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Research Article

Binomial Transform of the Generalized Third Order Pell Sequence

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Abstract. In this paper, we define the binomial transform of the generalized third order Pell sequence and as special cases, the binomial transform of the third order Pell, third Order Pell-Lucas and modified third order Pell sequences will be introduced. We investigate their properties in details.

Keywords. Binomial transform; Third order Pell sequence; Third order Pell numbers; Third order Pell-Lucas sequence; Third order Pell-Lucas numbers

MSC. 11B39; 11B83

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1. Introduction and Preliminaries

In this paper, we introduce the binomial transform of the generalized third order Pell sequence and we investigate, in detail, three special cases which we call them third order Pell, third order Pell-Lucas and modified third order Pell sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized Tribonacci sequence and generalized third order Pell sequence.

The generalized Tribonacci sequence $\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}, \qquad W_0=a, W_1=b, W_2=c, \; n\geq 3 \tag{1.1}$$

where W_0,W_1,W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers.

This sequence has been studied by many authors, see e.g., [2–6, 13, 14, 16, 18, 20, 23, 25, 26].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1,2,3,\ldots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\alpha = \alpha(r,s,t) = \frac{r}{3} + A + B,$$
$$\beta = \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B,$$
$$\gamma = \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B,$$

where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \qquad B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3},$$
$$\Delta = \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

If $\Delta(r,s,t) > 0$, then eq. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{c_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{c_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{c_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.3}$$

where

$$c_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \quad c_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \quad c_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [9]. This result of Howard and Saidak [9] is even true in the case of higher-order recurrence relations.

Now we consider the case $r = 2$, $s = t = 1$ and in this case we write $V_n = W_n$. A generalized third order Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} \tag{1.4}$$

with the initial values $V_0 = c_0$, $V_1 = c_1$, $V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - 2V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence (1.4) holds for all integer n.

Next, we define three special case of the sequence $\{V_n\}$. Third-order Pell sequence $\{P_n^{(3)}\}_{n\geq 0}$, third-order Pell-Lucas sequence $\{Q_n^{(3)}\}_{n\geq 0}$ and modified third-order Pell sequence $\{E_n^{(3)}\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$P_{n+3}^{(3)} = 2P_{n+2}^{(3)} + P_{n+1}^{(3)} + P_n^{(3)}, \quad P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2, \tag{1.5}$$

$$Q_{n+3}^{(3)} = 2Q_{n+2}^{(3)} + Q_{n+1}^{(3)} + Q_n^{(3)}, \quad Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6, \tag{1.6}$$

$$E_{n+3}^{(3)} = 2E_{n+2}^{(3)} + E_{n+1}^{(3)} + E_n^{(3)}, \quad E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1. \tag{1.7}$$

The sequences $\{P_n^{(3)}\}_{n\geq 0}$, $\{Q_n^{(3)}\}_{n\geq 0}$ and $\{E_n^{(3)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n}^{(3)} = -P_{-(n-1)}^{(3)} - 2P_{-(n-2)}^{(3)} + P_{-(n-3)}^{(3)}, \tag{1.8}$$

$$Q_{-n}^{(3)} = -Q_{-(n-1)}^{(3)} - 2Q_{-(n-2)}^{(3)} + Q_{-(n-3)}^{(3)}, \tag{1.9}$$

$$E_{-n}^{(3)} = -E_{-(n-1)}^{(3)} - 2E_{-(n-2)}^{(3)} + E_{-(n-3)}^{(3)} \tag{1.10}$$

for $n = 1,2,3,\ldots$ respectively. Therefore, recurrences (1.8), (1.9) and (1.10) hold for all integer n.

In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(3)}, Q_n^{(3)}$ and $E_n^{(3)}$, respectively. Note that P_n is the sequence $A077939$ in [17] associated with the expansion of $1/(1-2x-x^2-x^3)$, Q_n is the sequence $A276225$ in [17] and E_n is the sequence $A077997$ in [17].

For more details for the generalized third order Pell numbers, see Soykan [21].

2. Binomial Transform of the Generalized Third Order Pell Sequence V_n

In [12, p. 137], Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ may be defined by the rule

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} \hat{a}_i.$$

For more information on binomial transform, see, e.g., [7, 8, 15, 24] and references therein.

