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Research Article

About Riemann's Zeta-Function and Applications

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Abstract. In this paper we give some remarks on the Riemann's zeta-function related to theoretic arithmetic functions and some applications.

Keywords. Theoretic arithmetic function; Riemann zeta-function; Functional equation; Applications

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1. Introduction

In this work, we give some remarks on the Riemann's zeta function and its extension and certain links with the arithmetic functions and applications.

Leonhard Euler proved the Euler product formula for the Riemann's zeta function in his thesis ([5]) (Various Observations about Infinite Series), published by St. Petersburg Academy in 1737.

Indeed, the theory of the Riemann's zeta function and its generalizations represents an important part in the old and recent mathematical developments. It played a fundamental role for the interest of the prime number theory.

2. Euler's Product Formula and Applications

Theorem 2.1 (Euler's formula (General case))**.**

(i) *Let $f : \mathbb{N}^* \to \mathbb{C}$ be a multiplicative arithmetical function not identically zero such that*

$$\sum_{n\geq 1} |f(n)| < +\infty.$$

Then we have the following Euler's general product formula

$$\sum_{n\geq 1} f(n) = \prod_p (1+f(p)+f(p^2)+\dots), \tag{1}$$

where the infinite product to the second member spread in all first numbers $p \geq 2$ is absolutely converging.

(ii) *If f is completely multiplicative then* (1) *is reduced to*

$$\sum_{n\geq 1} f(n) = \prod_p (1-f(p))^{-1}. \tag{2}$$

Proof. (i) Let us put

$$P(x) = \prod_{2\leq p\leq x} (1+f(p)+f(p^2)+\dots), \quad x \text{ real}.$$

$P(x)$ is the product of a finite number of absolutely converging series.

We can thus multiply them term by term and we obtain, because f is multiplicative,

$$P(x) = \sum_{k>x} f(k),$$

where k runs over all integers ≥ 1 which have no prime factor strictly $> x$. Let us put

$$S = \sum_{n\geq 1} f(n)$$

and train

$$S - P(x) = \sum_p f(s),$$

where s runs over all integers ≥ 1 which have at least a prime factor $> x$. We have naturally $s > x$, from where

$$|S-P(x)| \leq \sum_s |f(s)| \leq \sum_{n\geq x} |f(n)|.$$

Let us make $x \to +\infty$, then $\sum_{n\geq x} |f(n)| \to 0$ (because the series of general term $f(n)$ is convergent and $\sum_{n>x} |f(n)|$ is the rest of order x). Hence $P(x) \to S$ which proves (1).

The infinite product in the second member of (1) is absolutely convergent.

$$\left(\prod_n (1+\alpha_n) \text{ is convergent if } \sum_n |\alpha_n| \text{ converges}\right)$$

because

$$\begin{aligned}\sum_{p\leq x} \left|f(p)+f(p^2)+\dots\right| &\leq \sum_{p\leq x} (|f(p)|+|f(p^2)|+\dots)\\ &\leq \sum_{n\geq 2} |f(n)| \overset{\text{assumption}}{<} +\infty \end{aligned} \tag{3}$$

(ii) Let us suppose f completely multiplicative, then it results from the formula (3) that for every $p \geq 2$ the series $(|f(p)|+|f(p^2)|+\dots)$ is convergent but in our case

$$\forall\ n \geq 1, \quad f(p^n) = (f(p))^n$$

$$\implies \text{ the series } (|f(p)|+|f(p)|^2+\dots) \text{ is convergent}$$

$$\implies \forall\ p \geq 2,\ |f(p)| < 1.$$

But then (1) spells

$$\begin{aligned}\sum_{n\geq 1} f(n) &= \prod_p (1+f(p)+f(p^2)+\dots)\\ &= \prod_p (1+f(p)+(f(p))^2+\dots)\\ &= \prod_p \frac{1}{1-f(p)}\end{aligned}$$

(it is the translation of the unique factorization theorem). □

As applications of this theorem, we have the following propositions:

Proposition 2.2. *Let f be an arithmetic function such that*

$$\sum_{n\geq 1}\left|\frac{f(n)}{n^s}\right| < +\infty, \quad s>1.$$

Then

(i) *if f is multiplicative one has*

$$\sum_{n\geq 1}\frac{f(n)}{n^s} = \prod_p \left(\sum_{m\geq 0}\frac{f(p^m)}{p^{ms}}\right); \tag{4}$$

(ii) *if f is strongly multiplicative one has*

$$\sum_{n\geq 1}\frac{f(n)}{n^s} = \prod_p \left(1+\frac{f(p)}{p^s-1}\right);$$

(iii) *if f is completely multiplicative one has*

$$\sum_{n\geq 1}\frac{f(n)}{n^s} = \prod_p \left(1-\frac{f(p)}{p^s}\right)^{-1}.$$

Proof. Let us apply the expression (1) to the function $\frac{f(n)}{n^s}$, $n\geq 1$, $s>1$ and such that

$$\sum_n \left|\frac{f(n)}{n^s}\right| < +\infty.$$

Then

(i) if f is multiplicative then it is of the same of $\frac{f(n)}{n^s}$, thus the identity

$$\sum_{n\geq 1}\frac{f(n)}{n^s} = \prod_p \left(\sum_{m\geq 0}\frac{f(p^m)}{p^{ms}}\right); \tag{5}$$

