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The Proper Elements and Simple Invariant Subspaces

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Abstract. A proper element of X is a triple (λ, L, A) composed by an eigenvalue λ, an invariant subspace of an operator A in $B(X)$ generated by one eigenvector of λ and the operator A. For $(\lambda_0, L_0, A_0) \in Eig(X)$, where $L_0 = \mathscr{L}(\{x_0\})$, the operator A_0 induces an operator $\widehat{A_0}$ from the quotient X/L_0 into itself, i.e. $\widehat{A_0}(x + L_0) = A_0(x) + L_0$.

In paper we show that λ_0 is a simple pole of A_0 if and only if $\lambda_0 \notin \sigma(\widehat{A_0})$. Follow this concept we can define simple invariant subspaces of linear operator T like invariant subspace E such that $\sigma(T_E) \cap \sigma(\widehat{T_E}) = \emptyset$, where $T_E : E \to E$ is the restriction of T on E, $\widehat{T_E}$ is the operator $\widehat{T_E}(\pi(y)) = \pi(T(y))$ on the quotient space X/E and π is the natural homoeomorphism between X and X/E. Also, we give some properties of stability of simple invariant subspaces.

1. Introduction

Let X be a Banach space, then $\mathscr{B}(X)$ denotes the space of all bounded linear operators from X to X. For $T \in \mathscr{B}(X)$, let $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T. Let $n(T)$ and $d(T)$ be the nullity and the deficiency of T defined by

$$n(T) = \dim N(T) \quad \text{and} \quad d(T) = \operatorname{codim} R(T).$$

Let $\pi_0(T)$ denote the set of Riesz points of T (i.e., the set of isolated eigenvalues of T of finite algebraic multiplicity). $\lambda \in \pi_0(T)$ is called a simple eigenvalue (pole) of T if its algebraic multiplicity is 1. Let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of T of finite geometric multiplicity (i.e. $0 < n(T - \lambda) < \infty$).

The *ascent*, notated by $\operatorname{asc}(T)$, and the *descent*, notated by $\operatorname{dsc}(T)$, of T are given by

$$\operatorname{asc}(T) = \inf\{n : N(T^n) = N(T^{n+1})\},$$
$$\operatorname{dsc}(T) = \inf\{n : R(T^n) = R(T^{n+1})\};$$

if no such n exists, then $\operatorname{asc}(T) = \infty$, respectively $\operatorname{dsc}(T) = \infty$.

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One of the oldest problem in the linear algebra is determinate (all) eigenvalues and corresponding eigenvectors of finite dimension matrices. Today, we can find many methods that give us partially or completely solution of this problem. We can extended some of those methods to the case of infinity dimensional matrices, or more general, to linear bounded operators between (Banach) vector spaces. The principal limitation of almost all of such methods is that they find only isolated eigenvalues of finite algebraic multiplicity. In the second section of the manuscript we gave necessary and sufficient condition such that a point in the spectrum of a linear bounded operator is a simple pole. Moreover, we extend the results from [4] obtaining solution for finding arbitraries eigenvalues and eigenvectors of an operator resolving a system of operators equations. In the third section, following ideas from previous one, we introduced the concept of simple invariant subspace of linear operator (Definition 3.1) and we give some basic properties.

2. Manifold of proper elements and pols of a linear operator

Let $P_1(X)$ denote the collection of all subspaces of X of dimension 1. The *manifold of proper elements of* X (see [4]) is the set

$$Eig(X) = \{(\lambda, L, A) \in \mathbf{C} \times P_1(X) \times \mathscr{B}(X) : A(L) \subset L \text{ and } A_{|L} = \lambda I\}.$$

In other words, the proper elements of X are triples consisting of an eigenvalue λ, an invariant subspace generated by one eigenvector of λ and an appropriate operator A. Now, fix one proper element of X, say $(\lambda_0, L_0, A_0) \in Eig(X)$.

For many practical reasons, it is important that the eigenvalue in the chosen proper element be a (simple) pole of A_0. For example, it is known that if $\lambda_0 \in \pi_0(A_0)$, then for any sequence $\{A_n\}$ in $\mathscr{B}(X)$ that converges in norm to A_0 there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \in \pi_0(A_n)$ and $\lambda_n \to \lambda_0$. Moreover, if λ_0 is a simple pole, then for almost all positive integer n, λ_n is a simple pole of A_n, and the corresponding eigenvectors x_n converge to x_0, i.e. we have that $(\lambda_n, L_n, A_n) \to (\lambda_0, L_0, A_0)$ (here L_n is the linear span of x_n, that is $L_n = \mathscr{L}(\{x_n\})$). (For the previous see [2, Theorem 2.17].)

