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Research Article

On Hyperbolic Numbers With Generalized Fibonacci Numbers Components

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Abstract. In this paper, we introduce the generalized hyperbolic Fibonacci numbers over the bidimensional Clifford algebra of hyperbolic numbers. As special cases, we deal with hyperbolic Fibonacci and hyperbolic Lucas numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we give Catalan's, Cassini's, d'Ocagne's, Gelin-Cesàro's, Melham's identities and present matrices related with these sequences.

Keywords. Hyperbolic numbers, Hyperbolic Fibonacci numbers, Fibonacci numbers, Lucas numbers, Cassini identity

Mathematics Subject Classification (2020). 11B39; 11B83

1. Introduction

It has been proven (see, for example, [12]) that there exist essentially three possible ways to generalize real numbers into real algebras of dimension 2. In fact, each possible system can be reduced to one of the following:

- numbers $a+bi$ with $i^2=-1$ (complex numbers);
- numbers $a+bh$ with $h^2=1$, (hyperbolic numbers);
- numbers $a+b\varepsilon$ with $\varepsilon^2=0$, (dual numbers).

There are also other generalizations (extensions) of real numbers into real algebras of higher dimension. The hypercomplex numbers systems [12], are extensions of real numbers.

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Some commutative examples of hypercomplex number systems are complex numbers, hyperbolic numbers [18], and dual numbers [8]. Some non-commutative examples of hypercomplex number systems are quaternions [9], octonions [3] and sedenions [20]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [4], [10], [15]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [9] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [6]. H. H. Cheng and S. Thompson [5] introduced dual numbers with complex coefficients and called complex dual numbers. Akar et al. [2] introduced dual hyperbolic numbers.

Here we use the set of hyperbolic numbers. The set of hyperbolic numbers $\mathbb{H}$ can be described as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

The hyperbolic ring $\mathbb{H}$ is a bidimensional Clifford algebra (see [13] for details). Hyperbolic numbers has been called in the mathematical literature with different names: Lorentz numbers, double numbers, duplex numbers, split complex numbers and perplex numbers. Hyperbolic numbers are useful for measuring distances in the Lorentz space-time plane (see Sobczyk [18]). For more information on hyperbolic numbers (see also [11], [14], [16], [19]).

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2),$$

$$z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h(x_1y_2 + y_1x_2).$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2} = \frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{x_1y_2 + y_1x_2}{x_2^2 - y_2^2}.$$

It is easy to see that this algebra of hyperbolic numbers is commutative and contains zero divisors. The hyperbolic conjugation of $z = x + hy$ is defined by

$$\overline{z} = z^{\dagger} = x - hy.$$

Note that $\overline{\overline{z}} = z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2},$$

$$\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2},$$

$$\|z\|^2 = z \times \overline{z} = x^2 - y^2.$$

Now let us recall the definition of generalized Fibonacci numbers.

A generalized Fibonacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = V_{n-1} + V_{n-2};\quad V_0 = a,\ V_1 = b,\quad (n \geq 2) \tag{1.1}$$

with the initial values V_0, V_1 not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} + V_{-(n-2)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Fibonacci numbers with positive subscript and negative subscript are given in Table 1.

Table 1. A few generalized Fibonacci numbers

n	V_n	V_{-n}
0	V_0	$\ldots$
1	V_1	$-V_0 + V_1$
2	$V_0 + V_1$	$2V_0 - V_1$
3	$V_0 + 2V_1$	$-3V_0 + 2V_1$
4	$2V_0 + 3V_1$	$5V_0 - 3V_1$
5	$3V_0 + 5V_1$	$-8V_0 + 5V_1$
6	$5V_0 + 8V_1$	$13V_0 - 8V_1$

If we set $V_0 = 0, V_1 = 1$ then $\{V_n\}$ is the well-known Fibonacci sequence and if we set $V_0 = 2, V_1 = 1$ then $\{V_n\}$ is the well-known Lucas sequence. In other words, Fibonacci sequence $\{F_n\}_{n\geq 0}$ (OEIS: A000045, [17]) and Lucas sequence $\{L_n\}_{n\geq 0}$ (OEIS: A000032, [17]) are defined by the second-order recurrence relations

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1 \tag{1.2}$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1. \tag{1.3}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$F_{-n} = -F_{-(n-1)} + F_{-(n-2)}$$

and

$$L_{-n} = -L_{-(n-1)} + L_{-(n-2)}$$

for $n = 1,2,3,\ldots$, respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Fibonacci numbers that are needed.

- Binet formula of generalized Fibonacci sequence can be calculated using its characteristic equation which is given as

  $$t^2 - t - 1 = 0.$$

  The roots of characteristic equation are

  $$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

  Using these roots and the recurrence relation, Binet formula can be given as

  $$V_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.4}$$

  where $A = V_1 - V_0\beta$ and $B = V_1 - V_0\alpha$.