(ii) if f is strongly multiplicative, one has

$$\forall\ m\geq 1,\quad f(p^m)=f(p)$$

and the formula (4) spells

$$\begin{aligned}\sum_{n\geq 1}\frac{f(n)}{n^s} &= \prod_p \left(1+f(p)\sum_{m\geq 1}\frac{1}{(p^s)^m}\right)\\ &= \prod_p \left(1+f(p)\left(\frac{1}{1-\frac{1}{p^s}}-1\right)\right) = \prod_p\left(1+\frac{f(p)}{p^s-1}\right);\end{aligned}$$

(iii) if f is completely multiplicative then it is of the same of $\frac{f(n)}{n^s}$, thus
$$\sum_{n\geq 1}\frac{f(n)}{n^s}=\prod_p\left(1-\frac{f(p)}{p^s}\right)^{-1}$$
namely, we obtain
$$f\to\widehat{f}(s)=\sum_{n\geq 1}\frac{f(n)}{n^s},\quad s>1$$

Dirichlet's transform of arithmetical function of major interest in probabilistic number theory. □

Remark 2.1. In Proposition 2.2, let us take the function $f(n)\equiv 1$, it is completely multiplicative. On the other hand, we have
$$\sum_{n\geq 1}\frac{f(n)}{n^s}=\sum_{n\geq 1}\frac{1}{n^s}<+\infty \quad \text{for } s>1,$$
thus the classical Euler product formula:
$$\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},\quad \forall\ s>1.$$

3. ζ Function for Real Arguments >1

Definition 3.1. The Riemann zeta function is the function defined simply by
$$\forall\ \ s \text{ real}>1,\ \zeta(s):=\sum_{n\geq 1}\frac{1}{n^s},$$
it is a function defined on $]1,+\infty[$, continuous and decreasing.

Theorem 3.1. *The Riemann zeta function ζ is derivable for every s real >1 and its derivative can be obtained by derivation term by term in all the interval of definition by*
$$\zeta'(s)=-\sum_{n\geq 1}\frac{\log n}{n^s},\ \forall\ s \textit{ real}>1.$$

Remark 3.1. We have the following important numerical value:
$$\zeta(2)=\sum_{n\geq 1}\frac{1}{n^s}=\frac{\pi^2}{6}.$$

Theorem 3.2 (Euler's product formula (appearance of prime numbers))**.** *We have*
$$\forall\ s>1,\ \ \zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},$$
where the infinite product is to be taken on all the prime numbers $p\geq 2$.

Remark 3.2. The Euler's product formula can be considered as analytical expression of the fundamental theorem of arithmetic.

Proposition 3.3. *There is an infinity of prime numbers.*

Proof. Let us show that there is an infinity of prime numbers by using the fact that $\zeta(2) = \frac{\pi^2}{6}$ is irrational. Let us suppose that there is only a finished number r of prime numbers $p_1, \dots, p_r$. By making $s = 2$ in the Euler's product formula, then

$$\zeta(2) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^2}\right)^{-1} = \frac{\pi^2}{6}$$

contradiction (it is impossible). □

Theorem 3.4 (Connection with Möbius function)**.** *We have*

$$\forall\ s > 1,\ \frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

Proof. We have

- the first equality
$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right)$$
holds from Euler's identity;

- the second equality
$$\prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$$
holds from the Euler's generalized product formula (1). Indeed, let us apply this product formula to the multiplicative function
$$f(n) = \frac{\mu(n)}{n^s}, \quad s > 1,$$
which gives
$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \frac{\mu(p^3)}{p^{3s}} + \dots\right)$$
$$= \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$
□

Remark 3.3. The functions identically equal to 1 or μ verify

$$\forall\ s > 1, \quad \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s),$$

$$\forall\ s > 1, \quad \sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

The functions 1 and μ are inverse of what we shall call "Dirichlet Algebra".

4. Asymptotic Behavior of $\zeta(s)$, $\log\zeta(s)$ and $\zeta'(s)$, when $(s \to 1^{+0})$

For later needs, we look how behave the zeta function ζ and its derivative ζ' in the neighborhood of 1.

Theorem 4.1 (Asymptotic behavior of $\zeta(s)$ and $\log\zeta(s)$ when $(s \to 1^{+0})$). *We have*

(a) $\zeta(s) = \frac{1}{s-1} + O(1)$ *and* $\zeta(s) \sim \frac{1}{s-1}$, *when* $(s \to 1^{+0})$;

(b) $\log\zeta(s) = \log\frac{1}{s-1} + O(s-1)$.

Proof. (a) For every $s > 1$, the function $\frac{1}{x^s}$, $x \geq 1$ is decreasing. We have,

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \int_1^{+\infty} \frac{1}{x^s} dx + \sum_{n=1}^{+\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx = S(s) + R(s).$$

- We have
$$S(x) = \int_1^{+\infty} \frac{1}{x^s} dx = \frac{1}{s-1}.$$
- Let us show that $R(s)$ is positive and bounded. Indeed, for every $n \geq 1$, for all $x \in [n, n+1]$, we have
$$0 < \frac{1}{n^s} - \frac{1}{x^s} = \int_n^x \frac{s}{t^{s+1}} dt < \frac{s}{n^2}$$
which implies
$$0 < \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx < \frac{s}{n^2}.$$
Then,
$$0 < \sum_{n=1}^{+\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx < s\sum_{n=1}^{+\infty} \frac{1}{n^2}.$$
Hence
$$0 < R(s) < s\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty \implies \frac{R(s)}{s} < \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty.$$
Then
$$\zeta(s) = \frac{1}{s-1} + O(1) \text{ and } \zeta(s) \sim \frac{1}{s-1}, \quad \text{as } (s \to 1^{+0}).$$

