Communications in Mathematics and Applications
Vol. 12, No. 1, pp. 29–50, 2021
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v12i1.1401

Research Article

Convergence and Stability of a Perturbed Mann Iterative Algorithm with Errors for a System of Generalized Variational-Like Inclusion Problems in *q*-uniformly smooth Banach Spaces

Jong Kyu Kim*[1], Mohammad Iqbal Bhat[2] and Sumeera Shafi[3]

[1] *Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 51767, Korea*
[2] *Department of Mathematics, South Campus University of Kashmir, Anantnag 192101, India*
[3] *Department of Mathematics, University of Kashmir, Srinagar 190006, India*
***Corresponding author:** jongkyuk@kyungnam.ac.kr

Abstract. In this paper, we introduce a class of $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operators in a real q-uniformly smooth Banach space. We define the resolvent operator associated with $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator and prove that it is single-valued and Lipschitz continuous. Moreover, we propose a perturbed Mann iterative method with errors for approximating the solution of the system of generalized variational-like inclusion problems and discuss the convergence and stability of the iterative sequences generated by the algorithm. Our results presented in this paper generalize and unify many known results in the literature.

Keywords. System of generalized variational-like inclusion problem; $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator; q-uniformly smooth Banach spaces; Resolvent operator technique; Perturbed Mann iterative method with errors; Convergence analysis; Stability analysis

MSC. 47H04; 47H05; 47H06; 47H10; 49J40

Received: April 17, 2020 **Accepted:** December 6, 2020

1. Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. For more details, see for example [6–10, 12, 13, 17, 19, 24] and the references there in. Among the methods for solving variational inclusion problems, resolvent operator technique has been widely used. In particular, the applications of the resolvent operator technique have been explored and improved recently. For instance, Fang and Huang [7] introduced a class of H-monotone operators and defined the associated class of resolvent operators, which extends the classes of the resolvent operators associated with η-subdifferential operators of Ding and Lou [6], maximal monotone operators of Liu, Agarwal and Kang [19] and maximal monotone operators of Huang and Fang [12], respectively.

In 2001, Huang and Fang [11] introduced the generalized m-accretive mapping and give the definition of resolvent operator for the generalized m-accretive mapping in Banach spaces. Since then a number of researchers investigated several classes of generalized m-accretive mappings such as H-accretive, $H(\cdot,\cdot)$-accretive and (H,η)-accretive, (A,η)-accretive mappings, see for example [4–6, 8, 11, 14–16, 26].

Motivated and inspired by the work going on in this direction, in this paper, we introduce and study a new class of variational inclusions called system of generalized variational-like inclusion problem involving $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator in real q-uniformly smooth Banach spaces. We define the resolvent operator associated with $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator and prove its single-valuedness and Lipschitz continuity. By using resolvent operator technique, we prove the existence of solution for this system of inclusions. Further, we suggest a perturbed Mann iterative scheme with errors for approximating the solution of this system of generalized variational-like inclusion problem. Furthermore, we discuss the convergence and stability analysis of the iterative sequence generated by the iterative algorithm.

2. Resolvent Operator and Formulation of Problem

We need the following definitions and results from the literature.

Let X be a real Banach space equipped with norm $\|\cdot\|$ and $X^\star$ be the topological dual space of X. Let $\langle\cdot,\cdot\rangle$ be the dual pair between X and $X^\star$, and 2^X be the power set of X.

Definition 2.1 ([25]). For $q>1$, the generalized duality mapping $J_q : X \to 2^{X^\star}$ is defined by

$$J_q(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^q, \|x\|^{q-1} = \|f\|\}, \quad \forall\, x \in X.$$

In particular, J_2 is the usual normalized duality mapping on X. It is well known (see, e.g., [25]) that

$$J_q(x) = \|x\|^{q-2} J_2(x), \quad \forall\, x(\neq 0) \in X.$$

Note that if $X \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2 ([25]). A Banach space X is said to be *smooth* if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.
The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$, defined by

$$\rho_X(\sigma) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \sigma\right\}.$$

Definition 2.3 ([25])**.** A Banach space X is said to be

(i) uniformly smooth if $\lim\limits_{\sigma\to 0}\frac{\rho_X(\sigma)}{\sigma}=0$,

(ii) q-uniformly smooth, for $q>1$, if there exists a constant $c>0$ such that $\rho_X(\sigma)\le c\sigma^q$, $\sigma\in[0,\infty)$.

It is well known (see, e.g., [26]) that

$$L_q \text{ (or } l_q) \text{ is } \begin{cases} q\text{-uniformly smooth}, & \text{if } 1<q\le 2,\\ 2\text{-uniformly smooth}, & \text{if } q\ge 2.\end{cases}$$

Note that if X is uniformly smooth, J_q becomes single-valued. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [25] established the following lemma.

Lemma 2.4. *Let $q>1$ be a real number and let X be a smooth Banach space. Then the following statements are equivalent:*

(i) *X is q-uniformly smooth.*

(ii) *There is a constant $c_q>0$ such that for every $x,y\in X$, the following inequality holds:*

$$|x+y\|^q\le\|x\|^q+q\langle y,J_q(x)\rangle+c_q\|y\|^q.$$

Definition 2.5. Let X be a q-*uniformly smooth* Banach space. Let $A,B:X\to X$, $\mathcal{H},\eta:X\times X\to X$ be single-valued mappings and $M:X\times X\to 2^X$ be a multi-valued mapping. Then

(i) A is said to be η-*accretive*, if

$$\langle Ax-Ay,J_q(\eta(x,y))\rangle\ge 0,\quad \forall\ x,y\in X.$$

(ii) A is said to be *strictly* η-*accretive*, if A is η-*accretive* and equality holds if and only if $x=y$.

(iii) $\mathcal{H}(A,\cdot)$ is said to be α-*strongly* η-*accretive* with respect to A if there exists a constant $\alpha>0$ such that

$$\langle\mathcal{H}(Ax,z)-\mathcal{H}(Ay,z),J_q(\eta(x,y))\rangle\ge\alpha\|x-y\|^q,\quad\forall\ x,y,z\in X.$$

(iv) $\mathcal{H}(\cdot,B)$ is said to be β-*relaxed* η-*accretive* with respect to B if there exists a constant $\beta>0$ such that

$$\langle\mathcal{H}(z,Bx)-\mathcal{H}(z,By),J_q(\eta(x,y))\rangle\ge-\beta\|x-y\|^q,\quad\forall\ x,y,z\in X.$$

(v) $\mathcal{H}(\cdot,\cdot)$ is said to be d_1-Lipschitz continuous with respect to A if there exists a constant $d_1>0$ such that

$$\|\mathcal{H}(Ax,z)-\mathcal{H}(Ay,z)\|\le d_1\|x-y\|,\quad\forall\ x,y,z\in X.$$

In a similar way, we can define the Lipschitz continuity of the mapping $\mathcal{H}(\cdot,\cdot)$ with respect to second argument.

(vi) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau>0$ such that

$$\|\eta(x,y)\|\le\tau\|x-y\|,\quad\forall\ x,y\in X.$$

(vii) M is said to be η-*accretive* if

$$\langle u-v,J_q(\eta(x,y))\rangle\ge 0,\quad\forall\ x,y\in X,\ \forall u\in M(x,z),v\in M(y,z),$$

for each fixed $z\in X$.

(viii) M is said to be *strictly η-accretive*, if M is *η-accretive* and equality holds if and only if $x = y$.

Definition 2.6. $\mathcal{H}(\cdot,\cdot)$ is said to be $\alpha\beta$-symmetric *η-accretive* with respect to A and B, if $\mathcal{H}(A,\cdot)$ is *α-strongly η-accretive* with respect to A and $\mathcal{H}(\cdot,B)$ is *β-relaxed η-accretive* with respect to B with $\alpha \ge \beta$, and $\alpha = \beta$ if and only if $x = y$ for all $x, y \in X$.

Definition 2.7. Let X be a q-uniformly smooth Banach space. Let $A : X \to X$, $\mathcal{H} : X \times X \to X$ be single-valued mappings. Then

(i) A is said to be δ-strongly η-accretive if there exists a constant $\delta > 0$ such that
$$\langle Ax - Ay, J_q(\eta(x,y))\rangle \ge \delta\|x-y\|^q, \quad \forall\, x,y \in X.$$

(ii) A is said to be λ-Lipschitz continuous if there exists a constant $\lambda > 0$ such that
$$\|Ax - Ay\| \le \lambda\|x-y\|, \quad \forall\, x,y \in X.$$

(iii) $\mathcal{H}(\cdot,\cdot)$ is said to be (ν,ξ)-mixed Lipschitz continuous if there exist constants $\nu > 0$, $\xi > 0$ such that
$$\|\mathcal{H}(x,s) - \mathcal{H}(y,t)\| \le \nu\|x-y\| + \xi\|z-t\|, \quad \forall\, x,y,s,t \in X.$$

Throughout the rest of the paper unless otherwise stated, we assume that X is a q-uniformly smooth Banach space.

Definition 2.8. Let $\phi, A, B : X \to X, \mathcal{H}, \eta : X \times X \to X$ be single-valued mappings, $M : X \times X \to 2^X$ be a multi-valued mapping. Then M is said to be a *$\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive* mapping with respect to A and B if for each fixed $x^\star \in X, \phi \circ M(x^\star,\cdot)$ is *η-accretive* in the second argument and $(\mathcal{H}(A,B) + \lambda\phi \circ M(x^\star,\cdot))X = X$, for all $\lambda > 0$.

Remark 2.9. If $\phi(x) = x$, for all $x \in X$, $M(\cdot,\cdot) = M(\cdot)$ and $\eta(x,y) = x - y$, then *$\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive* operator reduces to *$\mathcal{H}(\cdot,\cdot)$-accretive* operator, which was introduced and studied by Zou and Huang [27].

Example 2.10. Let $X = R$. Let $Az = 0$, $Bz = \sin z$, $\mathcal{H}(Az,Bz) = Az + Bz$ and $M(x^\star,z) = {x^\star}^2 + z^2$, for all $z \in X$ and for each fixed $x^\star \in X$. Let $\phi \circ M(x^\star,z) = \frac{\partial}{\partial z}[M(x^\star,z)] = 2z$ and $\eta(z_1,z_2) = \frac{z_1 - z_2}{2}$, for all $z_1, z_2 \in X$. Then
$$\begin{aligned}\langle \phi \circ M(x^\star,z_1) - \phi \circ M(x^\star,z_2), \eta(z_1,z_2)\rangle &= \left\langle 2z_1 - 2z_2, \frac{z_1 - z_2}{2}\right\rangle \\ &= (z_1 - z_2)^2 \ge 0,\end{aligned}$$
that is,
$$\langle \phi \circ M(x^\star,z_1) - \phi \circ M(x^\star,z_2), \eta(z_1,z_2)\rangle \ge 0,$$
which means that $\phi \circ M(x^\star,\cdot)$ is η-accretive in the second argument.
Also, for any $z \in X$, it follows from above that
$$\begin{aligned}(\mathcal{H}(A,B) + \lambda\phi \circ M(x^\star,\cdot))(z) &= \mathcal{H}(Az,Bz) + \lambda\phi \circ M(x^\star,z) \\ &= Az + Bz + \lambda\phi \circ M(x^\star,z) \\ &= 0 + \sin z + 2\lambda z \\ &= 2\lambda z + \sin z,\end{aligned}$$

which means that $(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))$ is surjective. Thus M is an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings A and B.

Example 2.11. Let $X,A,B,\mathcal{H},\eta$ and M be same as in above example. Let for each fixed $x^\star\in X$, $\phi\circ M(x^\star,z)=e^{x^{\star 2}+z^2}$, for all $z\in X$. Then

$$\begin{aligned}(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))(z)&=\mathcal{H}(Az,Bz)+\lambda\phi\circ M(x^\star,z)\\&=Az+Bz+\lambda\phi\circ M(x^\star,z)\\&=0+\sin z+\lambda e^{x^{\star 2}+z^2},\end{aligned}$$

which shows $0\notin(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))(X)$, that is $(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))$ is not surjective, hence M is not an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings A and B.