- Binet formula of Fibonacci and Lucas sequences are
$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$
and
$$L_n = \alpha^n + \beta^n$$
respectively.
- The generating function for generalized Fibonacci numbers is
$$g(t) = \frac{V_0 + (V_1 - V_0)t}{1 - t - t^2}. \tag{1.5}$$
- The Cassini identity for generalized Fibonacci numbers is
$$V_{n+1}V_{n-1} - V_n^2 = (V_0V_1 - V_1^2 - V_0^2). \tag{1.6}$$
- $$A\alpha^n = \alpha V_n + V_{n-1}, \tag{1.7}$$
$$B\beta^n = \beta V_n + V_{n-1}. \tag{1.8}$$

In this paper, we define the hyperbolic generalized Fibonacci numbers in the next section and give some properties of them.

2. Hyperbolic Generalized Fibonacci Numbers and their Generating Functions and Binet's Formulas

In this section, we define hyperbolic generalized Fibonacci numbers and present generating functions and Binet formulas for them. In [1], the author defined hyperbolic Fibonacci numbers and Dikmen [7] defined hyperbolic Jacobsthal numbers.

We now define hyperbolic generalized Fibonacci numbers over $\mathbb{H}$. The nth hyperbolic generalized Fibonacci number is
$$\widetilde{V}_n = V_n + hV_{n+1} \tag{2.1}$$
with initial conditions $\widetilde{V}_0 = V_0 + hV_1$, $\widetilde{V}_1 = V_1 + h(V_1 + V_0)$, where $h^2 = 1$. As special cases, the nth hyperbolic Fibonacci numbers and the nth hyperbolic Lucas numbers are given as
$$\widetilde{F}_n = F_n + hF_{n+1}$$
and
$$\widetilde{L}_n = L_n + hL_{n+1}$$
respectively. It can be easily shown that
$$\widetilde{V}_n = \widetilde{V}_{n-1} + \widetilde{V}_{n-2}. \tag{2.2}$$
The sequence $\{\widetilde{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining
$$\widetilde{V}_{-n} = -\widetilde{V}_{-(n-1)} + \widetilde{V}_{-(n-2)}$$
for $n = 1,2,3,\ldots$, respectively. Therefore, recurrence (2.2) holds for all integer n. Note that
$$\widetilde{V}_n h = V_{n+1} + V_n h.$$

The first few hyperbolic generalized Fibonacci numbers with positive subscript and negative subscript are given in Table 2.

Table 2. A few hyperbolic generalized Fibonacci numbers

n	$\widetilde{V}_n$	$\widetilde{V}_{-n}$
0	V_0+hV_1	$\dots$
1	$V_1+h(V_0+V_1)$	$V_1-V_0+hV_0$
2	$V_0+V_1+h(V_0+2V_1)$	$2V_0-V_1+h(-V_0+V_1)$
3	$V_0+2V_1+h(2V_0+3V_1)$	$2V_1-3V_0+h(2V_0-V_1)$
4	$2V_0+3V_1+h(3V_0+5V_1)$	$5V_0-3V_1+h(-3V_0+2V_1)$
5	$3V_0+5V_1+h(5V_0+8V_1)$	$5V_1-8V_0+h(5V_0-3V_1)$
6	$5V_0+8V_1+h(8V_0+13V_1)$	$13V_0-8V_1+h(-8V_0+5V_1)$

Note that

$$\widetilde{V}_0 = V_0+hV_1,$$
$$\widetilde{V}_1 = V_1+hV_2 = V_1+h(V_0+V_1).$$

For hyperbolic Fibonacci numbers (taking $V_n = F_n$, $F_0 = 0, F_1 = 1$), we get

$$\widetilde{F}_0 = h,$$
$$\widetilde{F}_1 = 1+h,$$

and for hyperbolic Lucas numbers (taking $V_n = L_n$, $L_0 = 2, L_1 = 1$), we get

$$\widetilde{L}_0 = 2h,$$
$$\widetilde{L}_1 = 1+3h.$$

A few hyperbolic Fibonacci numbers and hyperbolic Lucas numbers with positive subscript and negative subscript are given in Table 3 and Table 4.

Table 3. Hyperbolic Fibonacci numbers

n	$\widetilde{F}_n$	$\widetilde{F}_{-n}$
0	h	$\dots$
1	$1+h$	1
2	$1+2h$	$-1+h$
3	$2+3h$	$2-h$
4	$3+5h$	$-3+2h$
5	$5+8h$	$5-3h$
6	$8+13h$	$-8+5h$

Table 4. Hyperbolic Lucas numbers

n	$\widetilde{L}_n$	$\widetilde{L}_{-n}$
0	$2+h$	$\dots$
1	$1+3h$	$-1+2h$
2	$3+4h$	$3-h$
3	$4+7h$	$-4+3h$
4	$7+11h$	$7-4h$
5	$11+18h$	$-11+7h$
6	$18+29h$	$18-11h$