(b) According to (a), we have
$$(s-1)\zeta(s) = 1 + (s-1)O(1) = 1 + O(s-1).$$
Then,
$$\log(s-1) + \log\zeta(s) = \log(1 + O(s-1)) = O(s-1)$$
and thus,
$$\log\zeta(s) = \log\frac{1}{s-1} + O(s-1), \text{ as } (s \to 1^{+0}).$$
□

Theorem 4.2 (Asymptotic behavior of $\zeta'(s)$, when $(s \to 1^{+0})$). *We have*

$$\zeta'(s) = -\frac{1}{(s-1)^2} + O(1) \text{ and } \zeta'(s) \sim -\frac{1}{(s-1)^2}, \text{ as } (s \to 1^{+0}).$$

Proof. For all $s > 1$, the function $\frac{\log x}{x^s}$, with $x \geq e$, is decreasing, and

$$-\zeta'(s) = \sum_{n \geq 1} \frac{\log n}{n^s}$$

$$= \int_1^{+\infty} \frac{\log x}{x^s} dx + \sum_{n=1}^{+\infty} \int_n^{n+1} \left(\frac{\log n}{n^s} - \frac{\log x}{x^s} \right) dx$$
$$= S(s) + R(s).$$

- Let
$$S(s) = \int_1^{+\infty} \frac{\log x}{x^s} dx.$$
Let us make the change of variable $x^{s-1} = \exp(y)$, then,
$$S(s) = \frac{1}{(s-1)^2} \int_0^{+\infty} y \exp(-y) dy = \frac{1}{(s-1)^2} \quad \text{because} \quad \int_0^{+\infty} y \exp(-y) dy = 1.$$

- Let us show that
$$0 < \frac{R(s)}{s} < K < +\infty.$$
Indeed, for all $n \geq 3$, for all $x \in [n, n+1]$, we have
$$0 < \frac{\log n}{n^s} - \frac{\log x}{x^s} = \int_n^x \frac{s \overbrace{\log t}^{(<\log(n+1))} \overbrace{-1}^{\text{(negligible)}}}{t^{s+1}} dt < \frac{s}{n^2} \log(n+1)$$
and then,
$$0 < \int_n^{n+1} \left(\frac{\log n}{n^s} - \frac{\log x}{x^s} \right) dx < \frac{s}{n^2} \log(n+1)$$
which implies
$$0 < \sum_{n=3}^{+\infty} \int_n^{n+1} \left(\frac{\log n}{n^s} - \frac{\log x}{x^s} \right) dx < s \sum_{n=3}^{+\infty} \frac{\log(n+1)}{n^2}.$$
hence,
$$0 < R(s) < s \sum_{n=3}^{+\infty} \frac{\log(n+1)}{n^2},$$
namely
$$0 < \frac{R(s)}{s} < \sum_{n=3}^{+\infty} \frac{\log(n+1)}{n^2} < +\infty.$$ □

Theorem 4.3. *We have*
$$\log \zeta(s) = \sum_p \frac{1}{p^s} + O(1), \ \text{as} \ (s \to 1^{+0}).$$

Proof. According to the Theorem 3.2, we have
$$\forall \ s > 1, \ \ \zeta(s) = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}$$
$$\Longrightarrow \quad \log \zeta(s) = \sum_p -\log \left(1 - \frac{1}{p^s} \right) = \sum_p \left(\sum_{k \geq 1} \frac{1}{k \ p^{ks}} \right)$$
(according to Fubini we make interchange the signs of summation)
$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \sum_p \left(\sum_{k \geq 2} \frac{1}{k \ p^{ks}} \right).$$

or

$$\sum_{k\geq 2}\frac{1}{k\;p^{ks}} < \sum_{k\geq 2}\frac{1}{p^{ks}} = \frac{1}{p^{2s}(1-\frac{1}{p^s})} \quad\text{and}\quad \frac{1}{p^s}\leq\frac{1}{2^s}$$

$$\Longrightarrow\quad \sum_{k\geq 2}\frac{1}{k\;p^{ks}}\leq\frac{1}{p^{2s}}\cdot\left(1-\frac{1}{2^s}\right)^{-1}.$$

Then

$$\sum_p\left(\sum_{k\geq 2}\frac{1}{k\;p^{ks}}\right)\leq\left(1-\frac{1}{2^s}\right)^{-1}\cdot\sum_p\frac{1}{p^{2s}}<\left(1-\frac{1}{2^s}\right)^{-1}\cdot\zeta(2s)\leq 2\zeta(2)$$

(since $(1-\frac{1}{2^s})^{-1}<2$ and $\zeta(2s)<\zeta(2)$). □

Theorem 4.4 (Behavior of the sum $\sum_p\frac{1}{p^s}$ when $(s\to 1^{+0})$). *We have*

$$\sum_p\frac{1}{p^s}=\log\frac{1}{s-1}+O(1)\;\;\textit{and}\;\;\sum_p\frac{1}{p^s}\sim\log\frac{1}{s-1},\;\;\textit{as}\;\;(s\to 1^{+0}).$$

Proof. See (b) of Theorem 4.1 and also Theorem 4.3. □

Corollary 4.1. *The series $\sum_p\frac{1}{p}$ is divergent.*

Proof. One has,

$$\forall\; s>1,\;\;\frac{1}{p^s}<\frac{1}{p}\;\;\Longrightarrow\;\forall\; s>1,\;\;\sum_p\frac{1}{p^s}\leq\sum_p\frac{1}{p}.$$

Let us make $(s\to 1^{+0})$ according to the Theorem 4.3, the first member tends to $+\infty\Longrightarrow$ the second member $=+\infty\Longrightarrow\sum_p\frac{1}{p}=+\infty$. □

5. Extensions of the ζ Function

Let us consider the Riemann's zeta function

$$\zeta(s):=\sum_{n\geq 1}\frac{1}{n^s},\quad\forall\; s>1. \tag{6}$$

Let us put, from now on, $s=\sigma+it$, $\sigma,\;t\in\mathbb{R}$.

