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Research Article

# Common Fixed Point Results in *C*-Complete Complex Valued Metric Spaces

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**Abstract.** In this paper, we establish some common fixed point theorems involving two pairs of weakly compatible mappings satisfying rational inequality in the setting of $C$-complex valued metric space. The presented theorems generalize, extend and improve the known results in the literature.

**Keywords.** $C$-complex valued metric spaces; Common fixed point; Weakly compatible mappings

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## 1. Introduction

The concept of *Complex valued metric space* was introduced by Azam *et al*. [1], which is relatively more general than ordinary metric spaces. Since then, several authors [3, 5–9, 11] have been studying many different contractions condition and proved fixed point theorems in complex valued metric spaces. Recently, Sintunavarat *et al*. [9–11] introduced the concept of $C$-Cauchy sequence and $C$-complete complex valued metric spaces and proved the existence of common fixed theorems in $C$-complete complex valued metric spaces. Further, the authors in [2, 4] continue the study of common fixed point in $C$-complete complex valued metric spaces.

The aim of this manuscript is to establish the common fixed point theorems for two pairs of weakly compatible mappings satisfying rational inequality in the framework of $C$-complete complex valued metric spaces. Our results generalizes the results in [4, 10].

Before presenting our theorems, we discuss some concepts of the complex valued metric space due to Azam *et al*. [1] and give some definition, examples, applications in such spaces were introduced by Sintunavarat *et al*. [10].

Let $\mathbb{C}$ be the set of complex numbers. For $z_1, z_2 \in \mathbb{C}$ we will define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$.

We note that $z_1 \precsim z_2$ if one of the following holds:

(C1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;

(C2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;

(C3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;

(C4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we write $z_1 \prec z_2$ if and only if (C4) is satisfied.

**Remark 1.1.** We note that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \le b \to az \precsim bz$ for all $z \in \mathbb{C}$.

(ii) $0 \precsim z_1 \precnsim z_2 \to |z_1| < |z_2|$.

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \to z_1 \prec z_3$.

The following definitions and results will be needed in the sequel.

**Definition 1.2** ([1]). Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions;

(i) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a complex valued metric on $X$ and $(X, d)$ is called a complex valued metric space.

**Definition 1.3** ([1]). Let $(X, d)$ be a complex valued metric space.

(i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of $A$ whenever, for all $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A - X) \neq \phi.$$

(iii) A set $A \subseteq X$ is called open set whenever each element of $A$ is an interior point of $A$.

(iv) A set $A \subseteq X$ is called closed set whenever each limit point of $A$ belongs to $A$.

(v) A sub-basis for a Hausdorff topology $\tau$ on $X$ is the family

$$F = \{B(x,r) : x \in X \text{ and } 0 \prec r\}.$$

**Definition 1.4** ([1]). Let $(X,d)$ be a complex valued metric space, $\{x_n\}$ be a sequence in $X$ and let $x \in X$.

(i) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \prec c$, then $\{x_n\}$ is said to be convergent to a point $x \in X$ or $\{x_n\}$ converges to a point $x \in X$ and $x$ is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is called a Cauchy sequence in $X$.

(iii) If for every Cauchy sequence in $X$ is convergent, then $(X,d)$ is said to be complete complex valued metric space.

**Definition 1.5** ([9, 10]). Let $(X,d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in $X$.

(i) If, for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that for all $m,n > N$, $d(x_n,x_m) \prec c$, then $\{x_n\}$ is called a $C$-Cauchy sequence in $X$.

(ii) If every $C$-Cauchy sequence in $X$ is convergent, then $(X,d)$ is said to be a $C$-complete complex valued metric space.

For more concepts in $C$-complete complex valued metric space (see [9, 10]).

**Definition 1.6.** Let $S$ and $T$ be self mappings of a nonempty set $X$.

(i) A point $x \in X$ is called a fixed point of $T$ if $Tx = x$.

(ii) A point $x \in X$ is called a coincidence point of $S$ and $T$ if $Sx = Tx$ and the point $u \in X$ such that $u = Sx = Tx$ is called a point of coincidence of $S$ and $T$.

(iii) A point $x \in X$ is called a common fixed point of $S$ and $T$ if $x = Sx = Tx$.