Now, we will state Binet's formula for the hyperbolic generalized Fibonacci numbers and in the rest of the paper, we fix the following notations:

$$\widetilde{\alpha} = 1+\alpha h,$$
$$\widetilde{\beta} = 1+\beta h.$$

Note that we have the following identities:

$$\begin{aligned}
\widetilde{\alpha} &= 1+\alpha h, \\
\widetilde{\beta} &= 1+\beta h, \\
\widetilde{\alpha}\widetilde{\beta} &= h, \\
\widetilde{\alpha}^2 &= \alpha+2+2\alpha h, \\
\widetilde{\beta}^2 &= \beta+2+2\beta h, \\
\widetilde{\alpha}^2\widetilde{\beta} &= \alpha+h, \\
\widetilde{\alpha}\widetilde{\beta}^2 &= \beta+h, \\
\widetilde{\alpha}^2\widetilde{\beta}^2 &= 1.
\end{aligned}$$

Theorem 1 (Binet's formula). *For any integer n, the nth hyperbolic generalized Fibonacci number is*

$$\widetilde{V}_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha-\beta}. \tag{2.3}$$

Proof. Using Binet's formula

$$V_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta}$$

of the generalized Fibonacci numbers, we obtain

$$\begin{aligned}
\widetilde{V}_n &= V_n + hV_{n+1} \\
&= \frac{A\alpha^n - B\beta^n}{\alpha-\beta} + h\frac{A\alpha^{n+1} - B\beta^{n+1}}{\alpha-\beta} \\
&= \frac{A(1+\alpha h)\alpha^n - B(1+\beta h)\beta^n}{\alpha-\beta}.
\end{aligned}$$

This proves (2.3). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic Fibonacci number is

$$\widetilde{F}_n = \frac{\widetilde{\alpha}\alpha^n - \widetilde{\beta}\beta^n}{\alpha-\beta} \tag{2.4}$$

and the Binet's Formula of nth hyperbolic Lucas number is

$$\widetilde{L}_n = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n. \tag{2.5}$$

Next, we present generating function.

Theorem 2. *The generating function for the hyperbolic generalized Fibonacci numbers is*

$$\sum_{n=0}^{\infty} \widetilde{V}_n x^n = \frac{\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x}{1-x-x^2}. \tag{2.6}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widetilde{V}_n x^n$$

be generating function of the hyperbolic generalized Fibonacci numbers.

Then, using the definition of the hyperbolic generalized Fibonacci numbers, and subtracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-x-x^2)g(x) &= \sum_{n=0}^{\infty}\widetilde{V}_n x^n - x\sum_{n=0}^{\infty}\widetilde{V}_n x^n - x^2\sum_{n=0}^{\infty}\widetilde{V}_n x^n \\
&= \sum_{n=0}^{\infty}\widetilde{V}_n x^n - \sum_{n=0}^{\infty}\widetilde{V}_n x^{n+1} - \sum_{n=0}^{\infty}\widetilde{V}_n x^{n+2} \\
&= \sum_{n=0}^{\infty}\widetilde{V}_n x^n - \sum_{n=1}^{\infty}\widetilde{V}_{n-1} x^{n} - \sum_{n=2}^{\infty}\widetilde{V}_{n-2} x^{n} \\
&= (\widetilde{V}_0+\widetilde{V}_1x)-\widetilde{V}_0x+\sum_{n=2}^{\infty}(\widetilde{V}_n-\widetilde{V}_{n-1}-\widetilde{V}_{n-2})x^n \\
&= (\widetilde{V}_0+\widetilde{V}_1x)-\widetilde{V}_0x \\
&= \widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)x\,.
\end{aligned}$$

Note that we used the recurrence relation $\widetilde{V}_n=\widetilde{V}_{n-1}+\widetilde{V}_{n-2}$. Rearranging above equation, we get

$$g(x)=\frac{\widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)x}{1-x-x^2}.$$

□

As special cases, the generating functions for the hyperbolic Fibonacci and hyperbolic Lucas numbers are

$$\sum_{n=0}^{\infty}\widetilde{F}_n x^n=\frac{h+x}{1-x-x^2}$$

and

$$\sum_{n=0}^{\infty}\widetilde{L}_n x^n=\frac{(2+h)+(-1+2h)x}{1-x-x^2}$$

respectively.

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of hyperbolic generalized Fibonacci number $\{\widetilde{V}_n\}$ by the use of generating function for $\widetilde{V}_n$.

Theorem 3 (Binet formula of hyperbolic generalized Fibonacci numbers)**.**

$$\widetilde{V}_n=\frac{d_1\alpha^n}{(\alpha-\beta)}-\frac{d_2\beta^n}{(\alpha-\beta)} \tag{3.1}$$

where

$$\begin{aligned}
d_1 &= \widetilde{V}_0\alpha+(\widetilde{V}_1-\widetilde{V}_0), \\
d_2 &= \widetilde{V}_0\beta+(\widetilde{V}_1-\widetilde{V}_0).
\end{aligned}$$

Proof. Let

$$h(x)=1-x-x^2.$$

Then for some α and β we write

$$h(x)=(1-\alpha x)(1-\beta x)$$

i.e.,

$$1-x-x^2=(1-\alpha x)(1-\beta x) \tag{3.2}$$

Hence $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of $h(x)$. This gives α and β as the roots of

$$h\left(\frac{1}{x}\right)=1-\frac{1}{x}-\frac{1}{x^2}=0.$$

This implies $x^2-x-1=0$. Now, by (2.6) and (3.2), it follows that

$$\sum_{n=0}^{\infty}\widetilde{V}_n x^n=\frac{\widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)x}{(1-\alpha x)(1-\beta x)}.$$