(a) 1^{st} Extension to $\mathfrak{D}:=\{s\in\mathbb{C},\Re e(s)>1\}$

Let be

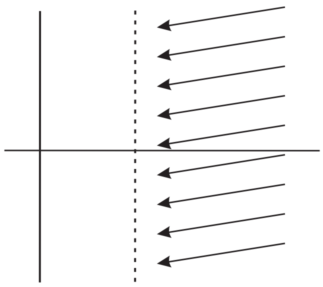

Figure 1. $\mathfrak{D}:=\{s\in\mathbb{C},\Re e(s)>1\}$

Theorem 5.1. *Let* $\mathfrak{D} := \{s \in \mathbb{C}, \Re e(s) > 1\}$. *We can extend the function given by the expression* (6) *in a holomorphic function in* $\mathfrak{D}$ *which we shall continue to call* ζ *and which still admits the representation*

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad s \in \mathfrak{D}.$$

Further, $\forall\ s \in \mathfrak{D}$, $\zeta(s) \neq 0$.

Proof. (i) Let us put $s = \sigma + it$, $\sigma > 1$, then

$$n^s = n^{\sigma+it} = n^{\sigma} e^{it\log n}.$$

Then

$$\sum_{n\geq 1} \frac{1}{|n^s|} = \sum_{n\geq 1} \frac{1}{n^{\sigma}} < +\infty, \quad \sigma > 1.$$

It holds that $\sum_{n\geq 1} \frac{1}{n^s}$ is absolutely convergent for $\Re e(s) > 1$ and uniformly convergent in any half-plan $\sigma \geq 1 + \varepsilon > 1$ where it defines a holomorphic function $\zeta(s)$.

Because $\varepsilon > 0$ is arbitrary, $\zeta(s)$ is holomorphic in all the open half-plan $\sigma > 1$.

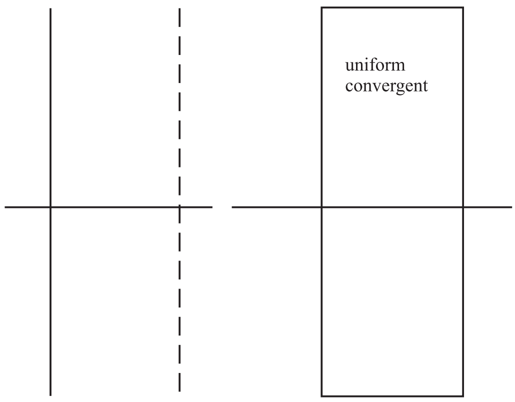

Figure 2

(ii) Because $\sum_{n\geq 1} \frac{1}{n^s}$ is absolutely convergent for $\sigma > 1$, Euler's product formula

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

remains valid for all $\sigma > 1$.

Besides, the infinite product in the second member is absolutely convergent. So for every s, with $\Re e(s) > 1$, $\zeta(s)$ is represented by an absolutely convergent infinite product of factor $(\neq 0) \Longrightarrow \zeta(s) \neq 0$. □

Remark 5.1. Theorems 3.1, 3.2, 4.1 and 4.2 with regard to the real case extend in case where s is complex with $\Re e(s) > 1$.

For needs in applications, we need to the following proposition:

Proposition 5.2 (Formula of integration by parts in Stieltjes integrals)**.** *Let $F(t) := [t]$ be a function of distribution of masses, namely, we have*

$$F(t) := \begin{cases} 1 \ 2 \ 3 \ 4 \ 5 \ldots \ n \\ 1 \ 1 \ 1 \ 1 \ 1 \ldots \ 1 \end{cases}$$

and let φ be of class $\mathscr{C}^1$:

$$\varphi : [1, +\infty[\to \mathbb{C}.$$

We have

$$\forall\, x > 0, \quad \sum_{n \le x} \varphi(n) = \int_{[1,x]} \varphi(t) dF(t) = -\int_1^x \varphi'(t) F(t) dt + \varphi(x) F(x).$$

As application of this proposition, we have the following proposition:

Proposition 5.3. *For $\Re e(s) > 1$, we have*

$$\sum_{n \ge 1} \frac{1}{n^s} = 1 + \frac{1}{s-1} - s \int_1^{+\infty} \frac{\{t\}}{t^{s+1}} dt,$$

where $\{t\}$ indicate the fractional part of t.

