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On Pullbacks of Abelian Groups and Pushouts of Rings

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Abstract. In this paper we generalize some results for pullbacks and pushouts known for C*-algebras to the cases of Abelian groups or rings.

1. **Introduction**

Both pullbacks and pushouts are classical constructions in category theory. These constructions have been applied successfully also in other fields of mathematics, like Kasparov's KK-Theory or theory of vector bundles (where pullback is called the Whitney sum).

The present paper has its roots in the paper [3], where pullbacks and pushouts were used in order to study C*-algebras. While studying the results and proofs of [3], it appeared, that the proofs were carried out in quite a general manner and, therefore, did not depend so much on the topological structure. With slight modifications, it was possible to provide pure algebraic proofs without any topology considered. Even the need for algebras was superfluous – in fact, groups or rings for pushouts and Abelian groups for pullbacks were sufficient.

Since the results, presented in this paper, are of algebraic nature, then one can easily get similar results for more specific algebraic or topological structures as corollaries by replacing "Abelian groups" everywhere with "algebras" or "topological algebras" or by replacing "rings" with "Banach algebras" or "C*-algebras" etc. One must just demand that all homomorphisms involved here should be from the appropriate category, i.e., homomorphisms of algebras in case of algebras or homomorphisms of Banach algebras in case of Banach algebras, as well as kernels and cokernels exist.

For some reasons unknown to the author, most of the papers in category theory dealing with pullbacks and pushouts, do not consider the extensions or kernels so

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much. They tend to stay in the category of sets, which is not narrow enough to get the results presented in this paper. On the other hand, papers, which do have analogous results, are mostly written by people dealing with funtional analysis and considering only C*-algebras or Banach algebras, although the results hold in much wider case. The aim of the present paper was to give and to prove the results in as general form as possible.

2. Some results on pullbacks of Abelian groups

Let A, B and C be Abelian groups and maps $\alpha : A \to C$, $\beta : B \to C$ homomorphisms of Abelian groups. Consider a diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

where X is an Abelian group and maps $\gamma : X \to B$ and $\delta : X \to A$ are homomorphisms of Abelian groups. If this diagram is commutative and for any Abelian group Y and Abelian group homomorphisms $\phi : Y \to A$, $\psi : Y \to B$ with $\alpha \circ \phi = \beta \circ \psi$ there exists unique Abelian group homomorphism $\sigma : Y \to X$ with $\phi = \delta \circ \sigma$ and $\psi = \gamma \circ \sigma$, then X (together with γ and δ) is called a pullback of A and B along α and β (for more general definition of a pullback, see [2], p. 71).

It can be shown (see, for example, [1, p. 123], or [2, p. 72, Exercise 1]) that if X is a pullback of A and B along α and β, then X is isomorphic to a canonical pullback of A and B along α and β, which is defined as $A \oplus_C B = \{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$ and γ and δ are restrictions of the canonical projection maps.

We start with a generalization of Proposition 3.1 of [3, p. 256]. As this result shows, it is enough to consider Abelian groups instead of C*-algebras.

Proposition 2.1. *In a commutative diagram of Abelian groups*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

X is a pullback of A and B along α and β if an only if the following conditions hold:

(i) $\ker\gamma \cap \ker\delta = \{\theta_X\}$;
(ii) $\beta^{-1}(\alpha(A)) = \gamma(X)$;
(iii) $\delta(\ker\gamma) = \ker\alpha$.

Proof. Suppose that X with δ and γ in the commutative diagram above is a pullback of A and B along α and β. Then $\alpha \circ \delta = \beta \circ \gamma$ and X is isomorphic to the canonical pullback $A \oplus_C B = \{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$. The isomorphism is realized through the map $\sigma : X \to A \oplus_C B$, which is defined as $\sigma(x) := (\delta(x), \gamma(x))$.