Then we write

$$\frac{\widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)x}{(1-\alpha x)(1-\beta x)}=\frac{A_1}{(1-\alpha x)}+\frac{A_2}{(1-\beta x)}. \tag{3.3}$$

So

$$\widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)x=A_1(1-\beta x)+A_2(1-\alpha x).$$

If we consider $x=\frac{1}{\alpha}$, we get $\widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)\frac{1}{\alpha}=A_1(1-\beta\frac{1}{\alpha})$. This gives

$$A_1=\frac{\widetilde{V}_0\alpha+(\widetilde{V}_1-\widetilde{V}_0)}{(\alpha-\beta)}=\frac{d_1}{(\alpha-\beta)}.$$

Similarly, we obtain

$$\begin{aligned}
&\widetilde{V}_0+(\widetilde{V}_1-\widetilde{V}_0)\frac{1}{\beta}=A_2\left(1-\alpha\frac{1}{\beta}\right)\\
\Rightarrow\quad &\widetilde{V}_0\beta+(\widetilde{V}_1-\widetilde{V}_0)=A_2(\beta-\alpha)
\end{aligned}$$

and so

$$A_2=-\frac{\widetilde{V}_0\beta+(\widetilde{V}_1-\widetilde{V}_0)}{(\alpha-\beta)}=-\frac{d_2}{(\alpha-\beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty}\widetilde{V}_n x^n=A_1(1-\alpha x)^{-1}+A_2(1-\beta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}\widetilde{V}_n x^n=A_1\sum_{n=0}^{\infty}\alpha^n x^n+A_2\sum_{n=0}^{\infty}\beta^n x^n=\sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$\widetilde{V}_n=A_1\alpha^n+A_2\beta^n$$

and then we get (3.1). □

Note that from (2.3) and (3.1) we have

$$\begin{aligned}
&(V_1-V_0\beta)\widetilde{\alpha}=\widetilde{V}_0\alpha+(\widetilde{V}_1-\widetilde{V}_0),\\
&(V_1-V_0\alpha)\widetilde{\beta}=\widetilde{V}_0\beta+(\widetilde{V}_1-\widetilde{V}_0).
\end{aligned}$$

Next, using Theorem 3, we present the Binet formulas of hyperbolic Fibonacci and hyperbolic Lucas numbers.

Corollary 4. *Binet formulas of hyperbolic Fibonacci and hyperbolic Lucas numbers are*

$$\widetilde{F}_n = \frac{\widetilde{\alpha}\alpha^n - \widetilde{\beta}\beta^n}{\alpha - \beta}$$

and

$$\widetilde{L}_n = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n,$$

respectively.

4. Some Identities

We now present a few special identities for the hyperbolic generalized Fibonacci sequence $\{\widetilde{V}_n\}$. The following theorem presents the Catalan's identity for the hyperbolic generalized Fibonacci numbers.

Theorem 5 (Catalan's identity)**.** *For all integers n and m, the following identity holds*

$$\widetilde{V}_{n+m}\widetilde{V}_{n-m} - \widetilde{V}_n^2 = \frac{(-1)^{n-m+1}((A+B)V_{2m-1} + (A\beta + B\alpha)V_{2m} - 2(-1)^m AB)}{5}h.$$

Proof. Using the Binet Formula

$$\widetilde{V}_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha - \beta}$$

and

$$A\alpha^n = \alpha V_n + V_{n-1},$$
$$B\beta^n = \beta V_n + V_{n-1},$$

we get

$$\begin{aligned}
\widetilde{V}_{n+m}\widetilde{V}_{n-m} - \widetilde{V}_n^2 &= \frac{(A\widetilde{\alpha}\alpha^{n+m} - B\widetilde{\beta}\beta^{n+m})(A\widetilde{\alpha}\alpha^{n-m} - B\widetilde{\beta}\beta^{n-m}) - (A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n)^2}{(\alpha-\beta)^2} \\
&= -AB\widetilde{\alpha}\widetilde{\beta}\frac{(\alpha^m - \beta^m)^2}{(\alpha-\beta)^2}\alpha^{n-m}\beta^{n-m} \\
&= \frac{(-1)^{n-m+1}AB(\alpha^m - \beta^m)^2}{5}\widetilde{\alpha}\widetilde{\beta} \\
&= \frac{(-1)^{n-m+1}((A+B)V_{2m-1} + (A\beta + B\alpha)V_{2m} - 2(-1)^m AB)}{5}h,
\end{aligned}$$

where $\alpha\beta = -1$ and $\widetilde{\alpha}\widetilde{\beta} = h$. □

As special cases of the above theorem, we give Catalan's identity of hyperbolic Fibonacci and hyperbolic Lucas numbers. Firstly, we present Catalan's identity of hyperbolic Fibonacci numbers.