Proposition 2.12. *Let $\phi,A,B:X\to X,\mathcal{H},\eta:X\times X\to X$ be single-valued mappings, $\mathcal{H}:X\times X\to X$ be an $\alpha\beta$-symmetric η-accretive mapping with respect to A and B ($\alpha>\beta$), and $M:X\times X\to 2^X$ be an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive mapping with respect to A and B. If the following inequality : $\langle u-v,J_q(\eta(x,y))\rangle\ge 0$, holds for all $(y,v)\in\mathrm{Graph}(\phi\circ M(x^\star,\cdot))$, then $(x,u)\in\mathrm{Graph}(\phi\circ M(x^\star,\cdot))$, where* $\mathrm{Graph}(\phi\circ M(x^\star,\cdot)):=\{(x,u)\in X\times X:u\in\phi\circ M(x^\star,x)\}$.

Proof. Suppose, on the contrary that there exists some $(x_0,u_0)\notin\mathrm{Graph}(\phi\circ M(x^\star,\cdot))$ such that

$$\langle u_0-v,J_q(\eta(x_0,y))\rangle\ge 0,\quad\forall\,(y,v)\in\mathrm{Graph}(\phi\circ M(x^\star,\cdot)).\tag{2.1}$$

Since M is an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator with respect to the mappings A and B, we have that $(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))(X)=X$ holds for every $\lambda>0$, and hence there exists $(x_1,u_1)\in\mathrm{Graph}(\phi\circ M(x^\star,\cdot))$ such that

$$\mathcal{H}(Ax_1,Bx_1)+\lambda u_1=\mathcal{H}(Ax_0,Bx_0)+\lambda u_0\in X.\tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}0&\le\lambda\langle u_0-u_1,J_q(\eta(x_0,x_1))\rangle\\&=-\langle\mathcal{H}(Ax_0,Bx_0)-\mathcal{H}(Ax_1,Bx_1),J_q(\eta(x_0,x_1)\rangle\\&\le-\langle\mathcal{H}(Ax_0,Bx_0)-\mathcal{H}(Ax_1,Bx_0),J_q(\eta(x_0,x_1))\rangle-\langle\mathcal{H}(Ax_1,Bx_0)-\mathcal{H}(Ax_1,Bx_1),J_q(\eta(x_0,x_1))\rangle\\&\le-(\alpha-\beta)\|x_0-x_1\|^q\le 0,\end{aligned}$$

which gives $x_1=x_0$, since $\alpha>\beta$. From (2.2), we have $u_1=u_0$. This is a contradiction. This completes the proof. □

Proposition 2.13. *Let $\phi,A,B:X\to X$, $\mathcal{H},\eta:X\times X\to X$ be single-valued mappings, $\mathcal{H}:X\times X\to X$ be an $\alpha\beta$-symmetric η-accretive mapping with respect to A and B ($\alpha>\beta$), and $M:X\times X\to 2^X$ be an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive mapping with respect to A and B. Then the mapping $(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))^{-1}$ is single-valued, for all $\lambda>0$.*

Proof. For any $z^\star\in X$, let $x,y\in(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))^{-1}(z^\star)$. It follows that

$$\frac{1}{\lambda}(z^\star-\mathcal{H}(A,B)(x))\in\phi\circ M(x^\star,x)$$

and

$$\frac{1}{\lambda}(z^\star-\mathcal{H}(A,B)(y))\in\phi\circ M(x^\star,y).$$

Since $\phi\circ M(x^\star,\cdot)$ is η-accretive in the second argument, $\mathcal{H}$ is $\alpha\beta$-symmetric η-accretive with respect to A and B, we have

$$\begin{aligned}0 &\le \left\langle \frac{1}{\lambda}(z^\star - \mathcal{H}(A,B)(x)) - \frac{1}{\lambda}(z^\star - \mathcal{H}(A,B)(y)), J_q(\eta(x,y))\right\rangle\\ &= -\frac{1}{\lambda}\langle \mathcal{H}(A,B)(x) - \mathcal{H}(A,B)(y), J_q(\eta(x,y))\rangle\\ &= -\frac{1}{\lambda}\langle \mathcal{H}(Ax,Bx) - \mathcal{H}(Ay,Bx), J_q(\eta(x,y))\rangle - \frac{1}{\lambda}\langle \mathcal{H}(Ay,Bx) - \mathcal{H}(Ay,By), J_q(\eta(x,y))\rangle\\ &\le -\frac{1}{\lambda}(\alpha-\beta)\|x-y\|^q,\end{aligned}$$

which implies that

$$\frac{1}{\lambda}(\alpha-\beta)\|x-y\|^q \le 0.$$

It follows from $\alpha > \beta$, that $x = y$ and so $(\mathcal{H}(A,B)+\lambda\phi\circ M(x^\star,\cdot))^{-1}$ is single-valued. This completes the proof. □

Definition 2.14. Let $\phi, A, B : X \to X$, $\mathcal{H}, \eta : X\times X \to X$ be single-valued mappings, $\mathcal{H} : X\times X \to X$ be an $\alpha\beta$-symmetric η-accretive mapping with respect to A and B ($\alpha > \beta$), and $M : X\times X \to 2^X$ be an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive mapping with respect to A and B. Then for each fixed $x^\star \in X$, the *resolvent* operator $R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda} : X \to X$ is defined by

$$R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x) = (\mathcal{H}(A,B) + \lambda\phi\circ M(x^\star,\cdot))^{-1}(x), \quad \forall\, x \in X.$$

Next, the following result gives the Lipschitz continuity of the resolvent operator.

Proposition 2.15. *Let $\phi, A, B : X \to X, \mathcal{H}, \eta : X\times X \to X$ be single-valued mappings, $\mathcal{H} : X\times X \to X$ be an $\alpha\beta$-symmetric η-accretive mapping with respect to A and B ($\alpha > \beta$), and $M : X\times X \to 2^X$ be an $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive mapping with respect to A and B. Let η be τ-Lipschitz continuous. Then for each fixed $x^\star \in X$, the resolvent operator $R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda} : X \to X$ is Lipschitz continuous with constant L, that is,*

$$\|R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x) - R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y)\| \le L\|x-y\|, \quad \forall x,y \in X,$$

where $L := \frac{\tau^{q-1}}{(\alpha-\beta)}$.

Proof. Let $x, y \in X$. Then by definition of resolvent operator, it follows that

$$R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x) = (\mathcal{H}(A,B) + \lambda\phi\circ M(x^\star,\cdot))^{-1}(x)$$

and

$$R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y) = (\mathcal{H}(A,B) + \lambda\phi\circ M(x^\star,\cdot))^{-1}(y).$$

and so, we have

$$\frac{1}{\lambda}\left(x - \mathcal{H}\left(A\left(R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x)\right), B\left(R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x)\right)\right)\right) \in \phi\circ M\left(x^\star, R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x)\right)$$

and

$$\frac{1}{\lambda}\left(y - \mathcal{H}\left(A\left(R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y)\right), B\left(R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y)\right)\right)\right) \in \phi\circ M\left(x^\star, R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y)\right).$$

For the sake of brevity, let $E(x) = R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x)$, $E(y) = R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y)$. Since $\phi\circ M(x^\star,\cdot)$ is an η-accretive operator in the second argument, we have

$$0 \le \left\langle \frac{1}{\lambda}(x-\mathcal{H}(A(E(x)),B(E(x)))) - \frac{1}{\lambda}(y-\mathcal{H}(A(E(y)),B(E(y)))), J_q\big(\eta\big(E(x),E(y)\big)\big)\right\rangle$$

which implies,

$$\big\langle x-y, J_q\big(\eta\big(E(x),E(y)\big)\big)\big\rangle \ge \big\langle \mathcal{H}\big(A(E(x)),B(E(x))\big)-\mathcal{H}\big(A(E(y)),B(E(y))\big), J_q\big(\eta\big(E(x),E(y)\big)\big)\big\rangle.$$

Since $\mathcal{H}$ is $\alpha\beta$-symmetric η-accretive with respect to A and B, we have

$$\begin{aligned}
\|x-y\|\,\big\|\eta\big(E(x),E(y)\big)\big\|^{q-1} &\ge \big\langle x-y, J_q\big(\eta\big(E(x),E(y)\big)\big)\big\rangle\\
&\ge \big\langle \mathcal{H}\big(A(E(x)),B(E(x))\big)-\mathcal{H}\big(A(E(y)),B(E(y))\big), J_q\big(\eta\big(E(x),E(y)\big)\big)\big\rangle\\
&= \big\langle \mathcal{H}\big(A(E(x)),B(E(x))\big)-\mathcal{H}\big(A(E(y)),B(E(x))\big), J_q\big(\eta\big(E(x),E(y)\big)\big)\big\rangle\\
&\quad + \big\langle \mathcal{H}\big(A(E(y)),B(E(x))\big)-\mathcal{H}\big(A(E(y)),B(E(y))\big), J_q\big(\eta\big(E(x),E(y)\big)\big)\big\rangle\\
&\ge \alpha\|E(x)-E(y)\|^q - \beta\|E(x)-E(y)\|^q\\
&\ge (\alpha-\beta)\,\|E(x)-E(y)\|^q,
\end{aligned}$$

and so

$$\|x-y\|\ \tau^{q-1}\ \|E(x)-E(y)\|^{q-1} \ge (\alpha-\beta)\,\|E(x)-E(y)\|^q.$$

This implies

$$\|E(x)-E(y)\| \le \frac{\tau^{q-1}}{(\alpha-\beta)}\|x-y\|$$

or

$$\big\|R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(x) - R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M(x^\star,\cdot),\lambda}(y)\big\| \le \frac{\tau^{q-1}}{(\alpha-\beta)}\|x-y\|.$$

This completes the proof. □

Definition 2.16. For $n\ge 0$, let $M^n, M : X\to 2^X$ be $\mathcal{H}^n(\cdot,\cdot)$-$\phi^n$-$\eta^n$, $\mathcal{H}(\cdot,\cdot)$-ϕ-η *accretive* mappings with respect to A and B, respectively. A sequence $\{\phi^n\circ M^n\}_{n\ge 0}$ is said to be graph convergent to $\phi\circ M$, denoted by $\phi^n\circ M^n \xrightarrow{\mathcal{G}} \phi\circ M$, if for each $(x,u)\in graph(\phi\circ M)$, there is a sequence $\{(x^n,u^n)\}_{n\ge 0}\subseteq graph(\phi^n\circ M^n)$ such that $x^n\to x$, $u^n\to u$ as $n\to\infty$.