Let $x \in \ker\gamma \cap \ker\delta$. Then $\delta(x) = \theta_A$ and $\gamma(x) = \theta_B$. Hence, $\sigma(x) = (\theta_A, \theta_B) = \theta_{A\oplus_C B}$. Since σ is an isomorphism, we get that $x = \theta_X$. Thus, (i) holds.

Take any $b \in \gamma(X)$. Then there exists $x \in X$ such that $\gamma(x) = b$. Now, $\beta(\gamma(x)) = \alpha(\delta(x)) \in \alpha(A)$. Thus, $b \in \beta^{-1}(\alpha(A))$. Consequently, $\gamma(X) \subseteq \beta^{-1}(\alpha(A))$.

Take any $b \in \beta^{-1}(\alpha(A))$. Then $\beta(b) \in \alpha(A)$ and there exists $a \in A$ such that $\beta(b) = \alpha(a)$. But now $(a, b) \in A \oplus_C B$, which means that there exists such $x \in X$ that $(a, b) = \sigma(x) = (\delta(x), \gamma(x))$. Hence, $b \in \gamma(X)$, which implies $\beta^{-1}(\alpha(A)) \in \gamma(X)$. Therefore, $\beta^{-1}(\alpha(A)) = \gamma(X)$ and (ii) holds.

Take $a \in \ker\alpha$. Then $\alpha(a) = \theta_C = \beta(\theta_B)$. Hence, $(a, \theta_B) \in A \oplus_C B$. Thus, there exists $x \in X$ such that $(a, \theta_B) = \sigma(x) = (\delta(x), \gamma(x))$. From this we see that $x \in \ker\gamma$. Hence, $a \in \delta(\ker\gamma)$. Thus, $\ker\alpha \subseteq \delta(\ker\gamma)$.

Take $a \in \delta(\ker\gamma)$. Then there exists $x \in X$ such that $a = \delta(x)$ and $\gamma(x) = \theta_B$. Since $(\delta(x), \gamma(x)) \in A \oplus_C B$, then $\alpha(a) = \alpha(\delta(x)) = \beta(\gamma(x)) = \beta(\theta_B) = \theta_C$. Therefore, $a \in \ker\alpha$ and $\delta(\ker\gamma) \subseteq \ker\alpha$. Consequently, (iii) holds.

Suppose that the conditions (i), (ii) and (iii) hold and that the diagram above is commutative. Since $A \oplus_C B$ is the canonical pullback of A and B along α and β, then, by the universal property of a pullback, there exists a homomorphism of Abelian groups σ, as defined above. Now, in order to show that X is a pullback of A and B along α and β, it is sufficient to show that σ is a bijection.

Suppose that there exist $x, y \in X$ such that $\sigma(x) = \sigma(y)$. Then $\delta(x) = \delta(y)$ and $\gamma(x) = \gamma(y)$. Hence, $x - y \in \ker\delta \cap \ker\gamma = \theta_X$, by (i). Therefore, $x = y$ and σ is one-to-one.

Take an element $(a, b) \in A \oplus_C B$. Then $\alpha(a) = \beta(b)$, which means that $b \in \beta^{-1}(\alpha(A)) = \gamma(X)$, by (ii). Thus, there exists $x \in X$ such that $b = \gamma(x)$. Now, by using the properties of maps α, β, γ and δ, we get

$$\alpha(a - \delta(x)) = \alpha(a) - \alpha(\delta(x)) = \beta(b) - \beta(\gamma(x)) = \beta(b) - \beta(b) = \theta_C.$$

Therefore, $a - \delta(x) \in \ker\alpha = \delta(\ker\gamma)$, by (iii). Thus, there exists $y \in X$ such that $\gamma(y) = \theta_B$ and $a - \delta(x) = \delta(y)$, whence $a = \delta(x + y)$. Using the linearity of δ and γ, we obtain

$$\begin{aligned}\sigma(x + y) &= (\delta(x + y), \gamma(x + y)) = (\delta(x) + \delta(y), \gamma(x) + \gamma(y))\\ &= (\delta(x) + a - \delta(x), b + \theta_B) = (a, b).\end{aligned}$$

Therefore, σ is onto, hence an isomorphism. Thus, X is a pullback of A and B along α and β. □

Remark 2.2. Similarly we can take instead of (ii) and (iii) the conditions (ii′) $\alpha^{-1}(\beta(B)) = \delta(X)$ and (iii′) $\gamma(\ker\delta) = \ker\beta$ in Proposition 1.