Corollary 6 (Catalan's identity for the hyperbolic Fibonacci numbers)**.** *For all integers n and m, the following identity holds*

$$\widetilde{F}_{n+m}\widetilde{F}_{n-m} - \widetilde{F}_n^2 = (-1)^{n-m+1}F_m^2 h.$$

Proof. Taking $V_n = F_n$ in Theorem 5 and using the identity
$$F_m^2 = \frac{2F_{2m-1} + F_{2m} - 2(-1)^m}{5},$$
we get the required result. □

Secondly, we give Catalan's identity of hyperbolic Lucas numbers.

Corollary 7 (Catalan's identity for the hyperbolic Lucas numbers)**.** *For all integers n and m, the following identity holds*
$$\begin{aligned}\widetilde{L}_{n+m}\widetilde{L}_{n-m} - \widetilde{L}_n^2 &= (-1)^{n-m}(L_{2m} - 2(-1)^m)h \\ &= (-1)^{n-m}(L_m^2 - 4(-1)^m)h.\end{aligned}$$

Proof. Taking $V_n = L_n$ in Theorem 5 and using the identity
$$L_m^2 = L_{2m} + 2(-1)^m,$$
we get the required result. □

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic generalized Fibonacci sequence.

Corollary 8 (Cassini's identity)**.** *For all integers n, the following identity holds*
$$\widetilde{V}_{n+1}\widetilde{V}_{n-1} - \widetilde{V}_n^2 = (-1)^n ABh.$$

As special cases of Cassini's identity, we give Cassini's identity of hyperbolic Fibonacci and hyperbolic Lucas numbers. Firstly, we present Cassini's identity of hyperbolic Fibonacci numbers.

Corollary 9 (Cassini's identity of hyperbolic Fibonacci numbers)**.** *For all integers n, the following identity holds*
$$\widetilde{F}_{n+1}\widetilde{F}_{n-1} - \widetilde{F}_n^2 = (-1)^n h.$$

Secondly, we give Cassini's identity of hyperbolic Lucas numbers.

Corollary 10 (Cassini's identity of hyperbolic Lucas numbers)**.** *For all integers n, the following identity holds*
$$\widetilde{L}_{n+1}\widetilde{L}_{n-1} - \widetilde{L}_n^2 = 5(-1)^{n+1} h.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using the Binet Formula of the hyperbolic generalized Fibonacci sequence:
$$\widetilde{V}_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha - \beta}.$$
The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of the hyperbolic generalized Fibonacci sequence $\{\widetilde{V}_n\}$.

Theorem 11. *Let n and m be any integers. Then the following identities are true:*

(a) (*d'Ocagne's identity*)

$$\widetilde{V}_{m+1}\widetilde{V}_n-\widetilde{V}_m\widetilde{V}_{n+1}=(V_nV_{m-1}-V_mV_{n-1})h.$$

(b) (*Gelin-Cesàro's identity*)

$$\widetilde{V}_{n+2}\widetilde{V}_{n+1}\widetilde{V}_{n-1}\widetilde{V}_{n-2}-\widetilde{V}_n^4=-A^2B^2.$$

(c) (*Melham's identity*)

$$\widetilde{V}_{n+1}\widetilde{V}_{n+2}\widetilde{V}_{n+6}-\widetilde{V}_{n+3}^3=(-1)^nAB\widetilde{V}_nh.$$

Proof. (a) Using (1.7) and (1.8) we obtain

$$\begin{aligned}
\widetilde{V}_{m+1}\widetilde{V}_n-\widetilde{V}_m\widetilde{V}_{n+1}&=\frac{AB\widetilde{\alpha}\widetilde{\beta}(-\alpha^{m+1}\beta^n-\alpha^n\beta^{m+1}+\alpha^m\beta^{n+1}+\alpha^{n+1}\beta^m)}{(\alpha-\beta)^2}\\
&=\frac{AB(\alpha^n\beta^m-\alpha^m\beta^n)}{(\alpha-\beta)}\widetilde{\alpha}\widetilde{\beta}\\
&=\frac{((\alpha V_n+V_{n-1})(\beta V_m+V_{m-1})-(\alpha V_m+V_{m-1})(\beta V_n+V_{n-1}))}{(\alpha-\beta)}h\\
&=(V_nV_{m-1}-V_mV_{n-1})h.
\end{aligned}$$

(b) $$\widetilde{V}_{n+2}\widetilde{V}_{n+1}\widetilde{V}_{n-1}\widetilde{V}_{n-2}-\widetilde{V}_n^4=-A^2B^2\widetilde{\alpha}^2\widetilde{\beta}^2=-A^2B^2.$$

(c) Using (1.7), (1.8) and

$$\widetilde{\alpha}^2\widetilde{\beta}=\alpha+h$$
$$\widetilde{\alpha}\widetilde{\beta}^2=\beta+h$$

we obtain

$$\begin{aligned}
\widetilde{V}_{n+1}\widetilde{V}_{n+2}\widetilde{V}_{n+6}-\widetilde{V}_{n+3}^3&=\frac{(-1)^{n+1}AB(-A\widetilde{\alpha}\alpha^n+B\widetilde{\beta}\beta^n)\widetilde{\alpha}\widetilde{\beta}}{(\alpha-\beta)}\\
&=\frac{(-1)^{n+1}AB(-(\alpha V_n+V_{n-1})\widetilde{\alpha}^2\widetilde{\beta}+(\beta V_n+V_{n-1})\widetilde{\alpha}\widetilde{\beta}^2)}{(\alpha-\beta)}\\
&=(-1)^nAB\widetilde{V}_nh.
\end{aligned}$$

□

As special cases of the above theorem, we give the d'Ocagne's, Gelin-Cesàro's and Melham' identities of hyperbolic Fibonacci and hyperbolic Lucas numbers. Firstly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of hyperbolic Fibonacci numbers.