In this section, we define the binomial transform of the generalized third order Pell sequence V_n and as special cases the binomial transform of the third order Pell, third Order Pell-Lucas and modified third order Pell sequences will be introduced.

Definition 2.1. The binomial transform of the generalized third order Pell sequence V_n is defined by

$$b_n = \widehat{V}_n = \sum_{i=0}^{n} \binom{n}{i} V_i.$$

The few terms of b_n are

$$b_0 = \sum_{i=0}^{0} \binom{0}{i} V_i = V_0,$$

$$b_1 = \sum_{i=0}^{1} \binom{1}{i} V_i = V_0 + V_1,$$
$$b_2 = \sum_{i=0}^{2} \binom{2}{i} V_i = V_0 + 2V_1 + V_2.$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the third order Pell, third order Pell-Lucas and modified third order Pell sequences are defined as follows: The binomial transform of the third order Pell sequence P_n is

$$\widehat{P}_n = \sum_{i=0}^{n} \binom{n}{i} P_i,$$

the binomial transform of the third order Pell-Lucas sequence Q_n is

$$\widehat{Q}_n = \sum_{i=0}^{n} \binom{n}{i} Q_i,$$

the binomial transform of the modified third order Pell sequence E_n is

$$\widehat{E}_n = \sum_{i=0}^{n} \binom{n}{i} E_i.$$

Lemma 2.2. *For $n \geq 0$, the binomial transform of the generalized third order Pell sequence V_n satisfies the following relation:*

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned} b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\ &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \end{aligned}$$

$$= V_0 + \sum_{i=1}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}$$
$$= \sum_{i=0}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}$$
$$= \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}).$$

This completes the proof. □

Remark 2.3. From the last Lemma we see that

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}.$$

The following Theorem gives recurrent relations of the binomial transform of the generalized third order Pell sequence.

Theorem 2.4. *For $n \geq 0$, the binomial transform of the generalized third order Pell sequence V_n satisfies the following recurrence relation:*

$$b_{n+3} = 5b_{n+2} - 6b_{n+1} + 3b_n. \tag{2.1}$$

Proof. To show (2.1), writing

$$b_{n+3} = A \times b_{n+2} + B \times b_{n+1} + C \times b_n$$

and taking the values $n = 0,1,2$ and then solving the system of equations

$$b_3 = A \times b_2 + B \times b_1 + C \times b_0,$$
$$b_4 = A \times b_3 + B \times b_2 + C \times b_1,$$
$$b_5 = A \times b_4 + B \times b_3 + C \times b_2.$$

We find that $A = 5$, $B = -6$, $C = 3$. □

Note that the recurrence relation (2.1) is independent from initial values. So

$$\widehat{P}_{n+3} = 5\widehat{P}_{n+2} - 6\widehat{P}_{n+1} + 3\widehat{P}_n,$$
$$\widehat{Q}_{n+3} = 5\widehat{Q}_{n+2} - 6\widehat{Q}_{n+1} + 3\widehat{Q}_n,$$
$$\widehat{E}_{n+3} = 5\widehat{E}_{n+2} - 6\widehat{E}_{n+1} + 3\widehat{E}_n.$$

The sequence $\{b_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = 2b_{-n+1} - \frac{5}{3}b_{-n+2} + \frac{1}{3}b_{-n+3}$$

for $n = 1,2,3,\ldots$. Therefore, recurrence (2.1) holds for all integer n.

The first few terms of the binomial transform of the generalized third order Pell sequence with positive subscript and negative subscript are given in Table 1.

Table 1. A few binomial transform (terms) of the generalized third order Pell sequence