An extension[1] of Abelian groups (or rings) is a short exact sequence

$$\theta_A \to A \xrightarrow{\alpha} X \xrightarrow{\beta} B \to \theta_B,$$

[1] Sometimes, the term "X is an extension of A along B" is used instead. For definitions of extensions of other types of algebraic structures, see, for example, [1, Definition 15.1.1, p. 121].

where A, B and C are Abelian groups (respectively, rings). Remember, that since this sequence is a short exact sequence, we get that α is injective, β is surjective and $\ker\beta = \mathrm{im}\,\alpha$.

The proofs of Proposition 3.4 and Proposition 3.6 of [3] are written in such general way that they actually do not depend on the structure of C*-algebras. What one needs in these proofs is actually the fact, that the structures under consideration are Abelian groups. Therefore, we can just reformulate the results and suggest the reader to see the proofs of [3], where one can just replace the word "C*-algebra" with term "Abelian group" without any difficulty.

Definition 2.3. We say that a class $\mathscr{C}$ of Abelian groups satisfies property P if the following conditions are fulfilled:

(i) if $A, B \in \mathscr{C}$ and $f : A \to B$ is a morphism of Abelian groups, then both $\ker f \in \mathscr{C}$ and $\mathrm{im} f = f(A) \in \mathscr{C}$.
(ii) $\mathscr{C}$ is closed under formation of extensions, i.e., if

$$\theta_A \to A \xrightarrow{\alpha} X \xrightarrow{\beta} B \to \theta_B,$$

is a short exact secuence of Abelian groups with $A, B \in \mathscr{C}$, then also $X \in \mathscr{C}$.

Similarly as in [3, p. 259, Proposition 3.4], we can prove the following lemma.

Lemma 2.4. *Let $\mathscr{C}$ be a class of Abelian groups which satisfies the property P. Then $\mathscr{C}$ is closed under formation of pullbacks.*

Again, following the proof of Proposition 3.6 of [3, pp. 260–261], we obtain more general result[2].

Proposition 2.5. *Given two extensions of additive groups*

$$\theta_{A_i} \to A_i \to X \to B_i \to \theta_{B_i},$$

where $i = 1, 2$, we obtain, by taking $C = X/(A_1 + A_2)$, a third extension of additive groups

$$\theta_{A_1 \cap A_2} \to A_1 \cap A_2 \to X \to B_1 \oplus_C B_2 \to \theta_{B_1 \oplus_C B_2}.$$

3. **Some results on pushouts of rings**

By a ring we will mean here an associative ring, which does not have to have a unital element.

[2]For the sake of correctness and in order to correct a misprint in the proof in [3], we mention, that the Noether Isomorphism Theorems for additive groups give us the isomorphisms between the additive groups $A_1/(A_1 \cap A_2)$ and $(A_1 + A_2)/A_2$ or between $A_2/(A_1 \cap A_2)$ and $(A_1 + A_2)/A_1$.

Let A, B and C be rings and maps $\alpha : C \to A$, $\beta : C \to B$ homomorphisms of rings. Consider a diagram

$$\begin{array}{ccc} C & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow{\delta} & X \end{array}$$

where X is a ring and maps $\delta : A \to X$ and $\gamma : B \to X$ are homomorphisms of rings. If this diagram is commutative and for any ring Y and ring homomorphisms $\phi : A \to Y$, $\psi : B \to Y$ with $\phi \circ \alpha = \psi \circ \beta$ there exists unique map $\sigma : X \to Y$ with $\phi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$, then X (together with δ and γ) is called a pushout of A and B along α and β. Equivalently, in this case it is also said that the square above is a pushout square.