Lemma 2.17. *For $n\ge 0$, let $\mathcal{H}^n : X\times X\to X$ be a (t^n,v^n)-mixed Lipschitz continuous and $\alpha^n\beta^n$-symmetric η^n-accretive mapping with respect to A and B, and $\mathcal{H} : X\times X\to X$ be a (t,v)-mixed Lipschitz continuous and $\alpha\beta$-symmetric η-accretive mapping with respect to A and B. Let η^n be τ^n-Lipschitz continuous and η be τ-Lipschitz continuous, $M^n : X\to 2^X$ be an $\mathcal{H}^n(\cdot,\cdot)$-$\phi^n$-$\eta^n$ accretive mapping with respect to A and B, and $M : X\to 2^X$ be a $\mathcal{H}(\cdot,\cdot)$-ϕ-η accretive mapping with respect to A and B. Assume that $\left\{\frac{(t^n+v^n)\ (\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right\}_{n\ge 0}$ and $\left\{\frac{(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right\}_{n\ge 0}$ are bounded and $\lim\limits_{n\to\infty}\mathcal{H}^n(A,B)(x) = \mathcal{H}(A,B)(x)$ for each $x\in X$. If $\phi^n\circ M^n \xrightarrow{\mathcal{G}} \phi\circ M$, then $\lim\limits_{n\to\infty} R^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}_{M^n,\lambda}(x) = R^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}_{M,\lambda}(x)$, for all $x\in X$ where $\alpha^n>\beta^n$.*

Proof. Since $(\mathcal{H}(A,B)+\lambda\phi\circ M)X = X$, for each $z^\star\in X$. Hence there exists $(x,u)\in graph(\phi\circ M)$ such that $z^\star = \mathcal{H}(A,B)(x)+\lambda u$. Further, $\phi^n\circ M^n \xrightarrow{\mathcal{G}} \phi\circ M$, it follows there exists a sequence $\{(x^n,u^n)\}_{n\ge 0}\subseteq graph(\phi^n\circ M^n)$ such that $x^n\to x$, $u^n\to u$ as $n\to\infty$. Put $z^{\star n} = \mathcal{H}^n(A,B)(x^n)+\lambda u^n$

and that

$$R_{M,\lambda}^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}(\mathcal{H}(A,B)(x)+\lambda u)=x \quad\text{and}\quad R_{M^n,\lambda}^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}(\mathcal{H}^n(A,B)(x^n)+\lambda u^n)=x^n.$$

Using Lipschitz continuity of $R_{M,\lambda}^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}$, we have

$$\begin{aligned}
&\|R_{M^n,\lambda}^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}(z^\star)-R_{M,\lambda}^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}(z^\star)\| \\
&\le \|R_{M^n,\lambda}^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}(z^{\star n})-R_{M,\lambda}^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}(z^\star)\|+\|R_{M^n,\lambda}^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}(z^{\star n})-R_{M^n,\lambda}^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}(z^\star)\| \\
&\le \|R_{M^n,\lambda}^{\mathcal{H}^n(\cdot,\cdot)\text{-}\phi^n\text{-}\eta^n}(\mathcal{H}^n(A,B)(x^n)+\lambda u^n)-R_{M,\lambda}^{\mathcal{H}(\cdot,\cdot)\text{-}\phi\text{-}\eta}(\mathcal{H}(A,B)(x)+\lambda u)\|+\left\{\frac{(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right\}\|z^{\star n}-z^\star\| \\
&\le \|x^n-x\|+\left\{\frac{(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right\}\Big[\|\mathcal{H}^n(A,B)(x^n)-\mathcal{H}(A,B)(x)\|+\lambda\|u^n-u\|\Big] \\
&\le \|x^n-x\|+\left\{\frac{(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right\}\Big[\|\mathcal{H}^n(A,B)(x^n)-\mathcal{H}^n(A,B)(x)\|+\|\mathcal{H}^n(A,B)(x)-\mathcal{H}(A,B)(x)\|+\lambda\|u^n-u\|\Big] \\
&\le \|x^n-x\|+\left\{\frac{(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right\}\Big[(t^n+v^n)\|x^n-x\|+\|\mathcal{H}^n(A,B)(x)-\mathcal{H}(A,B)(x)\|+\lambda\|u^n-u\|\Big] \\
&\le \left(1+\frac{(t^n+v^n)(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\right)\|x^n-x\|+\frac{(\tau^n)^{q-1}}{(\alpha^n-\beta^n)}\Big[\|\mathcal{H}^n(A,B)(x)-\mathcal{H}(A,B)(x)\|+\lambda\|u^n-u\|\Big] \\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

This completes the proof. □

Lemma 2.18 ([18]). *Let $\{\zeta^n\},\{\hbar^n\}$ and $\{c^n\}$ be nonnegative sequences satisfying*

$$\zeta^{n+1}\le(1-\omega^n)\zeta^n+\omega^n\hbar^n+c^n,\quad \forall\ n\ge 0,$$

where $\{\omega^n\}_{n=0}^{\infty}\subset[0,1]$, $\sum_{n=0}^{\infty}\omega^n=+\infty$, $\lim_{n\to\infty}\hbar^n=0$ and $\sum_{n=0}^{\infty}c^n<\infty$. Then $\lim_{n\to\infty}\zeta^n=0$.

Definition 2.19. The Hausdorff metric $\mathcal{D}(\cdot,\cdot)$ on $CB(X)$, is defined by

$$\mathcal{D}(S,T)=\max\left\{\sup_{u\in S}\inf_{v\in T}d(u,v),\ \sup_{v\in T}\inf_{u\in S}d(u,v)\right\},\quad S,T\in CB(X),$$

where $d(\cdot,\cdot)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 2.20 ([3]). A set-valued mapping $T:X\to CB(X)$ is said to be γ-$\mathcal{D}$-Lipschitz continuous, if there exists a constant $\gamma>0$ such that

$$\mathcal{D}(T(x),T(y))\le\gamma\|x-y\|,\quad \forall\ x,y\in X.$$

Theorem 2.21 (Nadler [22]). *Let $T:X\to CB(X)$ be a set-valued mapping on X and (X,d) be a complete metric space. Then, we have the following statements:*

(i) *For any given $\mu>0$, $x,y\in X$ and $u\in T(x)$, there exists $v\in T(y)$ such that*

$$d(u,v)\le(1+\mu)\mathcal{D}(T(x),T(y)).$$

(ii) *If $T:X\to C(X)$, then* (i) *holds for $\mu=0$, where $C(X)$ denotes the family of all nonempty compact subsets of X.*

Definition 2.22 ([23]). Let $A : X \to X$ be a single-valued mapping, $x^0 \in X$, $x^{n+1} = f(A,x^n)$ be an iteration procedure which yields a sequence of points $\{x^n\}_{n\geq 0} \subset X$, where f is a continuous mapping. Suppose that $\{x \in X : Ax = x\} \neq \emptyset$ and $\{x^n\}_{n\geq 0}$ converges to a fixed point $x^\star$ of A. Let $\{y^n\}_{n\geq 0} \subset X$, $h^n = \|y^{n+1} - f(A, y^n)\|$. If $\lim\limits_{n\to\infty} h^n = 0$ implies that $\lim\limits_{n\to\infty} y^n = x^\star$. Then the iteration procedure defined by $x^{n+1} = f(A,x^n)$ is said to be A-stable or stable with respect to A.

Now, we formulate our main problem.

Let for each $i = 1,2, j \in \{1,2\}\setminus i$, X_i be a q_i-uniformly smooth Banach space with norm $\|\cdot\|_i$. Let $\phi_i, A_i, B_i, f_i, g_i : X_i \to X_i$, $\mathcal{H}_i, \eta_i : X_i \times X_i \to X_i$, $N_i : X_i \times X_j \times X_i \to X_i$ be single-valued mappings and $M_i : X_j \times X_i \to 2^{X_i}$ be an $\mathcal{H}_i(\cdot,\cdot)$-$\phi_i$-$\eta_i$ accretive mapping with respect to A_i and B_i. Let $S_i, T_i, G_i : X_i \to C(X_i)$ be set-valued mappings.

We consider the following system of generalized variational-like inclusion problem (SGVLIP): Find (x_i,u_i,v_i,w_i) where $x_i \in X_i$, $u_i \in S_i(x_i)$, $v_i \in T_i(x_i)$, $w_i \in G_i(x_i)$ such that

$$\left.\begin{aligned} &\theta_1 \in \mathcal{H}_1(f_1(x_1),x_1) - \mathcal{H}_1(x_1,g_1(x_1)) + \lambda_1(N_1(u_1,v_2,w_1) + M_1(x_2,g_1(x_1))),\\ &\theta_2 \in \mathcal{H}_2(x_2,g_2(x_2)) - \mathcal{H}_2(f_2(x_2),x_2) + \lambda_2(N_2(u_2,v_1,w_2) + M_2(x_1,g_2(x_2))),\end{aligned}\right\} \tag{2.3}$$

where θ_1 and θ_2 are zero vectors of X_1 and X_2, respectively.

We remark that for appropriate and suitable choices of the above defined mappings, SGVLIP (2.1) includes a number of variational and variational-like inclusions as special cases, see for example [1, 2, 5, 7–10, 16, 28] and the related references cited therein.

3. Existence of Solution

First, we give the following lemma which guarantees the existence of solution of SGVLIP (2.1). The proof of this result can be followed by the definition of the resolvent operator $R^{\mathcal{H}_i(\cdot,\cdot)\text{-}\phi_i\text{-}\eta_i}_{M_i(x_j,\cdot),\lambda_i}(\cdot)$ and hence is omitted.

Lemma 3.1. *For $i = 1,2$, $j \in \{1,2\}\setminus i$, let $A_i,B_i,f_i,g_i : X_i \to X_i$, $\mathcal{H}_i,\eta_i : X_i \times X_i \to X_i$, $N_i : X_i \times X_j \times X_i \to X_i$ be single-valued mappings, let $\phi_i : X_i \to X_i$ be a mapping satisfying $\phi_i(x_i + x'_i) = \phi_i(x_i) + \phi_i(x'_i)$ for all $x_i, x'_i \in X_i$ and $Ker(\phi_i) = \{0\}$, where $Ker(\phi_i) = \{x_i \in X_i : \phi_i(x_i) = 0\}$ and $M_i : X_j \times X_i \to 2^{X_i}$ be a $\mathcal{H}_i(\cdot,\cdot)$-$\phi_i$-$\eta_i$-accretive mapping with respect to A_i and B_i. Then (x_i,u_i,v_i,w_i) is a solution of SGVLIP (2.1) where $x_i \in X_i$, $u_i \in S_i(x_i)$, $v_i \in T_i(x_i)$, $w_i \in G_i(x_i)$ if and only if it satisfies*

$$\left.\begin{aligned} g_1(x_1) = R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\big[&\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1)\\ &-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\big]\\ g_2(x_2) = R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}\big[&\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2) - \lambda_2\phi_2\circ N_2(u_2,v_1,w_2)\\ &-\phi_2\circ\mathcal{H}_2(x_2,g_2(x_2)) + \phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)\big],\end{aligned}\right\} \tag{3.1}$$

where λ_1, $\lambda_2 > 0$ are constants and

$$R^{\mathcal{H}_i(\cdot,\cdot)\text{-}\phi_i\text{-}\eta_i}_{M_i(x_j,\cdot),\lambda_i}(x_i) \equiv (\mathcal{H}_i(A_i,B_i) + \lambda_i\phi_i\circ M_i(x_j,\cdot))^{-1}(x_i), \quad \forall\ x_i \in X_i, x_j \in X_j.$$