Proof. Let us apply Proposition 5.2 to $\varphi(t) = \frac{1}{t^s}$. We have

$$\sum_{n \le x} \frac{1}{n^s} = s \int_1^x \frac{[t]}{t^{s+1}} dt + \frac{[x]}{x^s}.$$

Assume that $\Re e(s) > 1$, then $\sum_{n \ge 1} \frac{1}{n^s}$ is convergent and $\frac{[x]}{x^s} \to 0$ (when $x \to +\infty$) thus

$$\forall\, s, \ \Re e(s) > 1, \quad \sum_{n \ge 1} \frac{1}{n^s} = s \int_1^{+\infty} \frac{[t]}{t^{s+1}} dt.$$

Then since $[t] = t - \{t\}$, (where $[t]$ denotes the integer part of t and $\{t\}$ denotes fractional part of t), and then

$$\sum_{n \ge 1} \frac{1}{n^s} = s \int_1^{+\infty} \frac{1}{t^s} dt - s \int_1^{+\infty} \frac{\{t\}}{t^{s+1}} dt = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{t\}}{t^{s+1}} dt.$$

It holds that

$$\sum_{n \ge 1} \frac{1}{n^s} = 1 + \frac{1}{s-1} - s \int_1^{+\infty} \frac{\{t\}}{t^{s+1}} dt.$$

□

(b) 2^d Extension to $\mathfrak{D}' := \{s \in \mathbb{C}, \Re e(s) > 0\}$

Let be

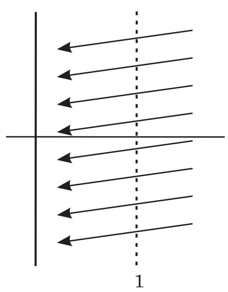

Figure 3. $\mathfrak{D}' := \{s \in \mathbb{C}, \Re e(s) > 0\}$

Theorem 5.4. *Let be* $\mathfrak{D}' := \{s \in \mathbb{C}, \Re e(s) > 0\}$. *We can extend the function* (6) *in an analytic function defined on* $\mathfrak{D}'$ *except for the real point* $s = 1$ *which is reduced to a simple pole of residue* 1. *We also appoint it* ζ.

Proof. One has

$$\sum_{n\geq 1} \frac{1}{n^s} = 1 + \frac{1}{s-1} - s\int_1^{+\infty} \frac{\{t\}}{t^{s+1}} dt, \quad \Re e(s) > 1.$$

- The first member of $\zeta(s)$ is defined on $\mathfrak{D} := \{s \in \mathbb{C} : \Re e(s) > 1\}$.
- But because $0 < \{t\} < 1$, thus it defines a holomorphic function on

  $$\mathfrak{D}' := \{s \in \mathbb{C},\ \Re e(s) > 0\}.$$

  The second member in its entirety defines an analytic function defined on $\mathfrak{D}'$ except for the real point $s = 1$ where it admits a simple pole of residue 1. Let us call $Z(s)$ this function.

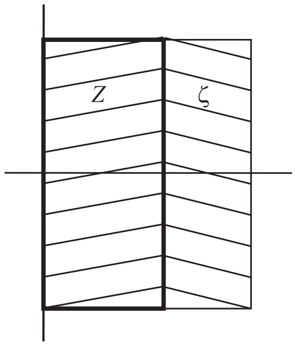

Figure 4

- We can see that $\forall\ s \in \mathfrak{D} := \{s \in \mathbb{C}, \Re e(s) > 1\}$, the function $\zeta(s) = Z(s)$ so that $Z(s)$ is an extension of $\zeta(s)$ on $\mathfrak{D}'$ we shall continue to call ζ the function Z. □

As application, we have the following proposition:

Proposition 5.5 (Application)(Hadammard-de La Vallée-Poussin theorem)**.** *One has*

$$\forall\ t \neq 0, \quad \zeta(1 + it) \neq 0.$$

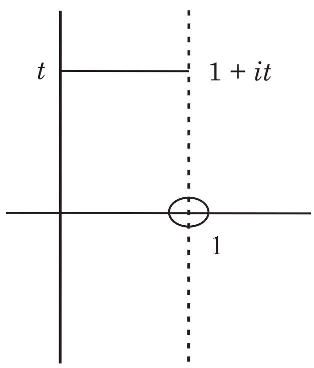

Figure 5

Proof. We have

$$\forall\ \sigma>1,\ \ \zeta(s)=\prod_p(1-p^{-s})^{-1}.$$

Then

$$\log\zeta(s)=\sum_{m\ge1,\ p\ge2}\frac{1}{m\ p^{ms}}\ \ \text{(expanded in entire series)}$$

$$\Longrightarrow\ \log|\zeta(s)|=\Re e(\log\zeta(s))=\Re e\left(\sum_{m,\ p}\frac{1}{m\ p^{ms}}\right)$$

or

$$\sum_{m,p}\frac{1}{m\ p^{ms}}=\sum_{n\ge2}\frac{c_n}{n^s},\ \ \text{where}\ \ c_n=\begin{cases}\frac{1}{m} & \text{if}\ \ n=p^m\\ 0 & \text{else}\end{cases}$$

$$\Longrightarrow\ \log|\zeta(s)|=\Re e\left(\sum_{m\ge2}\frac{c_n}{n^s}\right)\ \ \text{where}\ c_n\ge0$$

or

$$\frac{c_n}{n^s}=\frac{c_n}{n^\sigma}n^{-it}=\frac{c_n}{n^\sigma}e^{-it\log n}=\frac{c_n}{n^\sigma}(\cos(t\log n)-i\sin(t\log n))$$

$$\Longrightarrow\ \log|\zeta(s)|=\sum_{n\ge2}\frac{c_n}{n^\sigma}\cos(t\log n).$$

Then, one has

$$\begin{aligned}\log|\zeta^3(\sigma)\cdot\zeta^4(\sigma+it)\cdot\zeta(\sigma+2it)|&=3\log|\zeta(\sigma)|+4\log|\zeta(\sigma+it)|+\log|\zeta(\sigma+2it)|\\&=\sum_{n\ge2}\frac{c_n}{n^\sigma}(3+4\cos(t\log n)+\cos(2t\log n)).\end{aligned}$$