**Definition 1.7** ([8]). Let $X$ be a complex valued metric space. Then a pair of self mappings $S,T : X \to X$ is said to be weakly compatible if they commute at their coincidence points.

The following lemmas of [1] will be used in the sequel.

**Lemma 1.8** ([1]). *Let $(X,d)$ be a complex valued metric space, $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to a point $x \in X$ if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

**Lemma 1.9** ([1]). *Let $(X,d)$ be a complex valued metric space, $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

## 2. Main Results

Throughout this paper, we use the following notation,

$$\mathbb{C}_+ := \{x \in \mathbb{C} : x \succcurlyeq 0\}$$

and $\Gamma := \{\gamma : \mathbb{C}_+ \to [0,1) : \{x_n\} \subseteq \mathbb{C}_+ \text{ with } \gamma(x_n) \to 1 \Rightarrow |x_n| \to 0\}$.

This class was first introduced by Sintunavarat *et al*. [9] which is extension of the class of Geraghty's real valued mappings.

**Theorem 2.1.** *Let $(X,d)$ be a $C$-complete complex valued metric space and $f,g,S,T : X \to X$ be four self mappings. If there exists three mappings $\lambda_1,\lambda_2,\lambda_3 : \mathbb{C}_+ \to [0,1)$ such that the following conditions hold:*

(i) $\lambda_1(x)+\lambda_2(x)+\lambda_3(x) < 1$ *for all $x \in \mathbb{C}_+$ and the mapping $\gamma : \mathbb{C}_+ \to [0,1)$, which is defined by*

$$\gamma(x) := \frac{\lambda_1(x)}{1-[\lambda_2(x)+\lambda_3(x)]} \text{ belongs to } \Gamma.$$

(ii) *for each $x,y \in X$, we have*

$$d(Sx,Ty) \preccurlyeq \lambda_1(d(fx,gy))\frac{d(fx,Sx)d(fx,Ty)+d(gy,Ty)d(gy,Sx)}{d(fx,Ty)+d(gy,Sx)} + \lambda_2(d(fx,gy))d(gy,Ty)+\lambda_3(d(fx,gy))d(fx,gy).$$

*If $S(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$ and the pairs $(f,S)$ and $(g,T)$ are weakly compatible, then $f,g,S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Since $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$, we construct the two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Sx_{2n-2} = gx_{2n-1} = y_{2n-1}$ and

$$Tx_{2n-1} = fx_{2n} = y_{2n}, \quad \text{for all } n \geq 0. \tag{2.1}$$

For $n \geq 0$, we get

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &= d(Sx_{2n},Tx_{2n+1}) \\
&\preccurlyeq \lambda_1(d(fx_{2n},gx_{2n+1}))\frac{\begin{pmatrix} d(fx_{2n},Sx_{2n})d(fx_{2n},Tx_{2n+1}) \\ +d(gx_{2n+1},Tx_{2n+1})d(gx_{2n+1},Sx_{2n}) \end{pmatrix}}{d(fx_{2n},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n})} \\
&\quad + \lambda_2(d(fx_{2n},gx_{2n+1}))d(gx_{2n+1},Tx_{2n+1})+\lambda_3(d(fx_{2n},gx_{2n+1}))d(fx_{2n},gx_{2n+1}) \\
&= \lambda_1(d(y_{2n},y_{2n+1}))\frac{d(y_{2n},y_{2n+1})d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+2})d(y_{2n+1},y_{2n+1})}{d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})} \\
&\quad + \lambda_2(d(y_{2n},y_{2n+1}))d(y_{2n+1},y_{2n+2})+\lambda_3(d(y_{2n},y_{2n+1}))d(y_{2n},y_{2n+1}),
\end{aligned}$$

$$(1-\lambda_2(d(y_{2n},y_{2n+1})))d(y_{2n+1},y_{2n+2}) \preccurlyeq ((\lambda_1+\lambda_3)d(y_{2n},y_{2n+1}))d(y_{2n},y_{2n+1})$$