In this way, we will give necessary and sufficient conditions to obtain that λ_0 be a simple pole of $A_0 \in B(X)$. For this, we need some preliminary notations and results.

Let $(\lambda_0, L_0, A_0) \in Eig(X)$, where $L_0 = \mathscr{L}(\{x_0\})$. Then the operator A_0 induces the operator $\widehat{A_0}$ from the quotient X/L_0 into itself, i.e. $\widehat{A_0}(x + L_0) = A_0(x) + L_0$.

Proposition 2.1. *Let* $(\lambda_0, L_0, A_0) \in Eig(X)$. *Then* $\lambda_0 \notin \sigma_p(\widehat{A_0})$ *if and only if* $n(A_0 - \lambda_0) = 1$ *and* $asc(A_0 - \lambda_0) = 1$.

Proof. ($\Rightarrow$:) Let $\lambda_0 \notin \sigma_p(\widehat{A_0})$ and suppose that $n(A_0 - \lambda_0) > 1$. Then there exists $x_1 = kx_0 + h$, $h \neq 0$ such that $(A_0 - \lambda_0)x_1 = 0$ (with respect to the decomposition

$X = L_0 \oplus X_0$). Then,

$$(\widehat{A_0} - \lambda_0)[h_1] = [(A_0 - \lambda_0)h_1] = [0],$$

where $h_1 \notin L_0$. Hence, $\lambda_0 \in \sigma_p(\widehat{A_0})$.

Next, suppose that there exists $x \in N(A_0 - \lambda_0)^2 \setminus N(A_0 - \lambda_0)$. Then $(A_0 - \lambda_0)^2(x) = 0$ and $(A_0 - \lambda_0)x \neq 0$, or equivalently $x + L_0 \in N(\widehat{A_0} - \lambda_0)$ and $x + L_0 \neq L_0$ which contradicts with $\lambda_0 \notin \sigma_p(\widehat{A_0})$.

($\Leftarrow$:) Suppose now that $asc(A_0 - \lambda_0) = 1$. Then $L_0 + N(A_0 - \lambda_0) = N(A_0 - \lambda_0)$ and it is easy to see that $L_0 \subset (A_0 - \lambda_0)^{-1}(L_0)$. Let $y \in (A_0 - \lambda_0)^{-1}(L_0)$ or equivalently $(A_0 - \lambda_0)y \in L_0$. Then $(A_0 - \lambda_0)^2 y = 0$ and $y \in N(A_0 - \lambda_0)^2 = N(A_0 - \lambda_0)$. Hence, $L_0 + N(A_0 - \lambda_0) = (A_0 - \lambda_0)^{-1}(L_0)$ and by [3, Proposition 7] follows that

$$n(A_0 - \lambda_0) = n(A_{0|L_0} - \lambda_0) + n(\widehat{A_0} - \lambda_0).$$

Moreover, since $n(A_{0|L_0} - \lambda_0) = 1$, we have

$$n(A_0 - \lambda_0) = 1 + n(\widehat{A_0} - \lambda_0)$$

and, since $n(A_0 - \lambda_0) = 1$, we have that $\lambda_0 \notin \sigma_p(\widehat{A_0})$. □

Theorem 2.2. *Let $(\lambda_0, L_0, A_0) \in Eig(X)$. Then $\lambda_0 \notin \sigma(\widehat{A_0})$ if and only if the next conditions hold:*

(i) $n(A_0 - \lambda_0) = 1$;
(ii) $asc(A_0 - \lambda_0) = 1$;
(iii) $\lambda_0 \in iso\ \sigma(A_0)$.

Proof. ($\Leftarrow$:) By the proof of previous proposition, if $\mathrm{asc}(A_0 - \lambda_0) = 1$, then

$$n(A_0 - \lambda_0) = 1 + n(\widehat{A_0} - \lambda_0).$$

Since $d(A_{0|L_0} - \lambda_0) = 1$, applying [3, Proposition 7, (i) and (iii)], we have

$$d(A_0 - \lambda_0) = 1 + d(\widehat{A_0} - \lambda_0).$$

In the case of the isolated point λ_0 in $\sigma(A_0)$, the continuity of index implies

$$1 = n(A_0 - \lambda_0) = d(A_0 - \lambda_0),$$

and consequently $d(\widehat{A_0} - \lambda_0) = n(\widehat{A_0} - \lambda_0) = 0$, i.e. $\lambda_0 \notin \sigma(\widehat{A_0})$.