Corollary 12. *Let n and m be any integers. Then, for the hyperbolic Fibonacci numbers, the following identities are true:*

(a) (*d'Ocagne's identity*)

$$\widetilde{F}_{m+1}\widetilde{F}_n-\widetilde{F}_m\widetilde{F}_{n+1}=(-1)^mF_{n-m}h.$$

(b) (*Gelin-Cesàro's identity*)

$$\widetilde{F}_{n+2}\widetilde{F}_{n+1}\widetilde{F}_{n-1}\widetilde{F}_{n-2}-\widetilde{F}_n^4=-1.$$

(c) (*Melham's identity*)

$$\widetilde{F}_{n+1}\widetilde{F}_{n+2}\widetilde{F}_{n+6}-\widetilde{F}_{n+3}^3=(-1)^n\widetilde{F}_nh.$$

Secondly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of hyperbolic Lucas numbers.

Corollary 13. *Let n and m be any integers. Then, for the hyperbolic Lucas numbers, the following identities are true:*

(a) (*d'Ocagne's identity*)

$$\widetilde{L}_{m+1}\widetilde{L}_n - \widetilde{L}_m\widetilde{L}_{n+1} = (L_nL_{m-1} - L_mL_{n-1})h.$$

(b) (*Gelin-Cesàro's identity*)

$$\widetilde{L}_{n+2}\widetilde{L}_{n+1}\widetilde{L}_{n-1}\widetilde{L}_{n-2} - \widetilde{L}_n^4 = -25.$$

(c) (*Melham's identity*)

$$\widetilde{L}_{n+1}\widetilde{L}_{n+2}\widetilde{L}_{n+6} - \widetilde{L}_{n+3}^3 = 5(-1)^{n+1}\widetilde{V}_nh.$$

5. Linear Sums

In this section, we give the summation formulas of the hyperbolic generalized Fibonacci numbers with positive and negative subscripts. Now, we present the summation formulas of the generalized Fibonacci numbers.

Proposition 14. *For the generalized Fibonacci numbers, we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = V_{n+2} - V_1.$

(b) $\sum_{k=0}^{n} V_{2k} = V_{2n+1} - V_1 + V_0.$

(c) $\sum_{k=0}^{n} V_{2k+1} = V_{2n+2} - V_2 + V_1.$

Proof. For the proof, see Soykan [21]. □

Next, we present the formulas which give the summation of the hyperbolic generalized Fibonacci numbers.

Theorem 15. *For $n \geq 0$, hyperbolic generalized Fibonacci numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \widetilde{V}_k = \widetilde{V}_{n+2} - \widetilde{V}_1.$

(b) $\sum_{k=0}^{n} \widetilde{V}_{2k} = \widetilde{V}_{2n+1} - \widetilde{V}_1 + \widetilde{V}_0.$

(c) $\sum_{k=0}^{n} \widetilde{V}_{2k+1} = \widetilde{V}_{2n+2} - \widetilde{V}_0.$

Proof. (a) Note that using Proposition 14(a) we get

$$\sum_{k=0}^{n} V_k = V_{n+2} - V_1,$$

$$\sum_{k=0}^{n} V_{k+1} = V_{n+3} - (V_1 + V_0).$$

Then it follows that

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{V}_k &= \sum_{k=0}^{n}V_k + h\sum_{k=0}^{n}V_{k+1}\\ &= (V_{n+2}-V_1)+h(V_{n+3}-(V_1+V_0))\\ &= (V_{n+2}+hV_{n+3})-(V_1+h(V_0+V_1))\\ &= \widetilde{V}_{n+2}-(V_1+hV_2)\\ &= \widetilde{V}_{n+2}-\widetilde{V}_1.\end{aligned}$$

This proves (a).

(b) Note that using Proposition 14(b) and (c) we get

$$\sum_{k=0}^{n}V_{2k} = V_{2n+1}-V_1+V_0,$$

$$\sum_{k=0}^{n}V_{2k+1} = V_{2n+2}-V_0.$$

Then it follows that

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{V}_{2k} &= \sum_{k=0}^{n}V_{2k} + h\sum_{k=0}^{n}V_{2k+1}\\ &= (V_{2n+1}-V_1+V_0)+h(V_{2n+2}-V_0)\\ &= (V_{2n+1}+hV_{2n+2})+((-V_1+V_0)+h(-V_0))\\ &= (V_{2n+1}+hV_{2n+2})+((V_0-V_1)+h(V_1-V_2))\\ &= \widetilde{V}_{2n+1}-(V_1+hV_2)+(V_0+hV_1)\\ &= \widetilde{V}_{2n+1}-\widetilde{V}_1+\widetilde{V}_0.\end{aligned}$$