implies

$$|d(y_{2n+1},y_{2n+2})| \leq \frac{\lambda_1(d(y_{2n},y_{2n+1}))}{1-\lambda_2(d(y_{2n},y_{2n+1}))}|d(y_{2n},y_{2n+1})| + \frac{\lambda_3(d(y_{2n},y_{2n+1}))}{1-\lambda_2(d(y_{2n},y_{2n+1}))}|d(y_{2n},y_{2n+1})|$$

for all $n \in \mathbb{N}$. Applying condition (i) of Theorem 2.1, we get

$$|d(y_{2n+1},y_{2n+2})| \leq \gamma(d(y_{2n},y_{2n+1}))|d(y_{2n},y_{2n+1})|$$

for all $n \in \mathbb{N}$. Similarly, we obtain that

$$|d(y_{2n}, y_{2n+1})| \leq \gamma(d(y_{2n-1}, y_{2n}))|d(y_{2n-1}, y_{2n})|.$$

for all $n \in \mathbb{N}$. Consequently,

$$\begin{aligned} |d(y_n, y_{n+1})| &\leq \gamma(d(y_{n-1}, y_n))|d(y_{n-1}, y_n)| \\ &\leq |d(y_{n-1}, y_n)|, \quad \text{for all } n \in \mathbb{N}\backslash\{1\}. \end{aligned} \tag{2.2}$$

Thus the sequence $\{|d(y_n, y_{n+1})|\} n \in \mathbb{N}\backslash\{1\}$ is monotone non-increasing and bounded below. Therefore, $|d(y_n, y_{n+1})| \to l$ for some $l \geq 0$. Now, we claim that $l = 0$. On contrary assume that $l > 0$. Then taking limit as $n \to \infty$ in (2.2), we have

$$l \leq \lim_{n\to\infty} \gamma(d(y_{n-1}, y_n)) \leq 1 \;\Rightarrow\; \lim_{n\to\infty} \gamma(d(y_{n-1}, y_n)) = 1.$$

But $\gamma \in \Gamma$, so we can write $|d(y_{n-1}, y_n)| \to 0$, which is contradiction to the fact that $l > 0$. Thus $l = 0$ and hence

$$\lim_{n\to\infty} |d(y_{n-1}, y_n)| = 0. \tag{2.3}$$

Next, to show that $\{y_n\}$ is $C$-Cauchy sequence, it is enough to show that $\{y_{2n}\}$ is a $C$-Cauchy sequence. On contrary, suppose that $\{y_{2n}\}$ is not a $C$-Cauchy sequence. Then there exist $c \in \mathbb{C}$ with $0 \prec c$ for which, for all $k \in \mathbb{N}$ there exists $2m_k > 2n_k \geq k$ such that

$$d(y_{2n_k}, y_{2m_k}) \succcurlyeq c\,. \tag{2.4}$$

Now, corresponding to $n_k$, we can choose $m_k$ in such a way that it is the smallest integer with $2m_k > 2n_k \geq k$ satisfying (2.4). Then

$$d(y_{2n_k}, y_{2m_k-2}) \prec c. \tag{2.5}$$

From equation (2.4), (2.5) and triangular inequality, we have

$$\begin{aligned} c &\preccurlyeq d(y_{2n_k}, y_{2m_k}) \\ &\preccurlyeq d(y_{2n_k}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k}) \\ &\prec c + d(y_{2m_k-2}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k}) \end{aligned}$$

which implies that,

$$|c| \leq |d(y_{2n_k}, y_{2m_k})| \leq |c| + |d(y_{2m_k-2}, y_{2m_k-1})| + |d(y_{2m_k-1}, y_{2m_k})|.$$

Taking limit as $k \to \infty$ and using (2.3), we have

$$\begin{aligned} & |c| \leq \lim_{k\to\infty} |d(y_{2n_k}, y_{2m_k})| \leq |c| \\ \Rightarrow \quad & \lim_{k\to\infty} |d(y_{2n_k}, y_{2m_k})| = |c|. \end{aligned} \tag{2.6}$$

Now, using triangular inequality, we have

$$\begin{aligned} |d(y_{2n_k}, y_{2m_k})| &\leq |d(y_{2n_k}, y_{2m_k+1})| + |d(y_{2m_k+1}, y_{2m_k})| \\ &\leq |d(y_{2n_k}, y_{2m_k})| + |d(y_{2m_k}, y_{2m_k+1})| + |d(y_{2m_k+1}, y_{2m_k})|. \end{aligned}$$