This quantity is positive. Indeed,

$$c_n\ge0\ \ \text{and}\ \ \forall\ x\in\mathbb{R},\ \ 3+4\cos x+\cos 2x=2(1+\cos x)^2\ge0.$$

Thus

$$\forall\ \sigma>1,\ \ \left|\zeta^3(\sigma)\cdot\zeta^4(\sigma+it)\cdot\zeta(\sigma+2it)\right|\ge1$$

$$\Longleftrightarrow\ \ \forall\ \sigma>1,\ \ |(\sigma-1)\zeta(\sigma)|^3\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4|\zeta(\sigma+2it)|\ge\frac{1}{\sigma-1}$$

and

$$\forall\ t\ne0,\ \ \zeta(1+it)\ne0.$$

It holds that

$$\forall\ \sigma>1,\ \ |(\sigma-1)\zeta(\sigma)|^3\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4|\zeta(\sigma+2it)|\ge\frac{1}{\sigma-1}.\tag{7}$$

Let us show by the absurd that

$$\forall\ t\ne0,\ \ \zeta(1+it)\ne0.$$

Let us suppose, indeed, that it exists $t_0\ne0$ such that $\zeta(1+it_0)=0$ and let us consider the expression (7) written for $t=t_0$ then it results from the Theorem 3.4 that

$$\frac{\zeta(\sigma+it_0)}{\sigma-1}=\frac{\zeta(\sigma+it_0)-\zeta(1+it_0)}{(\sigma+it_0)-(1+it_0)}\to\zeta^{'}(1+it_0)\ \text{as}\ \sigma\to1^{+0}$$

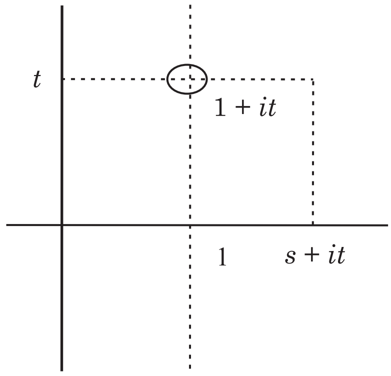

Figure 6

so that by making aim towards σ to 1^{+0} in (7) the first member tends to

$$\left|\zeta'(1+it_0)\right|^4 |\zeta(1+2it_0)| < +\infty.$$

The second member tends to $+\infty$, from where a contradiction because

$$\zeta(1+it_0) = 0 \Longrightarrow \forall\ t \neq 0,\ \ \zeta(1+it) \neq 0.$$ □

Theorem 5.6 (Fundamental Theorem (Riemann))**.** *Let us put*

$$\forall\ s > 1, \quad \zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

We can extend this function in an analytic function in all the complex plan $\mathbb{C}$ which admits a simple pole in the real point $s = 1$, the residue in this pole being equal to 1 we shall continue to call ζ this extension, and for s complex such that $\Re e(s) > 1$ this extension still admits the representation

$$\sum_{n\geq 1} \frac{1}{n^s}.$$

Proof. The basic idea of Riemann is to look for a formula representing the series $\sum_{n\geq 1} \frac{1}{n^s}$ and which remains valid for all the values of s.

(i) $\forall\ s$ real > 1 $\forall\ n$ integer ≥ 1 we have naturally

$$\int_0^{+\infty} x^{s-1} \exp(-nx)dx = \frac{\Gamma(s)}{n^s}.$$

(ii) Let us add for $n \geq 1$,

$$\forall\ s > 1, \ \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s) \sum_{n\geq 1} \frac{1}{n^s} = \Gamma(s)\zeta(s).$$

(iii) Calculation of the integral: Let us call $\mathscr{C}$ the outline, furthermore by writing $(-z)^s$ for $e^{s\log(-z)}$, and by defining $\log z$ in the complex plan deprived of its real negative half-line as the branch which is real for the positive real values we find

$$\int_{\mathscr{C}} \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = 2i \sin(\pi s) \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

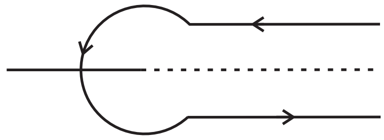

Figure 7. $\mathscr{C}$ Domain of integration

(iv) According to (ii) we find

$$\forall\ s>1,\ \int_{\mathscr{C}}\frac{(-z)^s}{e^z-1}\frac{dz}{z}=2i\sin(\pi s)\Gamma(s)\zeta(s).$$

Then by multiplying by $\Gamma(1-s)$ and by using what we call the complement formulae

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s},$$

hence

$$\forall\ s>1,\quad \zeta(s)=\frac{\Gamma(1-s)}{2i\pi}\int_{\mathscr{C}}\frac{(-z)^s}{e^z-1}\frac{dz}{z}$$

this is the formula which represents the formula of $\zeta(s)$.

(v) This formula is established for s real >1, or we have note that the second member defines an analytic function in $\mathscr{C}\setminus\{1\}$.

The real point 1 being a simple pole. Indeed,

(a) the integral

$$\int_{\mathscr{C}}\frac{(-z)^s}{e^z-1}\frac{dz}{z}$$

is convergent for all s complex and defines an analytic function of s complex.

(b) It holds that the second member is defined and analytic safe can be in the point $s=1,2,3,\dots$ which are simple poles of the function $\Gamma(1-s)$, or for $s=2,3,\dots$ the first member which is equal to

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$$

has not pole (we have concludes that the integral $\int_{\mathscr{C}}\frac{(-z)^s}{e^z-1}\frac{dz}{z}$ inevitably has to have 0 for $s=2,3,\dots$ to know the poles of $\zeta(1-s)$ in these points).