Last, we generalize Theorems 2.4 (part a) of Theorem 1) and 2.5 (part b) of Theorem 1) of [3], pp. 248–249, from the case of C*-algebras to the case of rings.

Theorem 3.1. *Consider a commutative diagram of extensions of rings*

$$\begin{array}{ccccccccc} \theta_C & \longrightarrow & C & \xrightarrow{\iota_C} & B & \xrightarrow{\beta} & D & \longrightarrow & \theta_D \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\epsilon} & & \\ \theta_A & \longrightarrow & A & \xrightarrow{\iota_A} & X & \xrightarrow{\delta} & E & \longrightarrow & \theta_E \end{array}$$

where ι_A and ι_C denote inclusions.

(a) *If $A = \alpha(C)A$, i.e., α is a proper morphism, $D = E$ and ϵ is the identity map, then the left square is a pushout square.*

(b) *$\alpha(C)$ generates A as an ideal if and only if the right square is a pushout square.*

Proof. (a) Since we have exact sequences in the diagram, we get that β is surjective. Moreover, $\iota_A(A) = \ker\delta$ is a two-sided ideal of X and $\iota_C(C) = \ker\beta$ is a two-sided ideal of B because the kernel of a morphism of rings is always a two-sided ideal. Take any $x \in X$. Then $\delta(x) \in E$. Since β is surjective, then there exists $b \in B$ such that $\beta(b) = \delta(x)$. Now, $\delta(\gamma(b)) = \epsilon(\beta(b)) = \beta(b) = \delta(x)$. Hence, $\delta(x - \gamma(b)) = \theta_E$. Thus, $x - \gamma(b) \in \ker\delta = \iota_A(A)$. Therefore, $x \in \iota_A(A) + \gamma(B)$. Hence, $X = \iota_A(A) + \gamma(B)$.

Suppose that there exist ring Y and ring homomorphisms $\phi : A \to Y$ and $\psi : B \to Y$ such that $\phi \circ \alpha = \psi \circ \iota_C$. We have to show that there exists unique ring homomorphism $\sigma : X \to Y$ which fulfils the conditions $\psi = \sigma \circ \gamma$ and $\phi = \sigma \circ \iota_A$.

Take $x \in X$. Since $X = \iota_A(A) + \gamma(B)$, then there exist $a \in A$ and $b \in B$ such that $x = \iota_A(a) + \gamma(b)$. Define $\sigma(x) := \phi(a) + \psi(b)$. Then $(\sigma \circ \iota_A)(a) = \phi(a)$ and $(\sigma \circ \gamma)(b) = \psi(b)$ for every $a \in A$ and $b \in B$.

Suppose that the exist $a, a' \in A$ and $b, b' \in B$ such that $x = \iota_A(a) + \gamma(b) = \iota_A(a') + \gamma(b')$. Then, using the linearity of ι_A and γ, we get $\theta_X = x - x = (\iota_A(a) + \gamma(b)) - (\iota_A(a') + \gamma(b')) = \iota_A(a - a') + \gamma(b - b')$. Using the linearity of ϕ and ψ and the fact obtained above, we see that $\theta_E = \phi(a - a') + \psi(b - b') =$

$(\phi(a)+\psi(b))-(\phi(a')+\psi(b'))=\sigma(\iota_A(a)+\gamma(b))-\sigma(\iota_A(a')+\gamma(b'))$. Thus, from $\iota_A(a)+\gamma(b)=x=\iota_A(a')+\gamma(b')$ follows $\sigma(\iota_A(a)+\gamma(b))=\sigma(\iota_A(a')+\gamma(b'))$. Hence, σ is well-defined.