n	b_n	b_{-n}
0	V_0	V_0
1	V_0+V_1	$\frac{2}{3}V_0-V_1+\frac{1}{3}V_2$
2	$V_0+2V_1+V_2$	$-\frac{5}{3}V_1+\frac{2}{3}V_2$
3	$2V_0+4V_1+5V_2$	$-\frac{7}{9}V_0-\frac{5}{3}V_1+\frac{7}{9}V_2$
4	$7V_0+11V_1+19V_2$	$-\frac{4}{3}V_0-\frac{8}{9}V_1+\frac{5}{9}V_2$
5	$26V_0+37V_1+68V_2$	$-\frac{37}{27}V_0+\frac{4}{9}V_1+\frac{1}{27}V_2$
6	$94V_0+131V_1+241V_2$	$-\frac{7}{9}V_0+\frac{49}{27}V_1-\frac{16}{27}V_2$
7	$335V_0+466V_1+854V_2$	$\frac{23}{81}V_0+\frac{70}{27}V_1-\frac{86}{81}V_2$
8	$1189V_0+1655V_1+3028V_2$	$\frac{38}{27}V_0+\frac{187}{81}V_1-\frac{91}{81}V_2$
9	$4217V_0+5872V_1+10739V_2$	$\frac{506}{243}V_0+\frac{73}{81}V_1-\frac{164}{243}V_2$
10	$14956V_0+20828V_1+38089V_2$	$\frac{155}{81}V_0-\frac{287}{243}V_1+\frac{41}{243}V_2$
11	$53045V_0+73873V_1+135095V_2$	$\frac{602}{729}V_0-\frac{752}{243}V_1+\frac{793}{729}V_2$
12	$188140V_0+262013V_1+479158V_2$	$-\frac{205}{243}V_0-\frac{2858}{729}V_1+\frac{1217}{729}V_2$
13	$667298V_0+929311V_1+1699487V_2$	$-\frac{5305}{2187}V_0-\frac{2243}{729}V_1+\frac{3460}{2187}V_2$

The first few terms of the binomial transform numbers of the third order Pell, Pell-Lucas and modified Pell sequences with positive subscript and negative subscript are given in Table 2.

Table 2. A few binomial transform (terms)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\widehat{P}_n$	0	1	4	14	49	173	613	2174	7711	27350	97006	344063	1220329	4328285
$\widehat{P}_{-n}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{14}{27}$	$\frac{17}{27}$	$\frac{38}{81}$	$\frac{5}{81}$	$-\frac{109}{243}$	$-\frac{205}{243}$	$-\frac{670}{729}$	$-\frac{424}{729}$	$\frac{191}{2187}$
$\widehat{Q}_n$	3	5	13	44	157	560	1990	7061	25045	88829	315058	1117451	3963394	14057438
$\widehat{Q}_{-n}$	3	2	$\frac{2}{3}$	-1	$-\frac{22}{9}$	-3	$-\frac{61}{27}$	$-\frac{1}{3}$	$\frac{170}{81}$	4	$\frac{1067}{243}$	$\frac{76}{27}$	$-\frac{259}{729}$	$-\frac{319}{81}$
$\widehat{E}_n$	0	1	3	9	30	105	372	1320	4683	16611	58917	208968	741171	2628798
$\widehat{E}_{-n}$	0	$-\frac{2}{3}$	-1	$-\frac{8}{9}$	$-\frac{1}{3}$	$\frac{13}{27}$	$\frac{11}{9}$	$\frac{124}{81}$	$\frac{32}{27}$	$\frac{55}{243}$	$-\frac{82}{81}$	$-\frac{1463}{729}$	$-\frac{547}{243}$	$-\frac{3269}{2187}$

Eq. (1.3) can be used to obtain Binet formula of the binomial transform of generalized third order Pell numbers. Binet formula of the binomial transform of generalized third order Pell numbers can be given as

$$b_n=\frac{C_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{C_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{C_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}, \tag{2.2}$$

where

$$C_1 = b_2 - (\theta_2+\theta_3)b_1 + \theta_2\theta_3 b_0 = (V_0+2V_1+V_2) - (\theta_2+\theta_3)(V_0+V_1) + \theta_2\theta_3 V_0,$$
$$C_2 = b_2 - (\theta_1+\theta_3)b_1 + \theta_1\theta_3 b_0 = (V_0+2V_1+V_2) - (\theta_1+\theta_3)(V_0+V_1) + \theta_1\theta_3 V_0,$$
$$C_3 = b_2 - (\theta_1+\theta_2)b_1 + \theta_1\theta_2 b_0 = (V_0+2V_1+V_2) - (\theta_1+\theta_2)(V_0+V_1) + \theta_1\theta_2 V_0.$$

Here, θ_1, θ_2 and θ_3 are the roots of the cubic equation $x^3-5x^2+6x-3=0$. Moreover

$$\theta_1 = \frac{5}{3} + \left(\frac{61}{54}+\sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54}-\sqrt{\frac{29}{36}}\right)^{1/3},$$
$$\theta_2 = \frac{5}{3} + \omega\left(\frac{61}{54}+\sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54}-\sqrt{\frac{29}{36}}\right)^{1/3},$$
$$\theta_3 = \frac{5}{3} + \omega^2\left(\frac{61}{54}+\sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54}-\sqrt{\frac{29}{36}}\right)^{1/3},$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\theta_1+\theta_2+\theta_3 = 5,$$
$$\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3 = 6,$$
$$\theta_1\theta_2\theta_3 = 3.$$