Theorem 3.2. *For $i = 1,2$, $j \in \{1,2\}\setminus i$, let $A_i,B_i,f_i,g_i : X_i \to X_i$, $\mathcal{H}_i,\eta_i : X_i \times X_i \to X_i$, $N_i : X_i \times X_j \times X_i \to X_i$ be single-valued mappings. Let $\phi_i : X_i \to X_i$ be a mapping satisfying $\phi_i(x_i + x'_i) = \phi_i(x_i) + \phi_i(x'_i)$ for all $x_i, x'_i \in X_i$ and $Ker(\phi_i) = \{0\}$, $\mathcal{H}_i : X_i \times X_i \to X_i$ be an $\alpha_i\beta_i$-*

symmetric η_i-accretive mapping with respect to $A_i(g_i(\cdot))$ and $B_i(g_i(\cdot))$, (ν_i,ξ_i)-mixed Lipschitz continuous and let η_i, f_i be τ_i, l_{f_i}-Lipschitz continuous, respectively and g_i be δ_i-strongly η_i-accretive and l_{g_i}-Lipschitz continuous. Let $\phi_i \circ N_i$ be a μ_i-strongly η_i-accretive mapping in the first argument and d_i, s_i and p_i-Lipschitz continuous in the first, second and third arguments, respectively and $\phi_i \circ \mathcal{H}_i$ be (ρ_i,γ_i)-mixed Lipschitz continuous. Let $M_i : X_j \times X_i \to 2^{X_i}$ be an $\mathcal{H}_i(\cdot,\cdot)$-$\phi_i$-$\eta_i$-accretive mapping with respect to A_i and B_i and $S_i, T_i, G_i : X_i \to C(X_i)$ be such that S_i is $\mathcal{L}_{S_i}$-$\mathcal{D}$-Lipschitz continuous, T_i is $\mathcal{L}_{T_i}$-$\mathcal{D}$-Lipschitz continuous and G_i is $\mathcal{L}_{G_i}$-$\mathcal{D}$-Lipschitz continuous. Suppose that there are constants $\lambda_1,\lambda_2,r_1,r_2 > 0$ satisfying the following conditions:

$$k_i := m_i + \{L_j\lambda_j s_j \mathcal{L}_{T_i} + r_j\} < 1, \tag{3.2}$$

where

$$\begin{aligned} m_i := \Big[&\big(1 - q_i\delta_i + q_i l_{g_i} \times (1+\tau_i^{q_i-1}) + c_{q_i} l_{g_i}^{q_i}\big)^{1/q_i} \\ &+ L_i\big\{\big(1 - q_i(\alpha_i-\beta_i) + q_i(\nu_i+\xi_i)\times(1+\tau_i^{q_i-1}) + c_{q_i}(\nu_i+\xi_i)^{q_i}\big)^{1/q_i} \\ &+ \big(1 - \lambda_i q_i \mu_i + \lambda_i q_i d_i \mathcal{L}_{S_i} \times (1+\tau_i^{q_i-1}) + \lambda_i^{q_i} c_{q_i} d_i^{q_i} \mathcal{L}_{S_i}^{q_i}\big)^{1/q_i} \\ &+ \lambda_i p_i \mathcal{L}_{G_i} + \rho_i(1+l_{f_i}) + \gamma_i(1+l_{g_i})\big\}\Big], \qquad L_i := \frac{\tau_i^{q_i-1}}{(\alpha_i-\beta_i)} \end{aligned}$$

and further assume that

$$\big\|R^{\mathcal{H}_i(\cdot,\cdot)\text{-}\phi_i\text{-}\eta_i}_{M_i(x_j,\cdot),\lambda_i}(x_i^\star) - R^{\mathcal{H}_i(\cdot,\cdot)\text{-}\phi_i\text{-}\eta_i}_{M_i(x_j',\cdot),\lambda_i}(x_i^\star)\big\| \le r_i\|x_j - x_j'\|_j, \quad \forall\ x_i^\star \in X_i, x_j, x_j' \in X_j. \tag{3.3}$$

Then SGVLIP (2.1) *has a solution.*

Proof. For each $(x_1,x_2) \in X_1\times X_2$, define a mapping $Q : X_1\times X_2 \to X_1 \times X_2$ by

$$Q(x_1,x_2) = (P_1(x_1,x_2), P_2(x_1,x_2)), \tag{3.4}$$

where $P_1 : X_1\times X_2 \to X_1$ and $P_2 : X_1 \times X_2 \to X_2$ are given by

$$\begin{aligned} P_1(x_1,x_2) = x_1 - g_1(x_1) + R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\big[&\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) \\ &- \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\big], \end{aligned} \tag{3.5}$$

$$\begin{aligned} P_2(x_1,x_2) = x_2 - g_2(x_2) + R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}\big[&\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2) \\ &- \lambda_2\phi_2\circ N_2(u_2,v_1,w_2) - \phi_2\circ\mathcal{H}_2(x_2,g_2(x_2)) + \phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)\big], \end{aligned} \tag{3.6}$$

for $\lambda_1,\lambda_2 > 0$, respectively. Then, for any (x_1,x_2), $(x_1',x_2') \in X_1\times X_2$, it follows from (3.5), (3.6) and the Lipschitz continuity of $R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}$ and $R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}$ that

$$\begin{aligned} &\|P_1(x_1,x_2) - P_1(x_1',x_2')\|_1 \\ &\le \Big\|x_1 - g_1(x_1) + R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) \\ &\quad -\lambda_1\phi_1\circ N_1(u_1,v_2,w_1) - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))] \\ &\quad - \big\{x_1' - g_1(x_1') + R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2',\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1') \\ &\quad - \lambda_1\phi_1\circ N_1(u_1',v_2',w_1') - \phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1') + \phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))\big]\big\}\Big\|_1 \\ &\le \|(x_1 - x_1') - (g_1(x_1) - g_1(x_1'))\|_1 \\ &\quad + \Big\|R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \end{aligned}$$

$$
\begin{aligned}
&-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\\
&-R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2',\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-\lambda_1\phi_1\circ N_1(u_1',v_2',w_1')\\
&-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')+\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))]\,\|_1. \qquad (3.7)
\end{aligned}
$$

Since g_i is δ_i-strongly η_i-accretive, l_{g_i}-Lipschitz continuous and η_i is τ_i-Lipschitz continuous, by using Lemma 2.4, we have

$$
\begin{aligned}
&\|(x_1-x_1')-(g_1(x_1)-g_1(x_1'))\|_1^{q_1}\\
&\le \|x_1-x_1'\|_1^{q_1}-q_1\langle g_1(x_1)-g_1(x_1'),J_{q_1}(\eta_1(x_1,x_1'))\rangle_1\\
&\quad -q_1\langle g_1(x_1)-g_1(x_1'),J_{q_1}(x_1-x_1')-J_{q_1}(\eta_1(x_1,x_1'))\rangle_1+c_{q_1}\|g_1(x_1)-g_1(x_1')\|_1^{q_1}\\
&\le \|x_1-x_1'\|_1^{q_1}-q_1\langle g_1(x_1)-g_1(x_1'),J_{q_1}(\eta_1(x_1,x_1'))\rangle_1\\
&\quad +q_1\|g_1(x_1)-g_1(x_1')\|_1\times\left(\|x_1-x_1'\|_1^{q_1-1}+\|\eta_1(x_1,x_1')\|_1^{q_1-1}\right)+c_{q_1}\|g_1(x_1)-g_1(x_1')\|_1^{q_1}\\
&\le \left(1-q_1\delta_1+q_1l_{g_1}\times(1+\tau_1^{q_1-1})+c_{q_1}l_{g_1}^{q_1}\right)\|x_1-x_1'\|_1^{q_1}.
\end{aligned}
$$

This implies that

$$
\|(x_1-x_1')-(g_1(x_1)-g_1(x_1'))\|_1\le\left(1-q_1\delta_1+q_1l_{g_1}\times(1+\tau_1^{q_1-1})+c_{q_1}l_{g_1}^{q_1}\right)^{1/q_1}\|x_1-x_1'\|_1. \qquad (3.8)
$$

Now, we have

$$
\begin{aligned}
&\|R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\lambda_1\phi_1\circ N_1(u_1,v_2,w_1)\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\\
&\quad-R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2',\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-\lambda_1\phi_1\circ N_1(u_1',v_2',w_1')\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')+\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))]\,\|_1\\
&\le \|R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\lambda_1\phi_1\circ N_1(u_1,v_2,w_1)\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\\
&\quad-R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-\lambda_1\phi_1\circ N_1(u_1',v_2',w_1')\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')+\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))]\,\|_1\\
&\quad+\|R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-\lambda_1\phi_1\circ N_1(u_1',v_2',w_1')\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')+\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))]\\
&\quad-R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2',\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-\lambda_1\phi_1\circ N_1(u_1',v_2',w_1')\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')+\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))]\,\|_1\\
&\le L_1\|[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\lambda_1\phi_1\circ N_1(u_1,v_2,w_1)\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\\
&\quad-[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-\lambda_1\phi_1\circ N_1(u_1',v_2',w_1')\\
&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')+\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))]\,\|_1+r_1\|x_2-x_2'\|_2\\
&\le L_1\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1')-(x_1-x_1')\|_1\\
&\quad+L_1\|(x_1-x_1')-\lambda_1(\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u_1',v_2',w_1'))\|_1\\
&\quad+L_1\|\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')\|_1
\end{aligned}
$$

$$
\begin{aligned}
&+L_1\|\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))-\phi_1\circ\mathcal{H}_1(x'_1,g_1(x'_1))\|_1+r_1\|x_2-x'_2\|_2\\
&\le L_1\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1)-(x_1-x'_1)\|_1\\
&\quad+L_1\|(x_1-x'_1)-\lambda_1(\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1))\|_1\\
&\quad+L_1\lambda_1\|\phi_1\circ N_1(u'_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v'_2,w_1)\|_1\\
&\quad+L_1\lambda_1\|\phi_1\circ N_1(u'_1,v'_2,w_1)-\phi_1\circ N_1(u'_1,v'_2,w'_1)\|_1\\
&\quad+L_1\|\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)-\phi_1\circ\mathcal{H}_1(f_1(x'_1),x'_1)\|_1\\
&\quad+L_1\|\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))-\phi_1\circ\mathcal{H}_1(x'_1,g_1(x'_1))\|_1+r_1\|x_2-x'_2\|_2. \qquad (3.9)
\end{aligned}
$$

Since $\mathcal{H}_i$ is an $\alpha_i\beta_i$-symmetric η_i-accretive mapping with respect to $A_i(g_i(\cdot))$ and $B_i(g_i(\cdot))$, (ν_i,ξ_i)-mixed Lipschitz continuous and η_i is τ_i-Lipschitz continuous, by using Lemma 2.4, we have

$$
\begin{aligned}
&\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1)-(x_1-x'_1)\|_1^{q_1}\\
&\le\|x_1-x'_1\|_1^{q_1}-q_1\langle\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1),J_{q_1}(\eta_1(x_1,x'_1))\rangle_1\\
&\quad-q_1\langle\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1),J_{q_1}(x_1-x'_1)-J_{q_1}(\eta_1(x_1,x'_1))\rangle_1\\
&\quad+c_{q_1}\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1)\|_1^{q_1}\\
&\le\|x_1-x'_1\|_1^{q_1}-q_1\langle\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1),J_{q_1}(\eta_1(x_1,x'_1))\rangle_1\\
&\quad+q_1\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1)\|_1\big(\|x_1-x'_1\|_1^{q_1-1}+\|\eta_1(x_1,x'_1)\|_1^{q_1-1}\big)\\
&\quad+c_{q_1}\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1)\|_1^{q_1}\\
&\le\big(1-q_1(\alpha_1-\beta_1)+q_1(\nu_1+\xi_1)\times(1+\tau_1^{q_1-1})+c_{q_1}(\nu_1+\xi_1)^{q_1}\big)\|x_1-x'_1\|_1^{q_1}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x'_1)-(x_1-x'_1)\|_1\\
&\quad\le\big(1-q_1(\alpha_1-\beta_1)+q_1(\nu_1+\xi_1)\times(1+\tau_1^{q_1-1})+c_{q_1}(\nu_1+\xi_1)^{q_1}\big)^{1/q_1}\|x_1-x'_1\|_1. \qquad (3.10)
\end{aligned}
$$