Taking limit as $k \to \infty$ and using (2.3) and (2.6), we get

$$\lim_{k\to\infty} |d(y_{2n_k}, y_{2m_k+1})| = |c|. \tag{2.7}$$

Next, we have

$$d(y_{2n_k}, y_{2m_k+1}) \precsim d(y_{2n_k}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2m_k+2}) + d(y_{2m_k+2}, y_{2m_k+1})$$
$$= d(y_{2n_k}, y_{2n_k+1}) + d(Sx_{2n_k}, Tx_{2m_k+1}) + d(y_{2m_k+2}, y_{2m_k+1}).$$

On using condition (ii) of Theorem 2.1 with $x = x_{2n_k}$ and $y = x_{2m_k+1}$, one can write

$$\begin{aligned} d(y_{2n_k}, y_{2m_k+1}) &\precsim d(y_{2n_k}, y_{2n_k+1}) \\ &\quad + \lambda_1(d(fx_{2n_k}, gx_{2m_k+1}))\left[\frac{\begin{pmatrix} d(fx_{2n_k}, Sx_{2n_k})d(fx_{2n_k}, Tx_{2m_k+1}) \\ +d(gx_{2m_k+1}, Tx_{2m_k+1})d(gx_{2m_k+1}, Sx_{2n_k}) \end{pmatrix}}{d(fx_{2n_k}, Tx_{2m_k+1}) + d(gx_{2m_k+1}, Sx_{2n_k})}\right] \\ &\quad + \lambda_2(d(fx_{2n_k}, gx_{2m_k+1}))d(gx_{2m_k+1}, Tx_{2m_k+1}) \\ &\quad + \lambda_3(d(fx_{2n_k}, gx_{2m_k+1}))d(fx_{2n_k}, gx_{2m_k+1}) + d(y_{2m_k+2}, y_{2m_k+1}). \end{aligned}$$

By using (i), we get

$$\begin{aligned} |d(y_{2n_k}, y_{2m_k+1})| &\le |d(y_{2n_k}, y_{2n_k+1})| \\ &\quad + \lambda_1(d(y_{2n_k}, y_{2m_k+1}))\left|\frac{\begin{pmatrix} d(y_{2n_k}, y_{2n_k+1})d(y_{2n_k}, y_{2m_k+2}) \\ +d(y_{2m_k+1}, y_{2m_k+2})d(y_{2m_k+1}, y_{2n_k+1}) \end{pmatrix}}{d(y_{2n_k}, y_{2m_k+2}) + d(y_{2m_k+1}, y_{2n_k+1})}\right| \\ &\quad + \lambda_2(d(y_{2n_k}, y_{2m_k+1}))|d(y_{2m_k+1}, y_{2m_k+2})| \\ &\quad + \lambda_3(d(y_{2n_k}, y_{2m_k+1}))|d(y_{2n_k}, y_{2m_k+1})| + |d(y_{2m_k+2}, y_{2m_k+1})|. \end{aligned}$$

In the sense of condition (i) of Theorem 2.1, we get

$$\begin{aligned} |d(y_{2n_k}, y_{2m_k+1})| &\le |d(y_{2n_k}, y_{2n_k+1})| \\ &\quad + \gamma(d(y_{2n_k}, y_{2m_k+1}))\left|\frac{\begin{pmatrix} d(y_{2n_k}, y_{2n_k+1})d(y_{2n_k}, y_{2m_k+2}) \\ +d(y_{2m_k+1}, y_{2m_k+2})d(y_{2m_k+1}, y_{2n_k+1}) \end{pmatrix}}{d(y_{2n_k}, y_{2m_k+2}) + d(y_{2m_k+1}, y_{2n_k+1})}\right| \\ &\quad + |d(y_{2m_k+1}, y_{2m_k+2})| + |d(y_{2n_k}, y_{2m_k+1})| + |d(y_{2m_k+2}, y_{2m_k+1})| \\ &\le |d(y_{2n_k}, y_{2n_k+1})| \\ &\quad + \left|\frac{d(y_{2n_k}, y_{2n_k+1})d(y_{2n_k}, y_{2m_k+2}) + d(y_{2m_k+1}, y_{2m_k+2})d(y_{2m_k+1}, y_{2n_k+1})}{d(y_{2n_k}, y_{2m_k+2}) + d(y_{2m_k+1}, y_{2n_k+1})}\right| \\ &\quad + |d(y_{2m_k+1}, y_{2m_k+2})| + |d(y_{2n_k}, y_{2m_k+1})| + |d(y_{2m_k+2}, y_{2m_k+1})|. \end{aligned}$$