For all integers n, (Binet formulas of) binomial transforms of third-order Pell, Pell-Lucas and modified Pell numbers (using initial conditions in (2.2) can be expressed using Binet's formulas as

$$\widehat{P}_n = \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$
$$\widehat{Q}_n = \theta_1^n+\theta_2^n+\theta_3^n$$

and

$$\widehat{E}_n = \frac{(-2+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-2+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-2+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$

respectively.

3. Generating Functions

The generating function of the binomial transform of the generalized third order Pell sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized third order Pell sequence.

Next, we give the ordinary generating function $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Lemma 3.1. *Suppose that* $f_{b_n}(x)=\sum_{n=0}^{\infty} b_n x^n$ *is the ordinary generating function of the binomial transform of the generalized third-order Pell sequence* $\{V_n\}_{n\geq 0}$. *Then,* $f_{b_n}(x)$ *is given by*

$$f_{b_n}(x)=\frac{V_0+(V_1-4V_0)x+(V_2-3V_1+2V_0)x^2}{1-5x+6x^2-3x^3}. \tag{3.1}$$

Proof. Using the definition of binomial transform, and subtracting $5x\sum_{n=0}^{\infty} b_n x^n$, $-6x^2\sum_{n=0}^{\infty} b_n x^n$ and $3x^3\sum_{n=0}^{\infty} b_n x^n$ from $\sum_{n=0}^{\infty} b_n x^n$ we obtain

$$\begin{aligned}
(1-5x+6x^2-3x^3)\sum_{n=0}^{\infty} b_n x^n &= (1-5x+6x^2-3x^3)\sum_{n=0}^{\infty} b_n x^n \\
&= \sum_{n=0}^{\infty} b_n x^n - 5x\sum_{n=0}^{\infty} b_n x^n + 6x^2\sum_{n=0}^{\infty} b_n x^n - 3x^3\sum_{n=0}^{\infty} b_n x^n \\
&= \sum_{n=0}^{\infty} b_n x^n - 5\sum_{n=0}^{\infty} b_n x^{n+1} + 6\sum_{n=0}^{\infty} b_n x^{n+2} - 3\sum_{n=0}^{\infty} b_n x^{n+3} \\
&= \sum_{n=0}^{\infty} b_n x^n - 5\sum_{n=1}^{\infty} b_{n-1} x^n + 6\sum_{n=2}^{\infty} b_{n-2} x^n - 3\sum_{n=3}^{\infty} b_{n-3} x^n \\
&= (b_0+b_1x+b_2x^2)-5(b_0x+b_1x^2)+6b_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(b_n-5b_{n-1}+6b_{n-2}-3b_{n-3})x^n \\
&= b_0+b_1x+b_2x^2-5b_0x-5b_1x^2+6b_0x^2 \\
&= b_0+(b_1-5b_0)x+(b_2-5b_1+6b_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$f_{b_n}(x)=\sum_{n=0}^{\infty} b_n x^n = \frac{b_0+(b_1-5b_0)x+(b_2-5b_1+6b_0)x^2}{1-5x+6x^2-3x^3}.$$

Using the values of b_0, b_1 and b_2, we obtain the desired result. □

Note that Barry shows in [1] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x)=\frac{1}{1-x}A\left(\frac{x}{1-x}\right)$$

is the generating function of the sequence $\{b_n\}$ with $b_n=\sum_{i=0}^{n}\binom{n}{i}a_i$. In our case, since

$$A(x)=\frac{V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2}{1-2x-x^2-x^3}. \quad \text{(see [21])}$$