Now, since $\phi_i\circ N_i$ is a μ_i-strongly η_i-accretive mapping in the first argument and is d_i,s_i and p_i-Lipschitz continuous in the first, second and third arguments, respectively and $S_i,T_i,G_i:X_i\to C(X_i)$ are $\mathcal{L}_{S_i}$-$\mathcal{D}$-Lipschitz continuous, $\mathcal{L}_{T_i}$-$\mathcal{D}$-Lipschitz continuous and $\mathcal{L}_{G_i}$-$\mathcal{D}$-Lipschitz continuous, respectively and η_i is τ_i-Lipschitz continuous, by using Lemma 2.4 and Nadler [22, Theorem 2.21], we have

$$
\begin{aligned}
&\|(x_1-x'_1)-\lambda_1(\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1))\|_1^{q_1}\\
&\le\|x_1-x'_1\|_1^{q_1}-\lambda_1q_1\langle\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1),J_{q_1}(\eta_1(x_1,x'_1))\rangle_1\\
&\quad-\lambda_1q_1\langle\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1),J_{q_1}(x_1-x'_1)-J_{q_1}(\eta_1(x_1,x'_1))\rangle_1\\
&\quad+\lambda_1^{q_1}c_{q_1}\|\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1)\|_1^{q_1}\\
&\le\|x_1-x'_1\|_1^{q_1}-\lambda_1q_1\mu_1\|x_1-x'_1\|_1^{q_1}+\lambda_1q_1\|\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1)\|_1\\
&\quad\times\big(\|x_1-x'_1\|_1^{q_1-1}+\|\eta_1(x_1,x'_1)\|_1^{q_1-1}\big)+\lambda_1^{q_1}c_{q_1}\|\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u'_1,v_2,w_1)\|_1^{q_1}\\
&\le\|x_1-x'_1\|_1^{q_1}-\lambda_1q_1\mu_1\|x_1-x'_1\|_1^{q_1}+\lambda_1q_1d_1\|u_1-u'_1\|_1\times\big(\|x_1-x'_1\|_1^{q_1-1}\\
&\quad+\tau_1^{q_1-1}\|x_1-x'_1\|_1^{q_1-1}\big)+\lambda_1^{q_1}c_{q_1}d_1^{q_1}\|u_1-u'_1\|_1^{q_1}\\
&\le\|x_1-x'_1\|_1^{q_1}-\lambda_1q_1\mu_1\|x_1-x'_1\|_1^{q_1}+\lambda_1q_1d_1\mathcal{D}(S_1(x_1),S_1(x'_1))_1\\
&\quad\times\big(\|x_1-x'_1\|_1^{q_1-1}+\tau_1^{q_1-1}\|x_1-x'_1\|_1^{q_1-1}\big)+\lambda_1^{q_1}c_{q_1}d_1^{q_1}\mathcal{D}(S_1(x_1),S_1(x'_1))_1^{q_1}
\end{aligned}
$$

$$\begin{aligned}&\leq \|x_1-x_1'\|_1^{q_1}-\lambda_1 q_1\mu_1\|x_1-x_1'\|_1^{q_1}+\lambda_1 q_1 d_1\mathcal{L}_{S_1}\|x_1-x_1'\|_1\\&\quad\times\left(\|x_1-x_1'\|_1^{q_1-1}+\tau_1^{q_1-1}\|x_1-x_1'\|_1^{q_1-1}\right)+\lambda_1^{q_1}c_{q_1}d_1^{q_1}\mathcal{L}_{S_1}^{q_1}\|x_1-x_1'\|_1^{q_1}\\&\leq\left(1-\lambda_1 q_1\mu_1+\lambda_1 q_1 d_1\mathcal{L}_{S_1}\times(1+\tau_1^{q_1-1})+\lambda_1^{q_1}c_{q_1}d_1^{q_1}\mathcal{L}_{S_1}^{q_1}\right)\|x_1-x_1'\|_1^{q_1}.\end{aligned}$$

This implies that

$$\begin{aligned}&\|(x_1-x_1')-\lambda_1(\phi_1\circ N_1(u_1,v_2,w_1)-\phi_1\circ N_1(u_1',v_2,w_1))\|_1\\&\quad\leq\left(1-\lambda_1 q_1\mu_1+\lambda_1 q_1 d_1\mathcal{L}_{S_1}\times(1+\tau_1^{q_1-1})+\lambda_1^{q_1}c_{q_1}d_1^{q_1}\mathcal{L}_{S_1}^{q_1}\right)^{1/q_1}\|x_1-x_1'\|_1.\end{aligned}\tag{3.11}$$

Also, we have

$$\begin{aligned}\|\phi_1\circ N_1(u_1',v_2,w_1)-\phi_1\circ N_1(u_1',v_2',w_1)\|_1&\leq s_1\|v_2-v_2'\|_2\\&\leq s_1\mathcal{D}(T_2(x_2),T_2(x_2'))_2\\&\leq s_1\mathcal{L}_{T_2}\|x_2-x_2'\|_2\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}\|\phi_1\circ N_1(u_1',v_2',w_1)-\phi_1\circ N_1(u_1',v_2',w_1')\|_1&\leq p_1\|w_1-w_1'\|_1\\&\leq p_1\mathcal{D}(G_1(x_1),G_1(x_1'))_1\\&\leq p_1\mathcal{L}_{G_1}\|x_1-x_1'\|_1.\end{aligned}\tag{3.13}$$

Again, since $\phi_i\circ\mathcal{H}_i$ is (ρ_i,γ_i)-mixed Lipschitz continuous, f_i,g_i is l_{f_i},l_{g_i}-Lipschitz continuous, respectively, we have

$$\begin{aligned}\|\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)-\phi_1\circ\mathcal{H}_1(f_1(x_1'),x_1')\|_1&\leq\rho_1\|f_1(x_1)-f_1(x_1')\|_1+\gamma_1\|x_1-x_1'\|_1\\&\leq\rho_1 l_{f_1}\|x_1-x_1'\|_1+\gamma_1\|x_1-x_1'\|_1\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\|\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))-\phi_1\circ\mathcal{H}_1(x_1',g_1(x_1'))\|_1&\leq\rho_1\|x_1-x_1'\|_1+\gamma_1\|g_1(x_1)-g_1(x_1')\|_1\\&\leq\rho_1\|x_1-x_1'\|_1+\gamma_1 l_{g_1}\|x_1-x_1'\|_1.\end{aligned}\tag{3.15}$$

From (3.5), (3.7)-(3.15), we have

$$\|P_1(x_1,x_2)-P_1(x_1',x_2')\|_1\leq m_1\|x_1-x_1'\|_1+\left(L_1\lambda_1 s_1\mathcal{L}_{T_2}+r_1\right)\|x_2-x_2'\|_2.\tag{3.16}$$

Similarly, we have

$$\|P_2(x_1,x_2)-P_2(x_1',x_2')\|_2\leq m_2\|x_2-x_2'\|_2+\left(L_2\lambda_2 s_2\mathcal{L}_{T_1}+r_2\right)\|x_1-x_1'\|_1.\tag{3.17}$$

From (3.16) and (3.17), we have

$$\begin{aligned}\|P_1(x_1,x_2)-P_1(x_1',x_2')\|_1+\|P_2(x_1,x_2)-P_2(x_1',x_2')\|_2&\leq k_1\|x_1-x_1'\|_1+k_2\|x_2-x_2'\|_2\\&\leq k\left\{\|x_1-x_1'\|_1+\|x_2-x_2'\|_2\right\},\end{aligned}\tag{3.18}$$

where $k=\max\{k_1,k_2\}$.

Now, define the norm $\|\cdot\|_\star$ on $X_1\times X_2$ by

$$\|(x_1,x_2)\|_\star=\|x_1\|_1+\|x_2\|_2,\quad\forall\,(x_1,x_2)\in X_1\times X_2.\tag{3.19}$$

Then we observe that $(X_1\times X_2,\|\cdot\|_\star)$ is a Banach space. Hence, it follows from (3.4), (3.18) and (3.19) that

$$\begin{aligned}\|Q(x_1,x_2)-Q(x_1',x_2')\|_\star&\leq\|(P_1(x_1,x_2),P_2(x_1,x_2))-(P_1(x_1',x_2'),P_2(x_1',x_2'))\|_\star\\&\leq\|P_1(x_1,x_2)-P_1(x_1',x_2'),P_2(x_1,x_2)-P_2(x_1',x_2')\|_\star\end{aligned}$$

$$\begin{aligned}&\le \left\|P_1(x_1,x_2)-P_1(x_1',x_2')\right\|_1+\left\|P_2(x_1,x_2)-P_2(x_1',x_2')\right\|_2\\&\le k\{\left\|(x_1-x_1'\right\|_1+\left\|x_2-x_2'\right\|_2\}.\end{aligned}\tag{3.20}$$

Since $k=\max\{k_1,k_2\}<1$ by (3.2), it follows from (3.20) that Q is a contraction mapping. Hence, by Banach contraction principle, it admits a unique fixed point $(x_1,x_2)\in X_1\times X_2$ such that $Q(x_1,x_2)=(x_1,x_2)$, which implies that

$$\begin{aligned}g_1(x_1)&=R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\lambda_1\phi_1\circ N_1(u_1,v_2,w_1)\\&\quad-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\big]\\g_2(x_2)&=R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}\big[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2)-\lambda_2\phi_2\circ N_2(u_2,v_1,w_2)\\&\quad-\phi_2\circ\mathcal{H}_2(x_2,g_2(x_2))+\phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)\big].\end{aligned}$$

It follows from Lemma 3.1, that (x_i,u_i,v_i,w_i) is a solution of SGVLIP (2.1). This completes the proof. □

4. Mann Type Perturbed Iterative Algorithm, Convergence and Stability Analysis

Lemma 3.1 is very important from the numerical point of view as it along with Nadler [22] allows us to suggest the following Mann type perturbed iterative algorithm (in short, MTPIA) for finding the approximate solution of SGVLIP (2.1).

Algorithm 4.1 (Mann Type Perturbed Iteration)**.** For each $i=1,2$, $j\in\{1,2\}\setminus i$, given $(x_i^0,u_i^0,v_i^0,w_i^0)$, where $x_i^0\in X_i$, $u_i^0\in S_i(x_i^0)$, $v_i^0\in T_i(x_i^0)$, $w_i^0\in G_i(x_i^0)$ such that $S_i,T_i,G_i:X_i\to C(X_i)$, compute the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{w_i^n\}$ by the iterative schemes:

$$\begin{aligned}x_1^{n+1}&=(1-a^n)x_1^n+a^n\Big\{x_1^n-g_1(x_1^n)+R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2^n,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1^n)\\&\quad-\lambda_1\phi_1\circ N_1(u_1^n,v_2^n,w_1^n)-\phi_1\circ\mathcal{H}_1(f_1(x_1^n),x_1^n)+\phi_1\circ\mathcal{H}_1(x_1^n,g_1(x_1^n))\big]\Big\}+a^ne_1^n,\\x_2^{n+1}&=(1-a^n)x_2^n+a^n\Big\{x_2^n-g_2(x_2^n)+R^{\mathcal{H}_2^n(\cdot,\cdot)\text{-}\phi_2^n\text{-}\eta_2^n}_{M_2^n(x_1^n,\cdot),\lambda_2}\big[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2^n)\\&\quad-\lambda_2\phi_2\circ N_2(u_2^n,v_1^n,w_2^n)-\phi_2\circ\mathcal{H}_2(x_2^n,g_2(x_2^n))+\phi_2\circ\mathcal{H}_2(f_2(x_2^n),x_2^n)\big]\Big\}+a^ne_2^n,\\u_i^n&\in S_i(x_i^n):\|u_i^{n+1}-u_i^n\|_i\le\mathcal{D}(S_i(x_i^{n+1}),S_i(x_i^n))_i,\\v_i^n&\in T_i(x_i^n):\|v_i^{n+1}-v_i^n\|_i\le\mathcal{D}(T_i(x_i^{n+1}),T_i(x_i^n))_i,\\w_i^n&\in G_i(x_i^n):\|w_i^{n+1}-w_i^n\|_i\le\mathcal{D}(G_i(x_i^{n+1}),G_i(x_i^n))_i,\end{aligned}$$

where $n=0,1,2,\cdots$, $\lambda_i>0$ are constants, M_i^n is an $\mathcal{H}_i^n(\cdot,\cdot)$-$\phi_i^n$-$\eta_i^n$-accretive mapping and $\{(e_1^n,e_2^n)\}_{n\ge0}$ is sequence in $X_1\times X_2$ introduced to take into account possible inexact computation which satisfies $\lim\limits_{n\to\infty}\|e_1^n\|=\lim\limits_{n\to\infty}\|e_2^n\|=0$ and $\{a^n\}$ is a sequence of real numbers such that $a^n\in[0,1]$ and $\sum\limits_{n=0}^{\infty}a^n=+\infty$.