Taking limit as $k \to \infty$ and using (2.3), (2.7), we get

$$|c| \le \lim_{k\to\infty} \gamma(d(y_{2n_k}, y_{2m_k+1}))|c| \le |c|$$
$$\Rightarrow \quad \lim_{k\to\infty} \gamma(d(y_{2n_k}, y_{2m_k+1})) = 1.$$

Since $\gamma \in \Gamma$, we obtain that $|d(y_{2n_k}, y_{2m_k+1})| \to 0$ as $k \to \infty$, which is a contradiction. Thus $\{y_{2n}\}$ is a $C$-Cauchy sequence and hence $\{y_n\}$ is a $C$-Cauchy sequence. As $X$ is $C$-complete, therefore there exists $t \in X$ such that $y_n \to t$ as $n \to \infty$.

Therefore, from equation (2.1) we get

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = t. \tag{2.8}$$

Next, since $S(x) \subseteq g(x)$, there exist $u \in X$ such that $g(u) = t$.

Thus equation (2.8) becomes

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = t = g(u). \tag{2.9}$$

We will show that $Tu = gu$, for this consider

$$\begin{aligned} d(t,Tu) &\preceq d(t,Sx_{2n}) + d(Sx_{2n},Tu) \\ &\preceq d(t,Sx_{2n}) + \lambda_1(d(fx_{2n},gu))\frac{d(fx_{2n},Sx_{2n})d(fx_{2n},Tu) + d(gu,Tu)d(gu,Sx_{2n})}{d(fx_{2n},Tu) + d(gu,Sx_{2n})} \\ &\quad + \lambda_2(d(fx_{2n},gu))d(gu,Tu) + \lambda_3(d(fx_{2n},gu))d(fx_{2n},gu). \end{aligned}$$

In the light of condition (i) of Theorem 2.1, we get

$$d(Tu,t) \preceq d(t,Sx_{2n}) + \frac{d(fx_{2n},Sx_{2n})d(fx_{2n},Tu) + d(gu,Tu)d(gu,Sx_{2n})}{d(fx_{2n},Tu) + d(gu,Sx_{2n})} + d(gu,Tu) + d(fx_{2n},gu).$$

Taking limit as $n \to \infty$ and using (2.9), we get $d(Tu,t) \preceq 0$, which is possible if $d(Tu,t) = 0$. Thus $Tu = t$ and hence from (2.8), we obtain that

$$Tu = gu = t. \tag{2.10}$$

Also, it is given that $T(x) \subseteq f(x)$, therefore there exist $v \in X$ such that $f(v) = t$. Thus from equation (2.8), we get

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = t = f(v). \tag{2.11}$$

Now, we will show that $Sv = fv$, for this, consider

$$d(Sv,t) \preceq d(Sv,Tx_{2n+1}) + d(Tx_{2n+1},t).$$

Setting $x = v$, $y = x_{2n+1}$ in condition (ii) of Theorem 2.1 and proceeding the same way as above, one can get $d(Sv,t) \preceq 0$.

It is possible if $d(Sv,t) = 0 \to Sv = t$ and hence from (2.11), we have

$$Sv = fv = t. \tag{2.12}$$

Therefore, from (2.10) and (2.12), we get

$$Tu = gu = Sv = fv = t. \tag{2.13}$$

Hence the pairs $(f,S)$ and $(g,T)$ are weakly compatible. Therefore from equation (2.13), we have

$$Sv = fv \Rightarrow fSv = Sfv \Rightarrow ft = St \tag{2.14}$$

and

$$Tu = gu \Rightarrow gTu = Tgu \Rightarrow gt = Tt \tag{2.15}$$

which implies that $t$ is a coincidence point of each pair $(f,S)$ and $(g,T)$ in $X$.