Take $x=\iota_A(a)+\gamma(b)\in X, y=\iota_A(a')+\gamma(b')\in X$ (in case we have algebras instead of rings, take also $\lambda\in\mathbb{R}$). Then using again the linearity of ι_A and γ, we get $x+y=\iota_A(a+a')+\gamma(b+b')$ (and, in case of algebras, $\lambda x=\iota_A(\lambda a)+\gamma(\lambda b)$). Now, the linearity of ϕ and ψ gives us $\sigma(x+y)=\phi(a+a')+\psi(b+b')=(\phi(a)+\psi(b))+(\phi(a')+\psi(b'))=\sigma(x)+\sigma(y)$ (and $\sigma(\lambda x)=\phi(\lambda a)+\psi(\lambda b)=\lambda\phi(a)+\lambda\psi(b)=\lambda(\phi(a)+\psi(b))=\lambda\sigma(x)$). Thus, σ is linear map.

Take again $x=\iota_A(a)+\gamma(b)\in X$, where $a\in A$ and $b\in B$. Since α is a proper map, there exist $a'\in A$ and $c\in C$ such that $a=a'\alpha(c)$. Since $\iota_C(c)$ is a two-sided ideal in B, then $\iota_C(c)b\in\iota_C(C)$. Thus, there exists $c'\in C$ such that $\iota_C(c)b=\iota_C(c')$. Now, using the multiplicativity of maps ι_A,γ and ι_C, we get

$$\begin{aligned}\iota_A(a)\gamma(b)&=\iota_A(a'\alpha(c))\gamma(b)=\iota_A(a')\iota_A(\alpha(c))\gamma(b)\\&=\iota_A(a')\gamma(\iota_C(c))\gamma(b)=\iota_A(a')\gamma(\iota_C(c)b)\\&=\iota_A(a')\gamma(\iota_C(c'))=\iota_A(a')\iota_A(\alpha(c'))\\&=\iota_A(a'\alpha(c')).\end{aligned}$$

Similarly, using in addition the multiplicativity of ϕ and ψ, we obtain

$$\begin{aligned}\phi(a)\psi(b)&=\phi(a'\alpha(c))\psi(b)=\phi(a')\phi(\alpha(c))\psi(b)\\&=\phi(a')\psi(\iota_C(c))\psi(b)=\phi(a')\psi(\iota_C(c)b)\\&=\phi(a')\psi(\iota_C(c'))=\phi(a')\phi(\alpha(c'))\\&=\phi(a'\alpha(c')).\end{aligned}$$

Next, using also the linearity of σ, we get

$$\begin{aligned}\sigma(x^2)&=\sigma([\iota_A(a)+\gamma(b)]^2)=\sigma([\iota_A(a)]^2+2\iota_A(a)\gamma(b)+[\gamma(b)]^2)\\&=\sigma(\iota_A(a^2))+2\sigma(\iota_A(a'\alpha(c')))+\sigma(\gamma(b^2))\\&=\phi(a^2)+2\phi(a'\alpha(c'))+\psi(b^2)\\&=[\phi(a)]^2+2\phi(a)\psi(b)+[\psi(b)]^2\\&=(\phi(a)+\psi(b))^2=[\sigma(x)]^2.\end{aligned}$$

Since it holds for any $x\in X$, we obtain

$$\begin{aligned}\sigma(xy)&=\sigma\left(\frac{1}{2}[(x+y)^2-x^2-y^2]\right)\\&=\frac{1}{2}[\sigma([x+y]^2)-\sigma(x^2)-\sigma(y^2)]\\&=\frac{1}{2}[(\sigma(x+y))^2-\sigma(x)^2-\sigma(y)^2]\end{aligned}$$

$$= \frac{1}{2}[(\sigma(x) + \sigma(y))^2 - \sigma(x)^2 - \sigma(y)^2]$$
$$= \sigma(x)\sigma(y) \text{ for any } x, y \in X.$$

Hence, σ is multplicative, as well. Therefore, σ is a homomorphism of rings.