(c) Note that using Proposition 14(b) and (c) we get

$$\sum_{k=0}^{n}V_{2k+2} = V_{2n+3}-V_1.$$

Then it follows that

$$\begin{aligned}\sum_{k=0}^{n}\widetilde{V}_{2k+1} &= \sum_{k=0}^{n}V_{2k+1} + h\sum_{k=0}^{n}V_{2k+2}\\ &= (V_{2n+2}-V_0)+h(V_{2n+3}-V_1)\\ &= (V_{2n+2}+hV_{2n+3})+((-V_0)+h(-V_1))\\ &= \widetilde{V}_{2n+2}-(V_0+hV_1)\\ &= \widetilde{V}_{2n+2}-\widetilde{V}_0.\end{aligned}$$

□

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Fibonacci numbers:

Corollary 16. *For $n \geq 0$, hyperbolic Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n}\widetilde{F}_k = \widetilde{F}_{n+2}-\widetilde{F}_1 = \widetilde{F}_{n+2}-(1+h).$

(b) $\sum_{k=0}^{n}\widetilde{F}_{2k} = \widetilde{F}_{2n+1}-\widetilde{F}_1+\widetilde{F}_0 = \widetilde{F}_{2n+1}-1.$

(c) $\sum\limits_{k=0}^{n} \widetilde{F}_{2k+1} = \widetilde{F}_{2n+2} - \widetilde{F}_0 = \widetilde{F}_{2n+2} - h.$

As a second special case of the above theorem, we have the following summation formulas for hyperbolic Lucas numbers:

Corollary 17. *For $n \geq 0$, hyperbolic Lucas numbers have the following properties:*

(a) $\sum\limits_{k=0}^{n} \widetilde{L}_k = \widetilde{L}_{n+2} - \widetilde{L}_1 = \widetilde{L}_{n+2} - (1+3h).$

(b) $\sum\limits_{k=0}^{n} \widetilde{L}_{2k} = \widetilde{L}_{2n+1} - \widetilde{L}_1 + \widetilde{L}_0 = \widetilde{L}_{2n+1} + (1-2h).$

(c) $\sum\limits_{k=0}^{n} \widetilde{L}_{2k+1} = \widetilde{L}_{2n+2} - \widetilde{L}_0 = \widetilde{L}_{2n+2} - (2+h).$

Now, we present the formula which give the summation formulas of the generalized Fibonacci numbers with negative subscripts.

Proposition 18. *For $n \geq 1$ we have the following formulas:*

(a) $\sum\limits_{k=1}^{n} V_{-k} = \ -2V_{-n-1} - V_{-n-2} + V_1.$

(b) $\sum\limits_{k=1}^{n} V_{-2k} = -V_{-2n-1} + V_1 - V_0.$

(c) $\sum\limits_{k=1}^{n} V_{-2k+1} = -V_{-2n} + V_0.$

Proof. This is given in Soykan [21]. □

Next, we present the formulas which give the summation of the hyperbolic generalized Fibonacci numbers with negative subscripts.

Theorem 19. *For $n \geq 1$, hyperbolic generalized Fibonacci numbers have the following formulas:*

(a) $\sum\limits_{k=1}^{n} \widetilde{V}_{-k} = \ -2\widetilde{V}_{-n-1} - \widetilde{V}_{-n-2} + \widetilde{V}_1.$

(b) $\sum\limits_{k=1}^{n} \widetilde{V}_{-2k} = -\widetilde{V}_{-2n-1} + \widetilde{V}_1 - \widetilde{V}_0.$

(c) $\sum\limits_{k=1}^{n} \widetilde{V}_{-2k+1} = -\widetilde{V}_{-2n} + \widetilde{V}_0.$

Proof. We prove (a), (b) and (c) can be proved similarly. Note that using Proposition 14(a) we get

$$\sum_{k=1}^{n} V_{-k} = -2V_{-n-1} - V_{-n-2} + V_1,$$

$$\sum_{k=1}^{n} V_{-k+1} = -2V_{-n} - V_{-n-1} + V_1 + V_0.$$

Then it follows that

$$\begin{aligned}\sum_{k=1}^{n} \widetilde{V}_{-k} &= \sum_{k=1}^{n} V_{-k} + h \sum_{k=1}^{n} V_{-k+1} \\ &= (-2V_{-n-1} - V_{-n-2} + V_1) + h(-2V_{-n} - V_{-n-1} + V_1 + V_0)\end{aligned}$$

$$
\begin{aligned}
&= -2(V_{-n-1}+hV_{-n})-(V_{-n-2}+hV_{-n-1})+(V_1+h(V_1+V_0))\\
&= -2\widetilde{V}_{-n-1}-\widetilde{V}_{-n-2}+(V_1+hV_2)\\
&= -2\widetilde{V}_{-n-1}-\widetilde{V}_{-n-2}+\widetilde{V}_1.
\end{aligned}
$$

This proves (a). □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Fibonacci numbers:

Corollary 20. *For $n \geq 1$, hyperbolic Fibonacci numbers have the following properties:*