Theorem 4.2. *For $i=1,2,j\in\{1,2\}\setminus i$, let $A_i,B_i,f_i,g_i:X_i\to X_i$, $\mathcal{H}_i,\eta_i:X_i\times X_i\to X_i$, $N_i:X_i\times X_j\times X_i\to X_i$ be single-valued mappings. Let $\phi_i:X_i\to X_i$ be a mapping satisfying $\phi_i(x_i+x_i')=\phi_i(x_i)+\phi_i(x_i')$ for all $x_i,x_i'\in X_i$ and $Ker(\phi_i)=\{0\}$, $\mathcal{H}_i:X_i\times X_i\to X_i$ be an $\alpha_i\beta_i$-symmetric η_i-accretive mapping with respect to $A_i(g_i(\cdot))$ and $B_i(g_i(\cdot))$, (ν_i,ξ_i)-mixed Lipschitz continuous and let η_i,f_i be τ_i,l_{f_i}-Lipschitz continuous, respectively and g_i be δ_i-strongly*

η_i-accretive and l_{g_i}-Lipschitz continuous. Let $\phi_i \circ N_i$ be a μ_i-strongly η_i-accretive mapping in the first argument and d_i, s_i and p_i-Lipschitz continuous in the first, second and third arguments, respectively and $\phi_i \circ \mathcal{H}_i$ be (ρ_i,γ_i)-mixed Lipschitz continuous. Let $M_i : X_j \times X_i \to 2^{X_i}$ be an $\mathcal{H}_i(\cdot,\cdot)$-$\phi_i$-$\eta_i$-accretive mapping with respect to A_i and B_i and $S_i, T_i, G_i : X_i \to C(X_i)$ be such that S_i is $\mathcal{L}_{S_i}$-$\mathcal{D}$-Lipschitz continuous, T_i is $\mathcal{L}_{T_i}$-$\mathcal{D}$-Lipschitz continuous and G_i is $\mathcal{L}_{G_i}$-$\mathcal{D}$-Lipschitz continuous. Let ϕ_i^n be a single-valued mapping, $\eta_i^n : X_i \times X_i \to X_i$ be τ_i^n-Lipschitz continuous and $\mathcal{H}_i^n : X_i \times X_i \to X_i$ be an $\alpha_i^n\beta_i^n$-symmetric η_i^n-accretive mapping. Further, let $M_i^n : X_j \times X_i \to 2^{X_i}$ satisfy the followings:

(i) *$M_i^n(x_j,\cdot) : X_i \to 2^{X_i}$ is a $\mathcal{H}_i^n(\cdot,\cdot)$-$\phi_i^n$-$\eta_i^n$-accretive mapping for each $x_j \in X_j$,*

(ii) *$\phi_i^n \circ M_i^n(x_j,\cdot) \xrightarrow{\mathcal{G}} \phi_i \circ M_i(x_j,\cdot)$, $\mathcal{H}_i^n(A_i,B_i)(x_i) \to \mathcal{H}_i(A_i,B_i)(x_i)$ as $n \to \infty$ for each $x_j \in X_j$, $x_i \in X_i$,*

(iii)
$$\left.\begin{aligned}
&\left\|R^{\mathcal{H}_i^n(\cdot,\cdot)-\phi_i^n-\eta_i^n}_{M_i^n(x_j,\cdot),\lambda_i}(x^\star) - R^{\mathcal{H}_i^n(\cdot,\cdot)-\phi_i^n-\eta_i^n}_{M_i^n(x_j,\cdot),\lambda_i}(y^\star)\right\|_i \le L_i^n \|x^\star - y^\star\|_i, \quad \forall\, x^\star, y^\star \in X_i,\\
&\left\|R^{\mathcal{H}_i^n(\cdot,\cdot)-\phi_i^n-\eta_i^n}_{M_i^n(x_j,\cdot),\lambda_i}(x^\star) - R^{\mathcal{H}_i^n(\cdot,\cdot)-\phi_i^n-\eta_i^n}_{M_i^n(x'_j,\cdot),\lambda_i}(x^\star)\right\|_i \le r_i^n \|x_j - x'_j\|_j, \quad \forall\, x^\star \in X_i,\ x_j, x'_j \in X_j,
\end{aligned}\right\} \tag{4.1}$$

(iv) *$\alpha_i^n \to \alpha_i$, $\beta_i^n \to \beta_i$, $r_i^n \to r_i$, $\tau_i^n \to \tau_i$ as $n \to \infty$ where, $L_i{}^n := \frac{(\tau_i^n)^{q_i-1}}{(\alpha_i{}^n - \beta_i{}^n)}$.*

Further, suppose that $\{(\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n)\}_{n\ge0}$ is a sequence in X_i and define $\epsilon^n = \epsilon_1^n + \epsilon_2^n$ for $n \ge 0$ by

$$\begin{aligned}
\epsilon_1^n &= \Big\|\bar{x}_1^{n+1} - \big[(1-a^n)\bar{x}_1^n + a^n\big\{\bar{x}_1^n - g_1(\bar{x}_1^n) + R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(\bar{x}_2^n,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(\bar{x}_1^n)\\
&\qquad -\lambda_1\phi_1 \circ N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n) - \phi_1 \circ \mathcal{H}_1(f_1(\bar{x}_1^n), \bar{x}_1^n) + \phi_1 \circ \mathcal{H}_1(\bar{x}_1^n, g_1(\bar{x}_1^n))\big]\big\} + a^n e_1^n\big]\Big\|_1,\\
\epsilon_2^n &= \Big\|\bar{x}_2^{n+1} - \big[(1-a^n)\bar{x}_2^n + a^n\big\{\bar{x}_2^n - g_2(\bar{x}_2^n) + R^{\mathcal{H}_2^n(\cdot,\cdot)\text{-}\phi_2^n\text{-}\eta_2^n}_{M_2^n(\bar{x}_1^n,\cdot),\lambda_2}\big[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(\bar{x}_2^n)\\
&\qquad -\lambda_2\phi_2 \circ N_2(\bar{u}_2^n, \bar{v}_1^n, \bar{w}_2^n) - \phi_2 \circ \mathcal{H}_2(\bar{x}_2^n, g_2(\bar{x}_2^n)) + \phi_2 \circ \mathcal{H}_2(f_2(\bar{x}_2^n), \bar{x}_2^n)\big]\big\} + a^n e_2^n\big]\Big\|_2,\\
\bar{u}_i^n &\in S_i(\bar{x}_i^n) : \left\|\bar{u}_i^{n+1} - \bar{u}_i^n\right\|_i \le \mathcal{D}(S_i(\bar{x}_i^{n+1}), S_i(\bar{x}_i^n))_i,\\
\bar{v}_i^n &\in T_i(\bar{x}_i^n) : \left\|\bar{v}_i^{n+1} - \bar{v}_i^n\right\|_i \le \mathcal{D}(T_i(\bar{x}_i^{n+1}), T_i(\bar{x}_i^n))_i,\\
\bar{w}_i^n &\in G_i(\bar{x}_i^n) : \left\|\bar{w}_i^{n+1} - \bar{w}_i^n\right\|_i \le \mathcal{D}(G_i(\bar{x}_i^{n+1}), G_i(\bar{x}_i^n))_i.
\end{aligned} \tag{4.2}$$

If there exist positive constants λ_1, λ_2 such that (3.2) *holds then we have the following statements:*

(a) *The iterative sequence $\{(x_i^n, u_i^n, v_i^n, w_i^n)\}_{n\ge0}$ generated by MTPIA* (4.1) *converges to the solution $\{(x_i, u_i, v_i, w_i)\}$ of SGVLIP* (2.1).

(b) *For any sequences $\{(\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n)\}_{n\ge0}$, $\lim_{n\to\infty}(\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n) = (x_i, u_i, v_i, w_i)$ if and only if $\lim_{n\to\infty}\epsilon^n = 0$, where $\epsilon^n = \epsilon_1^n + \epsilon_2^n$, for all $n \ge 0$.*

Proof. SGVLIP (2.1) has a solution (x_i, u_i, v_i, w_i) by Theorem 3.2. From Lemma 3.1, we have

$$\begin{aligned}
x_1 &= (1-a^n)x_1 + a^n\{x_1 - g_1(x_1) + R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)\\
&\qquad -\lambda_1\phi_1 \circ N_1(u_1, v_2, w_1) - \phi_1 \circ \mathcal{H}_1(f_1(x_1), x_1) + \phi_1 \circ \mathcal{H}_1(x_1, g_1(x_1))]\},\\
x_2 &= (1-a^n)x_2 + a^n\{x_2 - g_2(x_2) + R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2)\\
&\qquad -\lambda_2\phi_2 \circ N_2(u_2, v_1, w_2) - \phi_2 \circ \mathcal{H}_2(x_2, g_2(x_2)) + \phi_2 \circ \mathcal{H}_2(f_2(x_2), x_2)]\}.
\end{aligned} \tag{4.3}$$

Now, from MTPIA (4.1), (4.3) and using the same arguments used in estimating (3.7)-(3.15), we have

$$\begin{aligned}
\|x_1^{n+1}-x_1\|_1 = {} & \Big\|(1-a^n)x_1^n + a^n\{x_1^n - g_1(x_1^n) + R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2^n,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1^n) \\
& - \lambda_1\phi_1\circ N_1(u_1^n,v_2^n,w_1^n) - \phi_1\circ\mathcal{H}_1(f_1(x_1^n),x_1^n) + \phi_1\circ\mathcal{H}_1(x_1^n,g_1(x_1^n))]\} + a^n e_1^n \\
& - \Big[(1-a^n)x_1 + a^n\{x_1 - g_1(x_1) + R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) \\
& - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\}\Big]\Big\|_1 \\
\le {} & (1-a^n)\|x_1^n - x_1\|_1 + a^n\|(x_1^n - x_1) - (g_1(x_1^n) - g_1(x_1))\|_1 \\
& + a^n\Big\|R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2^n,\cdot),\lambda_1}\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1^n) - \lambda_1\phi_1\circ N_1(u_1^n,v_2^n,w_1^n) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1^n),x_1^n) + \phi_1\circ\mathcal{H}_1(x_1^n,g_1(x_1^n))\Big] \\
& - R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2^n,\cdot),\lambda_1}\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\Big]\Big\|_1 \\
& + a^n\Big\|R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2^n,\cdot),\lambda_1}\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\Big] \\
& - R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2,\cdot),\lambda_1}\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\Big]\Big\|_1 \\
& + a^n\Big\|R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2,\cdot),\lambda_1}\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\Big] \\
& - R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))\Big]\Big\|_1 + a^n\|e_1^n\|_1 \\
\le {} & (1-a^n)\|x_1^n - x_1\|_1 + a^n\|(x_1^n - x_1) - (g_1(x_1^n) - g_1(x_1))\|_1 \\
& + a^n L_1^n\Big\|\Big[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1^n) - \lambda_1\phi_1\circ N_1(u_1^n,v_2^n,w_1^n) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1^n),x_1^n) + \phi_1\circ\mathcal{H}_1(x_1^n,g_1(x_1^n))\Big] \\
& - [\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1\circ N_1(u_1,v_2,w_1) \\
& - \phi_1\circ\mathcal{H}_1(f_1(x_1),x_1) + \phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\Big\|_1 \\
& + a^n r_1^n\|x_2^n - x_2\|_2 + a^n b_{M_1}^n + a^n\|e_1^n\|_1 \\
\le {} & (1-a^n)\|x_1^n - x_1\|_1 + a^n\Big[m_1^n\|x_1^n - x_1\|_1 + \big(L_1^n\lambda_1 s_1\mathcal{L}_{T_2} + r_1^n\big)\|x_2^n - x_2\|_2\Big] \\
& + a^n b_{M_1}^n + a^n\|e_1^n\|_1,
\end{aligned} \tag{4.4}$$

where

$$b^n_{M_1} := \|R^{\mathcal{H}^n_1(\cdot,\cdot)\text{-}\phi^n_1\text{-}\eta^n_1}_{M^n_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\lambda_1\phi_1\circ N_1(u_1,v_2,w_1)$$
$$-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]$$
$$-R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}[\mathcal{H}_1(A_1(g_1(\cdot)),B_1(g_1(\cdot)))(x_1)-\lambda_1\phi_1\circ N_1(u_1,v_2,w_1)$$
$$-\phi_1\circ\mathcal{H}_1(f_1(x_1),x_1)+\phi_1\circ\mathcal{H}_1(x_1,g_1(x_1))]\|_1.$$