Next, we will show that $t$ is common fixed point of $f,g,S$ and $T$. For this assume that $St = t$. If not, then on using condition (ii) of Theorem 2.1 with $x = t$ and $y = u$, we have

$$\begin{aligned} d(St,Tu) &\preceq \lambda_1(d(ft,gu))\frac{d(ft,St)d(ft,Tu) + d(gu,Tu)d(gu,St)}{d(ft,Tu) + d(gu,St)} \\ &\quad + \lambda_2(d(ft,gu))d(gu,Tu) + \lambda_3(d(ft,gu))d(ft,gu). \end{aligned}$$

Using equation (2.13) and (2.14), we have

$$d(St,t) \precsim \lambda_1(d(St,t))\frac{d(St,St)d(St,t)+d(t,t)d(t,St)}{d(St,t)+d(t,St)} + \lambda_2(d(St,t))d(t,t)+\lambda_3(d(St,t))d(St,t)$$
$$\precsim \lambda_3(d(St,t))d(St,t)$$

implies

$$(1-\lambda_3(d(St,t))d(St,t) \precsim 0$$

therefore, $(1-\lambda_3(d(St,t))|d(St,t)| \le 0$.

Hence $d(St,t)=0$ i.e. $St=t$, therefore from equation (2.14), we get

$$ft = St = t. \tag{2.16}$$

Similarly assume that $Tt=t$, if not then using condition (ii) of Theorem 2.1 with $x=v$ and $y=t$, we have $Tt=t$, therefore from equation (2.15) we get

$$gt = Tt = t. \tag{2.17}$$

From equations (2.16) and (2.17), we have $ft=gt=St=Tt=t$.

Thus $t$ is a common fixed point of $f,g,S$ and $T$. To check uniqueness, assume that $t^* \neq t$ be another fixed point of $f,g,S$ and $T$. Let $x=t$ and $y=t^*$ in condition (ii) of Theorem 2.1, we get

$$\begin{aligned} d(t,t^*) &= d(St,Tt^*) \\ &\precsim \lambda_1(d(ft,gt^*))\frac{d(ft,St)d(ft,Tt^*)+d(gt^*,Tt^*)d(gt^*,St)}{d(ft,Tt^*)+d(gt^*,St)} \\ &\quad + \lambda_2(d(ft,gt^*))d(gt^*,Tt^*)+\lambda_3(d(ft,gt^*))d(ft,gt^*) \\ &= \lambda_1(d(t,t^*))\frac{d(t,t)d(t,t^*)+d(t^*,t^*)d(t^*,t)}{d(t,t^*)+d(t^*,t)}+\lambda_2(d(t,t^*))d(t^*,t^*)+\lambda_3(d(t,t^*))d(t,t^*) \\ &\precsim \lambda_3(d(t,t^*))d(t,t^*), \end{aligned}$$

implies

$$(1-\lambda_3(d(t,t^*))d(t,t^*) \precsim 0$$

therefore

$$(1-\lambda_3(d(t,t^*))|d(t,t^*)| \le 0$$

which implies that $d(t,t^*)=0$. Thus $t^*=t$ and so $t$ is a unique common fixed point of $f$, $g$, $S$ and $T$. □

The following corollaries are obtained from Theorem 2.1.

**Corollary 2.2.** *Let $(X,d)$ be a $C$-complete complex valued metric space and $f,g,S,T : X \to X$ be four mappings satisfying*

$$d(Sx,Ty) \precsim \lambda_1 \frac{d(fx,Sx)d(fx,Ty)+d(gy,Ty)d(gy,Sx)}{d(fx,Ty)+d(gy,Sx)} + \lambda_2 d(gy,Ty)+\lambda_3 d(fx,gy)$$

*for all $x,y \in X$, where $\lambda_1+\lambda_2+\lambda_3 \in \mathbb{R}_+$ with $\lambda_1+\lambda_2+\lambda_3 < 1$.*
*If $S(x) \subseteq g(x)$ and $T(x) \subseteq f(x)$ and the pairs $(f,S)$ and $(g,T)$ are weakly compatible then $f,g,S$ and $T$ have a unique common fixed point in $X$.*