We obtain

$$\begin{aligned}
S(x) &= \frac{1}{1-x}\,\frac{V_0+(V_1-2V_0)\frac{x}{1-x}+(V_2-2V_1-V_0)\left(\frac{x}{1-x}\right)^2}{1-2\frac{x}{1-x}-\left(\frac{x}{1-x}\right)^2-\left(\frac{x}{1-x}\right)^3} \\
&= \frac{V_0+(V_1-4V_0)x+(V_2-3V_1+2V_0)x^2}{1-5x+6x^2-3x^3}.
\end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 3.2. *Generated functions of the binomial transform of the third-order Pell, Pell-Lucas and modified Pell numbers are*

$$\sum_{n=0}^{\infty} \widehat{P}_n x^n = \frac{x-x^2}{1-5x+6x^2-3x^3},$$

$$\sum_{n=0}^{\infty} \widehat{Q}_n x^n = \frac{3-10x+6x^2}{1-5x+6x^2-3x^3}$$

and

$$\sum_{n=0}^{\infty} \widehat{E}_n x^n = \frac{x-2x^2}{1-5x+6x^2-3x^3},$$

respectively.

4. Obtaining Binet Formula of Binomial Transform From Generating Function

We next find Binet formula of the Binomial Transform of the generalized third order Pell numbers $\{V_n\}$ by the use of generating function for b_n.

Theorem 4.1 (Binet formula of the Binomial Transform of the generalized third order Pell numbers)**.**

$$b_n = \frac{d_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{d_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{d_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}, \tag{4.1}$$

where

$$d_1 = (V_0\theta_1^2 + (V_1-4V_0)\theta_1 + (V_2-3V_1+2V_0)),$$
$$d_2 = (V_0\theta_2^2 + (V_1-4V_0)\theta_2 + (V_2-3V_1+2V_0)),$$
$$d_3 = (V_0\theta_3^2 + (V_1-4V_0)\theta_3 + (V_2-3V_1+2V_0)).$$

Proof. Let

$$h(x) = 1-5x+6x^2-3x^3.$$

Then for some θ_1, θ_2 and θ_3 we write

$$h(x) = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)$$

i.e.,

$$1-5x+6x^2-3x^3 = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x). \tag{4.2}$$

Hence $\frac{1}{\theta_1}$, $\frac{1}{\theta_2}$ and $\frac{1}{\theta_3}$ are the roots of $h(x)$. This gives θ_1, θ_2 and θ_3 as the roots of

$$h\left(\frac{1}{x}\right) = 1 - \frac{5}{x} + \frac{6}{x^2} - \frac{3}{x^3} = 0.$$

This implies $x^3-5x^2+6x-3=0$. Now, by (3.1) and (4.2), it follows that

$$\sum_{n=0}^{\infty} b_n x^n = \frac{V_0 + (V_1-4V_0)x + (V_2-3V_1+2V_0)x^2}{1-5x+6x^2-3x^3}.$$

Then, we write

$$\frac{V_0+(V_1-4V_0)x+(V_2-3V_1+2V_0)x^2}{(1-\theta_1x)(1-\theta_2x)(1-\theta_3x)}=\frac{A_1}{(1-\theta_1x)}+\frac{A_2}{(1-\theta_2x)}+\frac{A_3}{(1-\theta_3x)}. \tag{4.3}$$

So

$$\begin{aligned}&V_0+(V_1-4V_0)x+(V_2-3V_1+2V_0)x^2\\&\quad=A_1(1-\theta_2x)(1-\theta_3x)+A_2(1-\theta_1x)(1-\theta_3x)+A_3(1-\theta_1x)(1-\theta_2x).\end{aligned}$$

If we consider $x=\frac{1}{\theta_1}$, we get

$$V_0+(V_1-4V_0)\frac{1}{\theta_1}+(V_2-3V_1+2V_0)\frac{1}{\theta_1^2}=A_1\left(1-\frac{\theta_2}{\theta_1}\right)\left(1-\frac{\theta_3}{\theta_1}\right).$$

This gives

$$A_1=\frac{\theta_1^2(V_0+(V_1-4V_0)\frac{1}{\theta_1}+(V_2-3V_1+2V_0)\frac{1}{\theta_1^2})}{(\theta_1-\theta_2)(\theta_1-\theta_3)}=\frac{(V_0\theta_1^2+(V_1-4V_0)\theta_1+(V_2-3V_1+2V_0))}{(\theta_1-\theta_2)(\theta_1-\theta_3)}.$$