(c) The only possible singular point is thus $s=1$. In this point the formula

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$$

shows that $\lim_{(s\to 1^+)}\zeta(s)=+\infty$. Thus actually is a pole. According to Theorem 5.4, it is a simple pole of residue 1 (see [6]). □

6. Bernoulli's Numbers

The function

$$f(x)=\frac{x}{e^x-1}$$

is analytic in the neighborhood of $x=0$. The singularity that closest to the being origin is $x=\pm 2i\pi$. We can develop it in entire series for x, $|x|<2\pi$.

$$\frac{x}{e^x-1}=\sum_{n\geq 0}B_n\frac{x^n}{n!}.$$

The coefficients B_n are called the Bernoulli's numbers, these numbers are rational.

In particular, we have

$$B_0=1,$$
$$B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{80},$$
$$B_1=-\frac{1}{2},\ B_3=0,\ B_5=0,\ \ldots$$

- The Bernoulli's numbers of odd indices are zeros except the first one.
- The Bernoulli's numbers of even indices alternate of sign from the second who is positive

7. Applications

Values of $\zeta(s)$ in points $s\in\mathbb{Z}$

The values of $\zeta(s)$ for the negative numbers and the even positive integers are given by mean of the Bernoulli's numbers.

Proposition 7.1. *The values of $\zeta(s)$ for integers ≤ 0 are of the shape*

$$\zeta(-n)=(-1)^n\frac{B_{n+1}}{n+1},\quad n=0,1,2,\ldots.$$

Proof. Using the formula which represents the function ζ for n integer ≤ 0

$$\begin{aligned}\zeta(-n)&=\frac{\Gamma(1+n)}{2i\pi}\int_{\mathscr{C}}\frac{(-z)^{-n}}{e^z-1}\frac{dz}{z}\\&=\frac{n!}{2i\pi}(-1)^n\lim_{(\varepsilon\to 0)}\int_{|z|=\varepsilon}\left(\frac{z}{e^z-1}\right)\frac{z^{-n}}{z}\frac{dz}{z}\\&=\frac{n!}{2i\pi}(-1)^n\lim_{(\varepsilon\to 0)}\int_{|z|=\varepsilon}\left(\sum_{k\geq 0}B_k\frac{z^k}{k!}\right)z^{-n-1}\frac{dz}{z}\\&=\frac{n!}{2i\pi}(-1)^n\lim_{(\varepsilon\to 0)}\sum_{k\geq 0}\frac{B_k}{k!}\int_{|z|=\varepsilon}z^{k-n-1}\frac{dz}{z}\end{aligned}$$

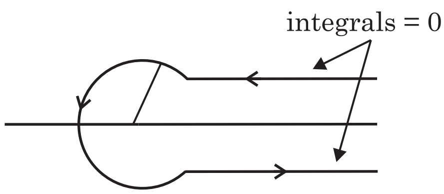

Figure 8

We make the change of variable

$$z=\varepsilon e^{i\theta}\Longrightarrow dz=id\theta$$

or

$$\int_{|z|=\varepsilon} z^{k-n-1}\frac{dz}{z} = \varepsilon^{k-n-1} i \int_0^{2\pi} e^{i(k-n-1)\cdot\theta} d\theta$$

$$= \begin{cases} 2i\pi & \text{if } k = n+1 \\ 0 & \text{else} \end{cases} \text{independent from } \varepsilon > 0.$$

$$\Longrightarrow \qquad \zeta(-n) = \frac{n!}{2i\pi}(-1)^n \frac{B_{n+1}}{(n+1)!} 2\pi = \frac{(-1)^n}{n+1} B_{n+1}. \qquad \square$$

Particular Values

$$\zeta(0) = -\frac{1}{2}, \;\; \zeta(-1) = -\frac{1}{12}, \;\; \zeta(-3) = \frac{1}{120}, \;\; \zeta(-2n) = 0, \;\; n = 1,2,\dots.$$

We notice that the function ζ takes zero on the even strictly negative integers. The even strictly negative integers are called trivial zeros of the function ζ.

Proposition 7.2. *The values of $\zeta(s)$ for even integers > 0 are the shape*

$$\zeta(2n) = (-1)^{n+1} 2^{2n-1} \pi^{2n} \frac{B_{2n}}{(2n)!}, \qquad n = 1,2,\dots.$$

Remark 7.1. This formula is called Euler formula (Not inevitably to know the extension because Euler lived before Riemann). Namely, because the argument is real > 1,

$$\sum_{k \ge 1} \frac{1}{k^{2n}} = (-1)^{n+1} 2^{2n-1} \pi^{2n} \frac{B_{2n}}{(2n)!}.$$

Particular Values

$$\zeta(2) = \frac{\pi^2}{6}, \;\; \zeta(4) = \frac{\pi^4}{90}, \;\; \zeta(6) = \frac{\pi^6}{945}, \;\; \zeta(8) = \frac{\pi^8}{9450}.$$

Remark 7.2. We do not know explicit expression for the values of the zeta function for odd integers > 0.

