If $g(z) \in \mathscr{L}_{\Sigma}(x,k)$,a subfamily of Σ satisfying*

$$\frac{2z(z(D^k g(z))')'}{(zD^k g(z))'} \prec \mathcal{G}(x,z)-1,\ z \in \mathfrak{D} \quad \text{and} \quad \frac{2\omega(\omega(D^k f(\omega))')'}{(\omega D^k f\omega))'} \prec \mathcal{G}(x,\omega)-1,\ \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4), then

$$|d_2| \le \frac{\sqrt{2}|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|6m^2(x)+25n(x)|}}, \qquad |d_3| \le \frac{1}{3^k}\left[\frac{4m^2(x)}{25}+\frac{|m(x)|}{7}\right]$$

and for $\delta \in \mathcal{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{7(3^k)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le \dfrac{1}{14}\left|6+25\left(\dfrac{n(x)}{m^2(x)}\right)\right| \\[2ex] \dfrac{2|m(x)|^3\left|1-\dfrac{3^k\delta}{2^{2k}}\right|}{3^k|6m^2(x)+25n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \dfrac{1}{14}\left|6+25\left(\dfrac{n(x)}{m^2(x)}\right)\right|. \end{cases}$$

Corollary 3.4 asserts an another interesting consequence of Theorem 2.1 for the family $\mathscr{P}_{\Sigma}(x,\mu,k)$, by putting $\gamma = 0$.

Corollary 3.4. *Let $\mu \ge 0$. If a function $g(z) \in \mathscr{P}_{\Sigma}(x,\mu,k)$, a subfamily of Σ satisfying*

$$\left(\frac{z(D^k g(z))'}{D^k g(z)}\right)\left(1+\mu\frac{z(D^k g(z))''}{(D^k g(z))'}\right) \prec \mathcal{G}(x,z)-1,\ z \in \mathfrak{D}$$

and

$$\left(\frac{\omega(D^k f(\omega))'}{D^k f(\omega)}\right)\left(1+\mu\frac{\omega(D^k f(\omega))''}{(D^k f(\omega))'}\right) \prec \mathcal{G}(x,\omega)-1,\ \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ is as in (1.2) and $\mathcal{G}$ is as in (1.4), then

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{[2|2\mu^2 m^2(x)+(1+2\mu)^2 n(x)|]}}, \qquad |d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{(1+2\mu)^2}+\frac{|m(x)|}{2(1+3\mu)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{2(3^k)(1+3\mu)}\,; & \left|1-\frac{3^k\delta}{2^{2k}}\right| \le \dfrac{1}{(1+3\mu)}\left|2\mu^2+(1+2\mu)^2\left(\dfrac{n(x)}{m^2(x)}\right)\right| \\ \dfrac{|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{2(3^k)[2\mu^2m^2(x)+(1+2\mu)^2n(x)]}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \dfrac{1}{(1+3\mu)}\left|2\mu^2+(1+2\mu)^2\left(\dfrac{n(x)}{m^2(x)}\right)\right|. \end{cases}$$

Setting $\delta = 1$ and $k = 0$ in Theorem 2.1, we arrive at the following:

Corollary 3.5. *Let* $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$ *and* $g(z)$ *of the form* (1.1) *be in* $\mathfrak{S}_{\sum}(x,\gamma,\mu,0)$. *Then*

$$|d_3 - d_2^2| \le \frac{|m(x)|}{2(1-\gamma+3\mu)}.$$

Remark 4. Corollary 3.5 reduces to Corollary 5, Corollary 6 and Corollary 7 of [3] when $\mu = \gamma$, $\mu = \gamma = 0$ and $\mu = \gamma = 1$, respectively.

Corollary 3.6 asserts immediate consequence of Theorem 2.2 for the family $\mathfrak{K}_{\sum}(x,\mu,k)$ when $\gamma = 0$.

Corollary 3.6. *If* $g(z) \in \mathfrak{K}_{\sum}(x,\mu,k)$, $\mu \ge 0$, *a subfamily of* $\sum$ *satisfying*

$$(D^k g(z))' + \mu z(D^k g(z))'' \prec \mathcal{G}(x,z) - 1 \ \text{ and } \ (D^k f(\omega))' + \mu\omega(D^k f(\omega))'' \prec \mathcal{G}(x,\omega) - 1,$$

where $z,\omega \in \mathfrak{D}$, $f(\omega) = g^{-1}(\omega)$ *is as in* (1.2) *and* $\mathcal{G}$ *is as in* (1.4), *then*

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|(4\mu^2+2\mu+1)m^2(x)+8(1+\mu)^2n(x)|}}, \qquad |d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{4(1+\mu)^2} + \frac{|m(x)|}{3(1+2\mu)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{3^{k+1}(1+2\mu)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le M_1 \\ \dfrac{|m(x)|^3|1-\frac{3^k\delta}{2^{2k}}|}{3^k|(4\mu^2+2\mu+1)m^2(x)+8(1+\mu)^2n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge M_1, \end{cases}$$

where $M_1 = \dfrac{1}{3(1+2\mu)}\left|(4\mu^2+2\mu+1)+8(1+\mu)^2\left(\dfrac{n(x)}{m^2(x)}\right)\right|$.

Corollary 3.7 asserts an another interesting consequence of Theorem 2.2 for the family $\mathfrak{L}_{\sum}(x,\mu,k)$ by putting $\gamma = 1$.