($\Rightarrow$:) Let $\lambda_0 \notin \sigma(\widehat{A_0})$, then by Proposition 2.1 the conditions (i) and (ii) hold. For (iii): suppose the contrary, i.e. there exists a sequence of different points $\{\lambda_n\}_{n=1}^{\infty}$ in $\sigma(A_0)$ such that $\lambda_n \to \lambda_0$. Since $\sigma(A_0) \subset \{\lambda_0\} \cup \sigma(\widehat{A_0})$ (see [5]) it follows that $\{\lambda_n\}_{n=1}^{\infty} \subset \sigma(\widehat{A_0})$ and consequently $\lambda_0 \in \sigma(\widehat{A_0})$, which is a contradiction. □

Corollary 2.3. *Let $(\lambda_0, L_0, A_0) \in Eig(X)$. Then $\lambda_0 \notin \sigma(\widehat{A_0})$ if and only if λ_0 is a simple pole of A_0.*

Proof. (⇒:) Let $\lambda_0 \notin \sigma(\widehat{A_0})$. Then conditions (i)-(iii) in the previous theorem hold. Then by [1, Theorem 3.4] follows that $\mathrm{dsc}(A_0 - \lambda_0) = \mathrm{asc}(A_0 - \lambda_0) = 1$ and consequently λ_0 is an simple pole of A_0.

(⇐:) If λ_0 is an simple pole of A_0, then $X = N(A_0 - \lambda_0) \oplus (A_0 - \lambda_0)(X)$ and, by this decomposition, A_0 has a representation $A_0 = \lambda_0 I \oplus A_1$, where $\lambda_0 \notin \sigma(A_1)$. By introduction of [3] it follows that $\sigma(A_1) = \sigma(\widehat{A_0})$ and this implies $\lambda_0 \notin \sigma(\widehat{A_0})$. □

If λ_0 is an eigenvalue (without any extra condition) of A_0 we can not claim that for every sequence of operators that converges to A_0 we will find a sequence of eigenvalues and eigenvectors such that $(\lambda_n, L_n, A_n) \to (\lambda_0, L_0, A_0)$. Moreover, the next theorem and corollary give us a method to construct a sequence of proper elements that converges to (λ_0, L_0, A_0). The ideas are in [4], but for the sake of completeness, we will give the proofs.

Theorem 2.4. *Let $(\lambda_0, L_0, A_0) \in Eig(X)$. Then $A \in B(X)$ has an eigenvalues λ_1 with eigenvector x_1 if and only if the next system of equations*

$$A_{12}h_1 = (\lambda_1 - A_{11})x_0$$
$$A_{21}x_0 = (\lambda_1 - A_{22})h_1$$

holds, where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is the matrix representation of the operator A with respect to the direct sum $L_0 \oplus X_0 = X$ where $L_0 = \mathscr{L}(\{x_0\})$ and $x_1 = x_0 + h_1$.

Proof. Let $L_0 = \mathscr{L}(\{x_0\})$. Since $\dim L_0 = 1$, there exists a closed subspace X_0 of X such that $X = L_0 \oplus X_0$. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in B(L_0 \oplus X_0)$ so that it has eigenvalue λ_1 and eigenvector $x_1 = x_0 + h_1$. Then

$$\lambda_1(x_0 + h_1) = Ax_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ h_1 \end{bmatrix} = (A_{11}x_0 + A_{12}h_1) + (A_{21}x_0 + A_{22}h_1) \Longleftrightarrow$$
$$A_{12}h_1 = (\lambda_1 - A_{11})x_0$$
$$A_{21}x_0 = (\lambda_1 - A_{22})h_1.$$

On the other side, if the equations holds for some $h_1 \in X_0$ and $\lambda_1 \in \mathbf{C}$, it is easy to see that λ_1 is an eigenvalue of A with eigenvector $x_0 + h_1$. □

Remark 2.5. In [4] is excluded the case when $h_1 = 0$, but, by the previous system of equations, then we have that $A_{21} = 0$, or L_0 is an invariant subspace of A that implies that x_0 is an eigenvector for the eigenvalue λ_1 of A.