8. Functional Equation verified by ζ

Having found the trivial zeros of ζ, Riemann proposes that when the real part of s is negative, the integral, instead of being taken in the positive sense around the assigned domain of sizes, can be taken in the negative sense around the domain of sizes which contains all the remaining complex sizes, because the integral, for values the module of which is infinitely big, is then infinitesimal. It holds the following functional equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-s/2}\Pi\left(\frac{s}{2}-1\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Pi\left(\frac{1-s}{2}-1\right)\zeta(1-s).$$

- The function $\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is invariant by the substitution $s \to (1-s)$. Namely, it is symmetric compared with $\Re e(s) = \frac{1}{2}$, furthermore it is analytic in $\mathbb{C}$ private of both points $\{0\}$, $\{1\}$ where it admits simple poles.
- We shall notice that points $s = -2n$, $n \ge 1$ are at the same time poles of $\Gamma\left(\frac{s}{2}\right)$ and zeros of $\zeta(s)$ what makes that by composition the function is still defined in these points.

Map of the Function ζ

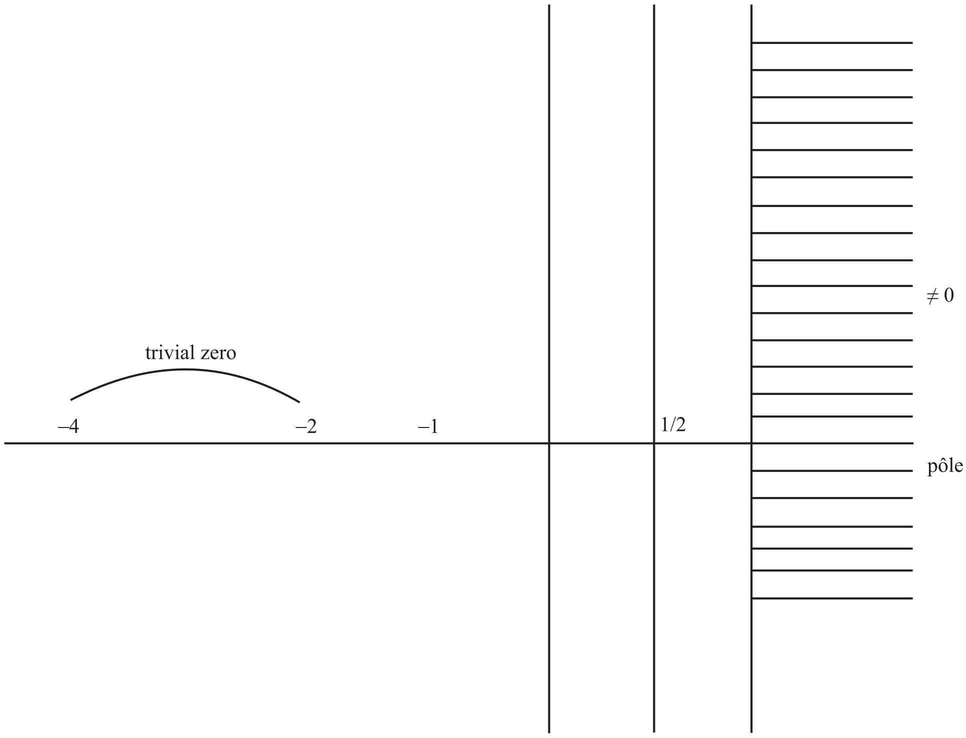

Figure 9

We have

(a) the only singular point is $s = 1$ it is a simple pole of the residue 1,

(b) for all s with $\Re e(s) > 1$, $\zeta(s) \neq 0$,

(c) for all real $t \neq 0$, $\zeta(1 + it) \neq 0$ it is the Hadammard-De La Vallée Poussin theorem ([4], [6]),

(d) the strictly negative even numbers are zeros of the function ζ, they are the trivial zeros. Other zeros are symmetric with regard to the line $\Re e(s) = \frac{1}{2}$ (see functional equation).

According to (b) and (c) ζ does not cancel for $\Re e(s) \geq 1$, other zeros are situated in the open band $0 < \Re e(s) < 1$.

9. The Riemann Hypothesis

Bernhard Riemann is one of the best mathematicians of his time. He knew well the domain of the Fourier transform and he had managed connect the number theory to that of the complex variable functions.

In a sense, he found that the Fourier transform of all the prime numbers, and it in its work ([8]) the only article of which is on the number theory which he wrote.

He understood that the zeta function had important links with prime numbers, his work resulted in an exact formula which allows to find the location of every prime number:

The function li, called logarithmic integral function or integral logarithmic, is defined by

$$li(x) := \int_0^\infty \frac{dt}{\ln t}.$$

s are the non-trival zeros of the Riemann zeta function

$$\forall s : \zeta(s) = 0, \quad 0 < \Re e(s) < 1.$$

Ψ is a function which allows to find the counting function π, we shall not need its precise definition

$$\Psi(x) = li(x) - \sum_s li(x^s) - \log(2) - \int_x^\infty \frac{dt}{t(t^2-1)\lg t}.$$

This formula has unfortunately an important problem: she contains a sum on the non trivial zeros s (namely the real part $0 < \Re e(s) < 1$) of the Riemann zeta function.

The study of the zeros of the Riemann zeta function takes then a lot of importance, the hypothesis that Riemann had made was that all zeros with their real part equals to $\frac{1}{2}$.

The Riemann Hypothesis ([1], [2]) consists in postulating that these zeros are all situated on this line $\Re e(s) = \frac{1}{2}$.

The Riemann Hypothesis is equivalent to the following result :

$$\pi(x) = Li(x) + O(\sqrt{x} \cdot \ln(x)),$$

where π is the counting function of the prime numbers and $Li(x) = li(x) - li(2)$ the Eulerian logarithmic integral.

The Riemann Hypothesis completes then the Theorem of Prime numbers by specifying the scale of the fluctuations between $\pi(x)$ and $li(x)$.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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