Corollary 3.7. *If* $g(z) \in \mathfrak{L}_{\sum}(x,\mu,k)$, $\mu \ge 1$, *a subfamily of* $\sum$ *satisfying*

$$1+\mu\left(\frac{z(D^k g(z))''}{(D^k g(z))'}\right) \prec \mathcal{G}(x,z)-1,\ z \in \mathfrak{D} \quad \text{and} \quad 1+\mu\left(\frac{\omega(D^k f(\omega))''}{(D^k f(\omega))'}\right) \prec \mathcal{G}(x,\omega)-1,\ \omega \in \mathfrak{D},$$

where $f(\omega) = g^{-1}(\omega)$ *is as in* (1.2) *and* $\mathcal{G}$ *is as in* (1.4), *then*

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{2\mu|(2\mu-1)m^2(x)+4\mu n(x)|}}, \quad |d_3| \le \frac{1}{2\mu(3^k)}\left[\frac{m^2(x)}{2\mu} + \frac{|m(x)|}{3}\right]$$

and for $\delta \in \mathcal{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{2\mu(3^{k+1})}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le \dfrac{1}{3}\left|(2\mu-1)+4\mu\left(\dfrac{n(x)}{m^2(x)}\right)\right| \\ \dfrac{|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{2\mu(3^k)|(2\mu-1)m^2(x)+4\mu n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \dfrac{1}{3}\left|(2\mu-1)+4\mu\left(\dfrac{n(x)}{m^2(x)}\right)\right|. \end{cases}$$

We conclude the below result for the family $\mathscr{M}_{\sum}(x,\xi,k)$ by putting $\tau=1$ in Theorem 2.3.

Corollary 3.8. *If* $g(z)\in\mathscr{M}_{\sum}(x,\xi,k)$, $\xi\ge 1$, *a subfamily of* $\sum$ *satisfying*

$$(1-\xi)\frac{1}{(D^kg(z))'}+\xi\left(1+\frac{z(D^kg(z))''}{(D^kg(z))'}\right) \prec \mathcal{G}(x,z)-1, \quad z\in\mathfrak{D}$$

and

$$(1-\xi)\frac{1}{(D^kf(\omega))'}+\xi\left(1+\frac{\omega(D^kf(\omega))''}{(D^kf(\omega))'}\right) \prec \mathcal{G}(x,\omega)-1, \quad \omega\in\mathfrak{D},$$

where $f(\omega)=g^{-1}(\omega)$ *is as in* (1.2) *and* $\mathcal{G}$ *is as in* (1.4), *then*

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|(\xi+1-4(2\xi-1)^2)m^2(x)-8(2\xi-1)^2n(x)|}},$$

$$|d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{4(2\xi-1)^2}+\frac{|m(x)|}{3(3\xi-1)}\right]$$

and for $\delta\in\mathcal{R}$

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{3^{k+1}(3\xi-1)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le \Omega_1 \\ \dfrac{2|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{3^k|(\xi+1-4(2\xi-1)^2)m^2(x)-8(2\xi-1)^2n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \Omega_1, \end{cases}$$

where $\Omega_1=\dfrac{1}{3(3\xi-1)}\left|(\xi+1-4(2\xi-1)^2)-8(2\xi-1)^2\left(\dfrac{n(x)}{m^2(x)}\right)\right|$.

Theorem 2.3 would yield the following corollary for the family $\mathfrak{N}_{\sum}(x,\tau,k)$, when $\xi=1$.

Corollary 3.9. *If* $g(z)\in\mathfrak{N}_{\sum}(x,\tau,k)$, $\tau\ge 1$, *a subfamily of* $\sum$ *satisfying*

$$\frac{[(z(D^kg(z))')']^\tau}{(D^kg(z))'} \prec \mathcal{G}(x,z)-1,\ z\in\mathfrak{D} \ \ \text{and} \ \ \frac{[(\omega(D^kf(\omega))')']^\tau}{(D^kf(\omega))'} \prec \mathcal{G}(x,\omega)-1, \quad \omega\in\mathfrak{D},$$

where $f(\omega)=g^{-1}(\omega)$ *is as in* (1.2) *and* $\mathcal{G}$ *is as in* (1.4), *then*

$$|d_2| \le \frac{|m(x)|\sqrt{|m(x)|}}{2^k\sqrt{|(8\tau^2-7\tau+1-4(2\tau-1)^2)m^2(x)-8(2\tau-1)^2n(x)|}},$$

$$|d_3| \le \frac{1}{3^k}\left[\frac{m^2(x)}{4(2\tau-1)^2}+\frac{|m(x)|}{3(3\tau-1)}\right]$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \le \begin{cases} \dfrac{|m(x)|}{3^{k+1}(3\tau-1)}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \le \Omega_2 \\ \dfrac{|m(x)|^3\left|1-\frac{3^k\delta}{2^{2k}}\right|}{3^k|((8\tau^2-7\tau+1)-4(2\tau-1)^2)m^2(x)-8(2\tau-1)^2n(x)|}\,; & \left|1-\dfrac{3^k\delta}{2^{2k}}\right| \ge \Omega_2, \end{cases}$$

where $\Omega_2 = \dfrac{1}{3(3\tau-1)}\left|(8\tau^2-7\tau+1-4(2\tau-1)^2)-8(2\tau-1)^2\left(\dfrac{n(x)}{m^2(x)}\right)\right|.$

4. Conclusion

Using the concept of subordination, we have introduced some special families of holomorphic and Sălăgean type bi-univalent functions in the open unit disc $\mathfrak{D}$ associated with (m,n)-Lucas polynomials. We have then derived the initial coefficient estimations and also Fekete-Szegö inequalities for functions belonging to these special families. Our main results are obtained in Theorem 2.1, Theorem 2.2 and Theorem 2.3. Further by specializing the parameters, several consequences of these new families are mentioned.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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