Theorem 2.6. *Let $(\lambda_0, L_0, A_0) \in Eig(X)$. Then there exists a transformation $F : U \to B(X)$ defined in a neighborhood U of (λ_0, L_0) such that $F(\lambda_0, L_0) = A_0$, $(\lambda, L, F(\lambda, L)) \in Eig(X)$, for every $(\lambda, L) \in U$, and F is continuous at (λ_0, L_0).*

Proof. Let $X = L_0 \oplus X_0$, $L_0 = \mathscr{L}(\{x_0\})$ and let A_0 such that it has matrix representation

$$A_0 = \begin{bmatrix} \lambda_0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix}$$

with respect to the decomposition of the space X. For $(\tilde{\lambda}, \tilde{L}) \in U$, $\tilde{L} = \mathscr{L}(\{x_0 + \tilde{h}\})$, we define $F(\tilde{\lambda}, \tilde{L}) \in B(X)$ using the operator matrix

$$F(\tilde{\lambda}, \tilde{L}) = \begin{bmatrix} \tilde{A}_{11} & A_{12}^0 \\ \tilde{A}_{21} & A_{22}^0 \end{bmatrix} : L_0 \oplus X_0 \to L_0 \oplus X_0,$$

where

$$\tilde{A}_{11}(\alpha x_0) = \alpha(\tilde{\lambda} x_0 - A_{12}^0 \tilde{h}) \quad \text{and}$$
$$\tilde{A}_{21}(\alpha x_0) = \alpha(\tilde{\lambda} - A_{22}^0)\tilde{h}.$$

It is easy to see that $F(\tilde{\lambda}, \tilde{L})(x_0 + \tilde{h}) = \tilde{\lambda}(x_0 + \tilde{h})$, i.e. $(\tilde{\lambda}, \tilde{L}, F(\tilde{\lambda}, \tilde{L})) \in Eig(X)$.

Without lost of generality we can suppose that $\|x_0\| = 1$ and let $\alpha x_0 + h \in X$ be an arbitrary norm one vector. Then

$$\|F(\tilde{\lambda}, \tilde{L})(\alpha x_0 + h) - A_0(\alpha x_0 + h)\| \leq |\alpha| \cdot (|\tilde{\lambda} - \lambda_0| + \|(A_0 - \tilde{\lambda})\| \cdot \|\tilde{h}\|)$$

i.e. $\|F(\tilde{\lambda}, \tilde{L}) - A_0\| \leq |\tilde{\lambda} - \lambda_0| + \|A_{22}^0 - \tilde{\lambda}\| \cdot \|\tilde{h}\|$ that converge to zero when $\tilde{\lambda} \to \lambda_0$ and $\tilde{h} \to 0$. □

By the previous theorem, for any two sequences $\{\lambda_n\}$ and $\{x_n\}$ that converges to λ_0 and x_0 respectively, the sequence of operators $\{F(\lambda_n, L_n)\}$ ($L_n = \mathscr{L}(x_n)$) converge to A_0. The operator $\tilde{A}_{21}$ in the matrix representation of the operator $F(\tilde{\lambda}, \tilde{L})$ has a crucial role with respect to its spectral properties and bounded condition. In general, let L_0 be a dimension 1 subspace of X and $X = L_0 \oplus X_0$. For a fixed operators $A \in B(L_0)$, $B \in B(X_0)$ and $C \in B(X_0, L_0)$, denote with M_D the matrix operator

$$M_D = \begin{bmatrix} A & C \\ D & B \end{bmatrix},$$

where $D \in B(L_0, X_0)$.

Theorem 2.7. *Let $X = L_0 \oplus X_0$, where $\dim L_0 = 1$ and let $\lambda \notin \sigma(A)$.*

(i) *If $C \neq 0$, then there exists a $D_\lambda \in B(L_0, X_0)$ such that $\lambda \in \sigma_p(M_{D_\lambda})$.*
(ii) *If $C = 0$, then for any $D \in B(L_0, X_0)$, $\lambda \in \sigma_p(M_D)$ if and only if $\lambda \in \sigma_p(B)$.*

Proof. (i) Let $L_0 = \mathscr{L}(\{x_0\})$. Since $C \neq 0$, there exits $y_0 \in X_0$ such that $Cy_0 \neq 0$. Let k be a complex non-zero number such that $-kx_0 = (A - \lambda)^{-1} C y_0$. Let $D_\lambda \in B(L_0, X_0)$ be define by

$$D_\lambda(x_0) = -\frac{1}{k}(B - \lambda) y_0.$$

Then for $x_\lambda = kx_0 + y_0$ we have

$$(M_{D_\lambda} - \lambda)x_\lambda = \begin{bmatrix} A-\lambda & C \\ D_\lambda & B-\lambda \end{bmatrix} \begin{bmatrix} kx_0 \\ y_0 \end{bmatrix} = 0,$$

i.e. λ is an eigenvalue of M_{D_λ} with eigenvector $x_\lambda = kx_0 + y_0$.