Similarly, we obtain

$$\|x^{n+1}_2-x_2\|_2\le(1-a^n)\|x^n_2-x_2\|_2+a^n[m^n_2\|x^n_2-x_2\|_2+(L^n_2\lambda_2s_2\mathcal{L}_{T_1}+r^n_2)\|x^n_1-x_1\|_1]$$
$$+a^nb^n_{M_2}+a^n\|e^n_2\|_2, \tag{4.5}$$

where

$$b^n_{M_2} := \|R^{\mathcal{H}^n_2(\cdot,\cdot)\text{-}\phi^n_2\text{-}\eta^n_2}_{M^n_2(x_1,\cdot),\lambda_2}[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2)-\lambda_2\phi_2\circ N_2(u_2,v_1,w_2)$$
$$-\phi_2\circ\mathcal{H}_2(x_2,g_2(x_2))+\phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)]$$
$$-R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2)-\lambda_2\phi_2\circ N_2(u_2,v_1,w_2)$$
$$-\phi_2\circ\mathcal{H}_2(x_2,g_2(x_2))+\phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)]\|_2.$$

It follows from (4.4) and (4.5) that

$$\|x^{n+1}_1-x_1\|_1+\|x^{n+1}_2-x_2\|_2\le[1-a^n(1-k^n_1)]\|x^n_1-x_1\|_1+[1-a^n(1-k^n_2)]\|x^n_2-x_2\|_2$$
$$+a^n(b^n_{M_1}+b^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2)$$
$$\le[1-a^n(1-k^n)](\|x^n_1-x_1\|_1+\|x^n_2-x_2\|_2)$$
$$+a^n(b^n_{M_1}+b^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2), \tag{4.6}$$

where

$$k^n_i := m^n_i+\{L^n_j\lambda_js_j\mathcal{L}_{T_i}+r^n_j\},$$
$$m^n_i := [(1-q_i\delta_i+q_il_{g_i}\times(1+\tau_i^{q_i-1})+c_{q_i}l^{q_i}_{g_i})^{1/q_i}$$
$$+L^n_i\{(1-q_i(\alpha_i-\beta_i)+q_i(\nu_i+\xi_i)\times(1+\tau_i^{q_i-1})+c_{q_i}(\nu_i+\xi_i)^{q_i})^{1/q_i}$$
$$+(1-\lambda_iq_i\mu_i+\lambda_iq_id_i\mathcal{L}_{S_i}\times(1+\tau_i^{q_i-1})+\lambda_i^{q_i}c_{q_i}d_i^{q_i}\mathcal{L}^{q_i}_{S_i})^{1/q_i}$$
$$+\lambda_ip_i\mathcal{L}_{G_i}+\rho_i(1+l_{f_i})+\gamma_i(1+l_{g_i})\}],$$

$k^n=\max(k^n_1,k^n_2)$ and from Lemma 2.17, $b^n_{M_i}\to 0$ as $n\to\infty$.

Clearly, for $k'=\frac{1}{2}(k+1)\in(k,1)$, there exists $N_0\ge 1$ such that $k^n<k'$, for all $n\ge N_0$. Therefore, for any $n\ge N_0$,

$$\|x^{n+1}_1-x_1\|_1+\|x^{n+1}_2-x_2\|_2\le[1-a^n(1-k')]\{\|x^n_1-x_1\|_1+\|x^n_2-x_2\|_2\}$$
$$+a^n\{b^n_{M_1}+b^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2\}$$
$$\le[1-a^n(1-k')]\{\|x^n_1-x_1\|_1+\|x^n_2-x_2\|_2\}$$
$$+a^n(1-k')\frac{\{b^n_{M_1}+b^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2\}}{(1-k')},\quad k'<1.$$

Let $\zeta^n=\|x^n_1-x_1\|_1+\|x^n_2-x_2\|_2$, $\hbar^n=\frac{\{b^n_{M_1}+b^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2\}}{(1-k')}$ and $\omega^n=a^n(1-k')$. Then, we have

$$\zeta^{n+1}\le(1-\omega^n)\zeta^n+\omega^n\hbar^n.$$

Using Lemma 2.18, we have $\zeta^n \to 0$ as $n \to \infty$ (since by Lemma 2.17 $b^n_{M_1}$ and $b^n_{M_2}$ both tend to 0 as $n \to \infty$). This implies $x_1^n \to x_1$, $x_2^n \to x_2$ as $n \to \infty$. Since S_i is $\mathcal{L}_{S_i} - \mathcal{D}$-Lipschitz continuous, it follows from MTPIA 4.1 that

$$\left\|u_i^n - u_i\right\|_i \le \mathcal{D}(S_i(x_i^n), S_i(x_i))_i \le \mathcal{L}_{S_i}\left\|x_i^n - x_i\right\|_i.$$

This implies that $u_i^n \to u_i$ as $n \to \infty$. Further, claim that $u_i \in S_i(x_i)$. We have

$$\begin{aligned} d(u_i, S_i(x_i)) &\le \left\|u_i - u_i^n\right\|_i + d(u_i^n, S_i(x_i))_i \\ &\le \left\|u_i - u_i^n\right\|_i + \mathcal{D}(S_i(x_i^n), S_i(x_i))_i \\ &\le \left\|u_i - u_i^n\right\|_i + \mathcal{L}_{S_i}\left\|x_i^n - x_i\right\|_i \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $S_i(x_i)$ is compact, we have $u_i \in S_i(x_i)$. Similarly, we can prove that $v_i \in T_i(x_i), w_i \in G_i(x_i)$. Thus the approximate solution $(x_i^n, u_i^n, v_i^n, w_i^n)$ generated by MTPIA (4.1) converges strongly to a solution (x_i, u_i, v_i, w_i) of (2.1).

Now, we demonstrate (b). By using (4.2) and (4.3), and proceeding as in (4.6) we deduce that

$$\begin{aligned} \left\|\bar{x}_1^{n+1} - x_1\right\|_1 &= \Big\|\bar{x}_1^{n+1} - \big[(1-a^n)\bar{x}_1^n + a^n\{\bar{x}_1^n - g_1(\bar{x}_1^n) \\ &\quad + R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(\bar{x}_2^n,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)), B_1(g_1(\cdot)))(\bar{x}_1^n) - \lambda_1\phi_1 \circ N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n) \\ &\quad - \phi_1 \circ \mathcal{H}_1(f_1(\bar{x}_1^n), \bar{x}_1^n) + \phi_1 \circ \mathcal{H}_1(\bar{x}_1^n, g_1(\bar{x}_1^n))\big]\} + a^n e_1^n\big]\Big\|_1 \\ &\quad + \Big\|\big[(1-a^n)\bar{x}_1^n + a^n\{\bar{x}_1^n - g_1(\bar{x}_1^n) \\ &\quad + R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(\bar{x}_2^n,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)), B_1(g_1(\cdot)))(\bar{x}_1^n) - \lambda_1\phi_1 \circ N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n) \\ &\quad - \phi_1 \circ \mathcal{H}_1(f_1(\bar{x}_1^n), \bar{x}_1^n) + \phi_1 \circ \mathcal{H}_1(\bar{x}_1^n, g_1(\bar{x}_1^n))\big]\} + a^n e_1^n\big] - x_1\Big\|_1 \\ &\le \epsilon_1^n + \Big\|\big[(1-a^n)\bar{x}_1^n + a^n\{\bar{x}_1^n - g_1(\bar{x}_1^n) \\ &\quad + R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(\bar{x}_2^n,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)), B_1(g_1(\cdot)))(\bar{x}_1^n) - \lambda_1\phi_1 \circ N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n) \\ &\quad - \phi_1 \circ \mathcal{H}_1(f_1(\bar{x}_1^n), \bar{x}_1^n) + \phi_1 \circ \mathcal{H}_1(\bar{x}_1^n, g_1(\bar{x}_1^n))\big]\} + a^n e_1^n\big] \\ &\quad - \big[(1-a^n)x_1 + a^n\{x_1 - g_1(x_1) \\ &\quad + R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)), B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1 \circ N_1(u_1, v_2, w_1) \\ &\quad - \phi_1 \circ \mathcal{H}_1(f_1(x_1), x_1) + \phi_1 \circ \mathcal{H}_1(x_1, g_1(x_1))\big]\}\big]\Big\|_1 \\ &\le \epsilon_1^n + (1-a^n)\left\|\bar{x}_1^n - x_1\right\|_1 + a^n\big[m_1^n\left\|\bar{x}_1^n - x_1\right\|_1 \\ &\quad + \left(L_1^n\lambda_1 s_1\mathcal{L}_{T_2} + r_1^n\right)\left\|\bar{x}_2^n - x_2\right\|_2\big] + a^n h^n_{M_1} + a^n\left\|e_1^n\right\|_1, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} h^n_{M_1} := \Big\|&R^{\mathcal{H}_1^n(\cdot,\cdot)\text{-}\phi_1^n\text{-}\eta_1^n}_{M_1^n(x_2,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)), B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1 \circ N_1(u_1, v_2, w_1) \\ &- \phi_1 \circ \mathcal{H}_1(f_1(x_1), x_1) + \phi_1 \circ \mathcal{H}_1(x_1, g_1(x_1))\big] \\ &- R^{\mathcal{H}_1(\cdot,\cdot)\text{-}\phi_1\text{-}\eta_1}_{M_1(x_2,\cdot),\lambda_1}\big[\mathcal{H}_1(A_1(g_1(\cdot)), B_1(g_1(\cdot)))(x_1) - \lambda_1\phi_1 \circ N_1(u_1, v_2, w_1) \\ &- \phi_1 \circ \mathcal{H}_1(f_1(x_1), x_1) + \phi_1 \circ \mathcal{H}_1(x_1, g_1(x_1))\big]\Big\|_1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \left\|\bar{x}_2^{n+1} - x_2\right\|_2 &\le \epsilon_2^n + (1-a^n)\left\|\bar{x}_2^n - x_2\right\|_2 + a^n\big[m_2^n\left\|\bar{x}_2^n - x_2\right\|_2 \\ &\quad + \left(L_2^n\lambda_2 s_2\mathcal{L}_{T_1} + r_2^n\right)\left\|\bar{x}_1^n - x_1\right\|_1\big] + a^n h^n_{M_2} + a^n\left\|e_2^n\right\|_2, \end{aligned} \tag{4.8}$$

where

$$
\begin{aligned}
h^n_{M_2} := \Big\| & R^{\mathcal{H}^n_2(\cdot,\cdot)\text{-}\phi^n_2\text{-}\eta^n_2}_{M^n_2(x_1,\cdot),\lambda_2}\big[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2)-\lambda_2\phi_2\circ N_2(u_2,v_1,w_2) \\
& -\phi_2\circ\mathcal{H}_2(x_2,g_2(x_2))+\phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)\big] \\
& - R^{\mathcal{H}_2(\cdot,\cdot)\text{-}\phi_2\text{-}\eta_2}_{M_2(x_1,\cdot),\lambda_2}\big[\mathcal{H}_2(A_2(g_2(\cdot)),B_2(g_2(\cdot)))(x_2)-\lambda_2\phi_2\circ N_2(u_2,v_1,w_2) \\
& -\phi_2\circ\mathcal{H}_2(x_2,g_2(x_2))+\phi_2\circ\mathcal{H}_2(f_2(x_2),x_2)\big]\Big\|_2.
\end{aligned}
$$