Suppose, that there is another homomorphism of rings $\omega : X \to Y$ such that $\phi = \omega \circ \iota_A$ and $\psi = \omega \circ \gamma$. Take any $x = \iota_A(a) + \gamma(b) \in X$. Then $\omega(x) = \omega(\iota_A(a) + \gamma(b)) = \omega(\iota_A(a)) + \omega(\gamma(b)) = \phi(a) + \psi(b) = \sigma(\iota_A(a)) + \sigma(\gamma(b)) = \sigma(\iota_A(a) + \gamma(b)) = \sigma(x)$. Hence, $\omega = \sigma$. Thus, we have shown that the left square is a pushout square.

(b) Suppose that $\mathrm{Id}(\alpha(C)) = A$. Since we deal with extensions, we have $C = \ker \beta$ and maps β, δ are onto. Suppose that there exist ring Y and ring homomorphisms $\phi : X \to Y$ and $\psi : D \to Y$ such that $\phi \circ \gamma = \psi \circ \beta$. Take any $c \in C$. Then $\psi(\beta(c)) = \psi(\theta_D) = \theta_E$. Hence, $C \subseteq \ker(\psi \circ \beta) = \ker(\phi \circ \gamma)$. Therefore, $\alpha(C) = \iota_A(\alpha(C)) = \gamma(\iota_C(C)) = \gamma(C) \subseteq \ker \phi$. Since $A = \mathrm{Id}(\alpha(C))$, then we get also that $\ker \delta = A \subseteq \ker \phi$.

Take any $e \in E$. Since δ is onto, there exists $x \in X$ such that $\delta(x) = e$. Define a map $\rho : E \to Y$ by $\rho(e) := \phi(x)$. Suppose that there is another $x' \in X$ such that $\delta(x') = e$. Then $x - x' \in \ker \delta \subset \ker \phi$. Hence, $\phi(x) = \phi(x')$ and ρ is well-defined. Obviously, $\phi = \rho \circ \delta$.

Take now any $d \in D$. Since β is onto, there exists $b \in B$ such that $\beta(b) = d$. Now,

$$\psi(d) = \psi(\beta(b)) = \phi(\gamma(b)) = \rho(\delta(\gamma(b))) = \rho(\epsilon(\beta(b))) = \rho(\epsilon(d)).$$

Hence, $\psi = \rho \circ \epsilon$.

Suppose, that there exists another ring homomorphism $\varpi : E \to Y$ such that $\varpi \circ \delta = \phi$ and $\varpi \circ \epsilon = \psi$. Take any $e \in E$. Since δ is onto, there exists $x \in X$ such that $\delta(x) = e$. Hence,

$$\varpi(e) = \varpi(\delta(x)) = \phi(x) = \rho(\delta(x)) = \rho(e).$$

Thus, $\varpi = \rho$, whence the right square is a pushout square.

Suppose now, that the right square is a pushout square and denote $A_0 = \mathrm{Id}(\alpha(C)) \subseteq A$. Since A_0 is a two-sided ideal of X, then X/A_0 is a ring. Take $Y = X/A_0$ and consider the quotient morphism $\phi : X \to Y$. Take any $d \in D$. Since β is onto, there exists $b \in B$ such that $d = \beta(b)$. Let $\psi : D \to Y$ be a morphism defined by $\psi(d) = \psi(\beta(b)) := \phi(\gamma(b))$. Then (because we have a pushout) there exists a unique map $\rho : E \to Y$, with $\phi = \rho \circ \delta$ and $\psi = \rho \circ \epsilon$. Moreover, it is clear that $\ker \phi = A_0$.

Now, $A = \ker \delta \subseteq \ker(\rho \circ \delta) = \ker \phi = A_0$ gives us $A = A_0 = \mathrm{Id}(\alpha(C))$, which proves the claim. □

Remark 3.2. When we study carefully the proof of part b) of Theorem 1, then we see that we can generalize part b) for the case of groups in a following way (in this case we have to substitute $\mathrm{Id}(\alpha(C))$ by the smallest normal subgroup of A containing $\alpha(C)$ everywhere in the proof):

(b′) *A is the smallest normal subgroup of A which contains the set $\alpha(C)$ if and only if the right square is a pushout square.*

In case of a rings, a normal subgroup is a two-sided ideal of that ring.

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