(a) $\sum\limits_{k=1}^{n} \widetilde{F}_{-k} = -2\widetilde{F}_{-n-1}-\widetilde{F}_{-n-2}+\widetilde{F}_1 = -2\widetilde{F}_{-n-1}-\widetilde{F}_{-n-2}+(1+h).$

(b) $\sum\limits_{k=1}^{n} \widetilde{F}_{-2k} = -\widetilde{F}_{-2n-1}+\widetilde{F}_1-\widetilde{F}_0 = -\widetilde{F}_{-2n-1}+1.$

(c) $\sum\limits_{k=1}^{n} \widetilde{F}_{-2k+1} = -\widetilde{F}_{-2n}+\widetilde{F}_0 = -\widetilde{F}_{-2n}+h.$

As a second special case of the above theorem, we have the following summation formulas for hyperbolic Lucas numbers:

Corollary 21. *For $n \geq 1$, hyperbolic Lucas numbers have the following properties.*

(a) $\sum\limits_{k=1}^{n} \widetilde{L}_{-k} = -2\widetilde{L}_{-n-1}-\widetilde{L}_{-n-2}+\widetilde{L}_1 = -2\widetilde{L}_{-n-1}-\widetilde{L}_{-n-2}+(1+3h).$

(b) $\sum\limits_{k=1}^{n} \widetilde{L}_{-2k} = -\widetilde{L}_{-2n-1}+\widetilde{L}_1-\widetilde{L}_0 = -\widetilde{L}_{-2n-1}+(-1+2h).$

(c) $\sum\limits_{k=1}^{n} \widetilde{L}_{-2k+1} = -\widetilde{L}_{-2n}+\widetilde{L}_0 = -\widetilde{L}_{-2n}+(2+h).$

6. Matrices Related with Hyperbolic Generalized Fibonacci Numbers

We define the square matrix C of order 2 as:

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

such that $\det C = -1$. Induction proof may be used to establish

$$C^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{6.1}$$

and (the matrix formulation of V_n)

$$\begin{pmatrix} V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_1 \\ V_0 \end{pmatrix}. \tag{6.2}$$

Now, we define the matrices C_V as

$$C_V = \begin{pmatrix} \widetilde{V}_3 & \widetilde{V}_2 \\ \widetilde{V}_2 & \widetilde{V}_1 \end{pmatrix}.$$

This matric C_V is called hyperbolic generalized Fibonacci matrix.

As special cases, hyperbolic Fibonacci matrix and hyperbolic Lucas matrix are

$$C_F = \begin{pmatrix} \widetilde{F}_3 & \widetilde{F}_2 \\ \widetilde{F}_2 & \widetilde{F}_1 \end{pmatrix}$$

and

$$C_L = \begin{pmatrix} \widetilde{L}_3 & \widetilde{L}_2 \\ \widetilde{L}_2 & \widetilde{L}_1 \end{pmatrix}$$

respectively.

Theorem 22. *For $n \geq 0$, the following is valid:*

$$C_V \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widetilde{V}_{n+3} & \widetilde{V}_{n+2} \\ \widetilde{V}_{n+2} & \widetilde{V}_{n+1} \end{pmatrix}. \tag{6.3}$$

Proof. We prove by mathematical induction on n. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$C_V C^k = \begin{pmatrix} \widetilde{V}_{k+3} & \widetilde{V}_{k+2} \\ \widetilde{V}_{k+2} & \widetilde{V}_{k+1} \end{pmatrix}.$$

If we use (2.1), then we have $\widetilde{V}_{k+2} = \widetilde{V}_{k+1} + \widetilde{V}_k$. Then, by induction hypothesis, we obtain

$$\begin{aligned} C_V C^{k+1} &= (C_V C^k)C \\ &= \begin{pmatrix} \widetilde{V}_{k+3} & \widetilde{V}_{k+2} \\ \widetilde{V}_{k+2} & \widetilde{V}_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{V}_{k+2} + \widetilde{V}_{k+3} & \widetilde{V}_{k+3} \\ \widetilde{V}_{k+1} + \widetilde{V}_{k+2} & \widetilde{V}_{k+2} \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{V}_{k+4} & \widetilde{V}_{k+3} \\ \widetilde{V}_{k+3} & \widetilde{V}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (6.3) holds for all non-negative integers n. □

Remark 23. *The above theorem is true for $n \leq -1$. It can also be proved by induction.*

Corollary 24. *For all integers n, the following holds:*

$$\widetilde{V}_{n+2} = \widetilde{V}_2 F_{n+1} + \widetilde{V}_1 F_n.$$

Proof. The proof can be seen by the coefficient of the matrix C_V and (6.1). □

Taking $V_n = F_n$ and $V_n = L_n$, respectively, in the above corollary, we obtain the following results.

Corollary 25. *For all integers n, the followings are true.*

(a) $\widetilde{F}_{n+2} = \widetilde{F}_2 F_{n+1} + \widetilde{F}_1 F_n$.

(b) $\widetilde{L}_{n+2} = \widetilde{L}_2 F_{n+1} + \widetilde{L}_1 F_n$.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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