By Lemma 2.17, $h^n_{M_1}$ and $h^n_{M_2}$ both tend to 0 as $n\to\infty$. Hence from (4.7) and (4.8), for all $n\geq N_0$, we have

$$
\begin{aligned}
\|\bar{x}^{n+1}_1-x_1\|_1+\|\bar{x}^{n+1}_2-x_2\|_2 \leq{} & [1-a^n(1-k')]\{\|\bar{x}^n_1-x_1\|_1+\|\bar{x}^n_2-x_2\|_2\} \\
& +a^n(1-k')\frac{\{h^n_{M_1}+h^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2\}}{(1-k')}+\epsilon^n. \qquad (4.9)
\end{aligned}
$$

Suppose that $\lim\limits_{n\to\infty}\epsilon^n=0$. Let $\zeta^n=\|\bar{x}^n_1-\bar{x}_1\|_1+\|\bar{x}^n_2-\bar{x}_2\|_2$, $\hbar^n=\frac{\{h^n_{M_1}+h^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2\}}{(1-k')}$ and $\omega^n=a^n(1-k')$. Then we have

$$
\zeta^{n+1}\leq(1-\omega^n)\zeta^n+\omega^n\hbar^n.
$$

Using Lemma 2.18, we have $\zeta^n\to 0$ as $n\to\infty$. This implies $\bar{x}^n_1\to x_1,\bar{x}^n_2\to x_2$ as $n\to\infty$. Proceeding as in the convergence of the sequence of $(u^n_i,\,v^n_i,\,w^n_i)$, it follows that $(\bar{u}^n_i,\bar{v}^n_i,\bar{w}^n_i)\to(u_i,v_i,w_i)$ as $n\to\infty$.

Conversely suppose that $(\bar{x}^n_i,\bar{u}^n_i,\bar{v}^n_i,\bar{w}^n_i)\to(x_i,u_i,v_i,w_i)$ as $n\to\infty$. In view of (4.9), we have

$$
\begin{aligned}
\epsilon^n &= \epsilon^n_1+\epsilon^n_2 \\
&\leq \|\bar{x}^{n+1}_1-x_1\|_1+\|\bar{x}^{n+1}_2-x_2\|_2+[1-a^n(1-k')]\{\|\bar{x}^n_1-x_1\|_1+\|\bar{x}^n_2-x_2\|_2\} \\
&\quad +a^n(1-k')\frac{\{h^n_{M_1}+h^n_{M_2}+\|e^n_1\|_1+\|e^n_2\|_2\}}{(1-k')},\quad \forall n\geq N_0.
\end{aligned}
$$

Therefore, we have $\lim\limits_{n\to\infty}\epsilon^n=0$. This completes the proof. □

Remark 4.3. Theorem 4.2 extend, improve and unify Theorem 3.1 of Fang and Huang [7], Theorem 4.1 of Fang *et al.* [8], Theorem 4.1 of Huang [10], Theorem 2.1 of He, Lou and He [9], Theorem 3.1 of Kazmi and Bhat [13], Theorem 3.5-3.8 of Liu, Kang and Ume [20], Theorem 4.3 of Bhat and Zahoor [1], Theorem 4.2 of Bhat and Zahoor [2], Theorem 4.1 of Liu *et al.* [21]. The class of $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator is much wider and more general than those of (A,η)-accretive operator in [5], (H,η)-monotone operator in [8] and $H(\cdot,\cdot)$-accretive operator in [16].

5. Conclusion

In this paper, we consider a new class of variational inclusions which is called a system of generalized variational-like inclusion problem involving $\mathcal{H}(\cdot,\cdot)$-ϕ-η-accretive operator in real q-uniformly smooth Banach spaces. By using the new resolvent operator technique, we prove the existence of solution for this system of inclusions. And also, we discuss the convergence and stability analysis of the iterative sequence generated by perturbed Mann iterative scheme with errors. The our main Theorem 4.2 is an extension, improvement and unification of the well-known results (see [7], [8], [9], [10], [13], [20], [21]).

Acknowledgments

This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. I. Bhat and B. Zahoor, $(H(\cdot,\cdot),\eta)$-monotone operator with an application to a system of set-valued variational-like inclusions, *Nonlinear Functional Analysis and Applications* **22**(3) (2017), 673 – 692, URL: `http://nfaa.kyungnam.ac.kr/journal-nfaa/index.php/NFAA/article/view/993/899`.

[2] M. I. Bhat and B. Zahoor, Existence of solution and iterative approximation of a system of generalized variational-like inclusion problems in semi-inner product spaces, *Filomat* **31**(19) (2017), 6051 – 6070, DOI: 10.2298/FIL1719051B.

[3] S. S. Chang, J. K. Kim and K. H. Kim, On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces, *Journal of Mathematical Analysis and Applications* **268** (2002), 89 – 108, DOI: 10.1006/jmaa.2001.7800.

[4] C. E. Chidume, K. R. Kazmi and H. Zegeye, Iterative approximation of a solution of a general variational-like inclusions in Banach spaces, *International Journal of Mathematics and Mathematical Sciences* **22** (2004), 1159 – 1168, DOI: 10.1155/S0161171204209395.

[5] X. P. Ding and H. R. Feng, The p-step iterative algorithms for a system of generalized mixed quasi-variational inclusions with (A,η)-accretive operators in q-uniformly smooth Banach spaces, *Journal of Computational and Applied Mathematics* **220** (2008), 163 – 174, DOI: 10.1016/j.cam.2007.08.003.

[6] X. P. Ding and C. L. Lou, Perturbed proximal point algorithms for generalized quasi-variational-like inclusions, *Journal of Computational and Applied Mathematics* **113**(1-2) (2000), 153 – 165, DOI: 10.1016/S0377-0427(99)00250-2.

[7] Y. P. Fang and N. P. Huang, H-monotone operator and resolvent operator technique for variational inclusions, *Applied Mathematics and Computation* **145** (2003), 795 – 803, DOI: 10.1016/S0096-3003(03)00275-3.

[8] Y. P. Fang, N. J. Huang and H. B. Thompson, A new system of variational inclusions with (H,η)-monotone operators in Hilbert spaces, *Computers & Mathematics with Applications* **49**(2-3) (2005), 365 – 374, DOI: 10.1016/j.camwa.2004.04.037.

[9] X. F. He, J. Lou and Z. He, Iterative methods for solving variational inclusions in Banach spaces, *Journal of Computational and Applied Mathematics* **203** (2007), 80 – 86, DOI: 10.1016/j.cam.2006.03.011.

[10] N. J. Huang, Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions, *Computers & Mathematics with Applications* **35**(10) (1998), 1 – 7, DOI: 10.1016/S0898-1221(98)00066-2.

[11] N. J. Huang and Y. P. Fang, Generalized m-accretive mappings in Banach spaces, *Journal of Sichuan University* **38**(4) (2001), 591 – 592.

[12] N. J. Huang and Y. P. Fang, A new class of general variational inclusions invloving maximal η-monotone mappings, *Publicationes Mathematicae Debrecen* **62**(1-2) (2003), 83 – 98, URL: `http://publi.math.unideb.hu/load_pdf.php?p=830`.

[13] K. R. Kazmi and M. I. Bhat, Iterative algorithm for a system of nonlinear variational-like inclusions, *Computers & Mathematics with Applications* **48**(12)(2004), 1929 – 1935, DOI: 10.1016/j.camwa.2004.02.009.

[14] K. R. Kazmi and M. I. Bhat, Convergence and stability of iterative algorithms for generalized set-valued variational-like inclusions in Banach spaces, *Applied Mathematics and Computation* **166** (2005), 164 – 180, DOI: 10.1016/j.amc.2004.04.057.

[15] K. R. Kazmi, M. I. Bhat and N. Ahmad, An iterative algorithm based on M-proximal mappings for a system of generalized implicit variational inclusions in Banach spaces, *Journal of Computational and Applied Mathematics* **233** (2009), 361 – 371, DOI: 10.1016/j.cam.2009.07.028.

[16] K. R. Kazmi, F. A. Khan and M. Shahzad, A system of generalized variational inclusions involving generalized $H(\cdot,\cdot)$-accretive mapping in real q-uniformly smooth Banach spaces, *Applied Mathematics and Computation* **217** (2011), 9679 – 9688, DOI: 10.1016/j.amc.2011.04.052.

[17] S. Li, B. Q. Liu, Y. J. Zhang and X. L. Feng, Iterative algorithm and convergence analysis for systems of variational inequalities, *Nonlinear Functional Analysis and Applications* **22**(5) (2017), 947 – 982, URL: `http://nfaa.kyungnam.ac.kr/journal-nfaa/index.php/NFAA/article/view/1023/918`.

[18] L. S. Liu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces, *Journal of Mathematical Analysis and Applications* **194** (1995), 114 – 125, DOI: 10.1006/jmaa.1995.1289.

[19] Z. Liu, R. P. Agarwal and S. M. Kang, On perturbed three-step iterative algorithm for completely generalized nonlinear mixed quasivariational inclusions, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **15**(2) (2008), 229 – 241, URL: `http://online.watsci.org/abstract_pdf/2008v15/v15n2a-pdf/5.pdf`.

[20] Z. Liu, S. M. Kang and J. S. Ume, The solvability of a class of general nonlinear implicit variational inequalities based on perturbed three-step iterative processes with errors, *Fixed Point Theory and Applications* **2008** (2008), Article ID 634921, 13 pages, DOI: 10.1155/2008/634921.

[21] Z. Liu, M. Liu, S. M. Kang and S. Lee, Perturbed Mann iterative method with errors for a new system of generalized nonlinear variational-like inclusions *Mathematical and Computer Modelling* **51** (2010), 63 – 71, DOI: 10.1016/j.mcm.2009.08.041.

[22] S. B. Nadler, Multivalued contraction mappings, *Pacific Journal of Mathematics* **30** (1969), 475 – 488, URL: `https://projecteuclid.org/euclid.pjm/1102978504`.

[23] O. Osilike, Stability results for the Ishikawa fixed point iteration procedure, *Indian Journal of Pure & Applied Mathematics* **26**(10) (1995), 937 – 945, URL: `https://insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20005a1b_937.pdf`.

[24] S. Q. Shan, Y. B. Xiao and N. J. Huang, A new systems of generalized implicit set-valued variational inclusions in Banach spaces, *Nonlinear Functional Analysis and Applications* **22**(5) (2017), 1091 – 1105, URL: `http://nfaa.kyungnam.ac.kr/journal-nfaa/index.php/NFAA/article/view/1031/926`.

[25] H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Analysis: Theory, Methods & Applications* **16**(12) (1991), 1127 – 1138, DOI: 10.1016/0362-546X(91)90200-K.

[26] L. C. Zeng, S. M. Guu and J. C. Yao, Characterization of H-monotone operators with applications to variational inclusions, *Computers & Mathematics with Applications* **50** (2005), 329 – 337, DOI: 10.1016/j.camwa.2005.06.001.

[27] Y. Z. Zou and N. J. Huang, $H(\cdot,\cdot)$-accretive operators with an application for solving variational inclusions in Banach spaces, *Applied Mathematics and Computation* **204** (2008), 809 – 816, DOI: 10.1016/j.amc.2008.07.024.

[28] Y. Z. Zou and N. J. Huang, A new system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces, *Applied Mathematics and Computation* **212** (2009), 135 – 144, DOI: 10.1016/j.amc.2009.02.007.