**Corollary 2.3.** *Let $S,T$ be two self-mappings on $C$-complete complex valued metric space $(X,d)$ and $\lambda_1,\lambda_2,\lambda_3:\mathbb{C}_+\to[0,1)$ be given mappings. Suppose that the following conditions hold:*

(i) $\lambda_1(x)+\lambda_2(x)+\lambda_3(x)<1$ *for all* $x\in\mathbb{C}_+$ *and the mapping* $\gamma:\mathbb{C}_+\to[0,1)$, *which is defined by*
$$\gamma(x):=\frac{\lambda_1(x)}{1-[\lambda_2(x)+\lambda_3(x)]} \quad \text{for all } x\in\mathbb{C}_+ \text{ belongs to } \Gamma.$$

(ii) *for each* $x,y\in X$, *we have*
$$d(Sx,Ty)\precsim\lambda_1(d(x,y))\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)} + \lambda_2(d(x,y))d(y,Ty)+\lambda_3(d(x,y))d(x,y).$$

*If the pair $(S,T)$ is weakly compatible, then $S$ and $T$ have a unique common fixed point in $X$.*

**Corollary 2.4.** *Let $S$ be a self-mapping on a $C$-complete complex valued metric space $(X,d)$ and $\lambda_1,\lambda_2,\lambda_3:\mathbb{C}_+\to[0,1)$ be given mappings. Suppose that the following conditions hold:*

(i) $\lambda_1(x)+\lambda_2(x)+\lambda_3(x)<1$ *for all* $x\in\mathbb{C}_+$ *and the mapping* $\gamma:\mathbb{C}_+\to[0,1)$, *which is defined by*
$$\gamma(x):=\frac{\lambda_1(x)}{1-[\lambda_2(x)+\lambda_3(x)]} \quad \text{for all } x\in\mathbb{C}_+ \text{ belongs to } \Gamma.$$

(ii) *for each* $x,y\in X$, *we have*
$$d(Sx,Sy)\precsim\lambda_1(d(x,y))\frac{d(x,Sx)d(x,Sy)+d(y,Sy)d(y,Sx)}{d(x,Sy)+d(y,Sx)} + \lambda_2(d(x,y))d(y,Sy)+\lambda_3(d(x,y))d(x,y).$$

*Then $S$ has a unique fixed point in $X$.*

**Corollary 2.5.** *Let $P$ be a self-mapping on $C$-complete complex valued metric space $(X,d)$ and $\lambda_1,\lambda_2,\lambda_3:\mathbb{C}_+\to[0,1)$ be given mappings. Suppose that the following conditions hold:*

(i) $\lambda_1(x)+\lambda_2(x)+\lambda_3(x)<1$ *for all* $x\in\mathbb{C}_+$ *and the mapping* $\gamma:\mathbb{C}_+\to[0,1)$, *which is defined by*
$$\gamma(x):=\frac{\lambda_1(x)}{1-[\lambda_2(x)+\lambda_3(x)]} \quad \text{for all } x\in\mathbb{C}_+ \text{ belongs to } \Gamma.$$

(ii) *for each* $x,y\in X$, *we have*
$$d(P^nx,P^ny)\precsim\lambda_1(d(x,y))\frac{d(x,P^nx)d(x,P^ny)+d(y,P^ny)d(y,P^nx)}{d(x,P^ny)+d(y,P^nx)} + \lambda_2(d(x,y))d(y,P^ny)+\lambda_3(d(x,y))d(x,y)$$

*for some* $n\in\mathbb{N}$.

*Then $P$ has a unique fixed point in $X$.*

## 3. Conclusion

In this paper, we generalize and extend the common fixed point theorem in $C$-complete complex valued metric spaces of Kumar *et al*. [4] and Sintunavarat *et al*. [10]. The future scope of our results, to obtain the existence and uniqueness of a common solution of the system of Urysohn integral equations. The integral equation plays very significant and important role in mathematical analysis and has various applications in real world problems.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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