(ii) Let $C = 0$. Then X_0 is an invariant subspace for M_D, $(\sigma(A) \cup \sigma(B)) \setminus \sigma(M_D) \subset \sigma(A) \cap \sigma(B)$ and $\dim N(B) \leq \dim N(M_D)$ (see [6]). Hence, if $\lambda \notin \sigma(A)$, then $\lambda \in \sigma(M_D)$ if and only if $\lambda \in \sigma(B)$ and if $\lambda \in \sigma_p(B)$, then $\lambda \in \sigma_p(M_D)$. Also, if $kx_0 + y$ is eigenvector for $\lambda \in \sigma_p(M_D)$, then since $\lambda \notin \sigma(A)$, follows that $k = 0$ and $(B - \lambda)y = 0$. Hence $\lambda \in \sigma_p(B)$. □

Remark 2.8. (i) It is easy to see that, in the case when $\lambda \in \sigma(A)$, then, for $D = 0$, λ is an eigenvalue of M_0 with eigenvector x_0 ($L_0 = \mathscr{L}(\{x_0\})$).
(ii) For a similar result see [6, Theorem 8].

3. Simple invariant subspace

Let $Inv(T)$ denote the set of non-trivial closed (in X) invariant subspaces of T. For $T \in B(X)$ and $E \in Inv(T)$, we shall denote by $T_E : E \to E$ the restriction of T on E, and by $\widehat{T_E}$ the operator $\widehat{T_E}(\pi(y)) = \pi(T(y))$ on the quotient space X/E, where π is the natural homoeomorphism between X and X/E.

Definition 3.1. Let $T \in B(X)$. We tell that $E \in Inv(T)$ is a *simple invariant subspace* if $\sigma(T_E) \cap \sigma(\widehat{T_E}) = \emptyset$.

Proposition 3.2. *Let $E \in Inv(T)$ be a simple invariant subspace. Then there exists a $\delta > 0$ such that any operator $S \in B(X)$ commuting with T and satisfying $\|T - S\| < \delta$ has a simple invariant subspace.*

Proof. Let $E \in Inv(T)$ be a simple invariant subspace and denote by $\sigma_1 = \sigma(T_E)$, $\sigma_2 = \sigma(\widehat{T_E})$ and $\epsilon = \frac{1}{3} dist(\sigma_1, \sigma_2)$.

Suppose the contrary (no such δ exists), then there exists a sequence of operators $\{S_n\} \subseteq B(X)$ such that $TS_n = S_n T$ and $\|T - S_n\| \to 0$. By [8, Theorem 4], we have that $\lim_{n\to\infty} \sigma(S_n) = \sigma(T)$, or equivalently, for any $\epsilon > 0$, there exists a positive integer n_0 such that, for every positive integer $n > n_0$, $\sigma(S_n) \subset (\sigma(T))_\epsilon$ and $\sigma(T) \subset (\sigma(S_n))_\epsilon$. Now it easy to see that, for every $n > n_0$, S_n has the spectrum separated in (last) two spectral sets and applying Cauchy projection we can find simple invariant subspaces for S_n. □

By the proof of the previous proposition it is easy to see that we use the commutation of T and S to have closedness (in Hausdorff metric sense) of spectrums of T and S. Of course, if the operator T is point of spectral continuity we have this property for any another operator that is close enough to T and then next corollary is clear.

Corollary 3.3. *Let $T \in B(X)$ be a point of spectrum continuity and $E \in Inv(T)$ be a simple invariant subspace. Then there exists a $\delta > 0$ such that any operator $S \in B(X)$, with $\|T - S\| < \delta$, has a simple invariant subspace.*

Theorem 3.4. *Let $E \in Inv(T)$ be a simple invariant subspace. Then there exists a simple invariant subspace F such that $\sigma(T_E) = \sigma(\widehat{T_F})$ and $\sigma(\widehat{T_E}) = \sigma(T_F)$.*

Proof. Let E be a simple invariant subspace for an operator T. Then, by [5, Corollary 2.2], it follows that $\sigma(T) = \sigma(T_E) \cup \sigma(\widehat{T_E})$ and both of $\sigma(T_E)$ and $\sigma(\widehat{T_E})$ are spectral set of T. Let Γ be a Cauchy curve such that $\sigma(\widehat{T_E})$ is inside and $\sigma(T_E)$ outside the curve. Let $F = P_T(X)$ and $G = N(P_T(X))$, where P_T is the Cauchy projection associated with T and Γ (see [7, p. 178]). Then $X = F \oplus G$, $\sigma(T_F) = \sigma(\widehat{T_E})$ and $\sigma(T_G) = \sigma(T_E)$. Moreover, by the introduction of [3], it easy to see that $\sigma(\widehat{T_F}) = \sigma(T_E)(= \sigma(T_G))$. □

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