Similarly, we obtain

$$A_2=\frac{(V_0\theta_2^2+(V_1-4V_0)\theta_2+(V_2-3V_1+2V_0)}{(\theta_2-\theta_1)(\theta_2-\theta_3)},\quad A_3=\frac{(V_0\theta_3^2+(V_1-4V_0)\theta_3+(V_2-3V_1+2V_0)}{(\theta_3-\theta_1)(\theta_3-\theta_2)}.$$

Thus (4.3) can be written as

$$\sum_{n=0}^{\infty}b_nx^n=A_1(1-\theta_1x)^{-1}+A_2(1-\theta_2x)^{-1}+A_3(1-\theta_3x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}b_nx^n=A_1\sum_{n=0}^{\infty}\theta_1^nx^n+A_2\sum_{n=0}^{\infty}\theta_2^nx^n+A_3\sum_{n=0}^{\infty}\theta_3^nx^n=\sum_{n=0}^{\infty}(A_1\theta_1^n+A_2\theta_2^n+A_3\theta_3^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$b_n=A_1\theta_1^n+A_2\theta_2^n+A_3\theta_3^n,$$

where

$$\begin{aligned}A_1&=\frac{(V_0\theta_1^2+(V_1-4V_0)\theta_1+(V_2-3V_1+2V_0))}{(\theta_1-\theta_2)(\theta_1-\theta_3)},\\A_2&=\frac{(V_0\theta_2^2+(V_1-4V_0)\theta_2+(V_2-3V_1+2V_0)}{(\theta_2-\theta_1)(\theta_2-\theta_3)},\\A_3&=\frac{(V_0\theta_3^2+(V_1-4V_0)\theta_3+(V_2-3V_1+2V_0)}{(\theta_3-\theta_1)(\theta_3-\theta_2)}.\end{aligned}$$

and then we get (4.1). □

Note that from (2.2) and (4.1), we have

$$\begin{aligned}&(V_0+2V_1+V_2)-(\theta_2+\theta_3)(V_0+V_1)+\theta_2\theta_3V_0=(V_0\theta_1^2+(V_1-4V_0)\theta_1+(V_2-3V_1+2V_0)),\\&(V_0+2V_1+V_2)-(\theta_1+\theta_3)(V_0+V_1)+\theta_1\theta_3V_0=(V_0\theta_2^2+(V_1-4V_0)\theta_2+(V_2-3V_1+2V_0),\\&(V_0+2V_1+V_2)-(\theta_1+\theta_2)(V_0+V_1)+\theta_1\theta_2V_0=(V_0\theta_3^2+(V_1-4V_0)\theta_3+(V_2-3V_1+2V_0).\end{aligned}$$

Next, using Theorem 4.1, we present the Binet formulas of binomial transform of third-order Pell, Pell-Lucas and modified Pell sequences.

Corollary 4.2. *Binet formulas of binomial transform of third-order Pell, Pell-Lucas and modified Pell sequences are*

$$\widehat{P}_n = \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},$$

$$\widehat{Q}_n = \theta_1^n + \theta_2^n + \theta_3^n$$

and

$$\widehat{E}_n = \frac{(-2+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-2+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-2+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}$$

respectively.

We can find Binet formulas by using matrix method which is given in [11]. Take $k = i = 3$ in Corollary 3.1 in [11]. Let

$$\Lambda = \begin{pmatrix} \theta_1^2 & \theta_1 & 1 \\ \theta_2^2 & \theta_2 & 1 \\ \theta_3^2 & \theta_3 & 1 \end{pmatrix}, \qquad \Lambda_1 = \begin{pmatrix} \theta_1^{n-1} & \theta_1 & 1 \\ \theta_2^{n-1} & \theta_2 & 1 \\ \theta_3^{n-1} & \theta_3 & 1 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} \theta_1^2 & \theta_1^{n-1} & 1 \\ \theta_2^2 & \theta_2^{n-1} & 1 \\ \theta_3^2 & \theta_3^{n-1} & 1 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \theta_1^2 & \theta_1 & \theta_1^{n-1} \\ \theta_2^2 & \theta_2 & \theta_2^{n-1} \\ \theta_3^2 & \theta_3 & \theta_3^{n-1} \end{pmatrix}.$$

Then the Binet formula for binomial transform of third-order Pell numbers is

$$\begin{aligned}
\widehat{P}_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} \det \widehat{P}_{4-j}(\Lambda_j) = \frac{1}{\Lambda}(\widehat{P}_3 \det(\Lambda_1) + \widehat{P}_2 \det(\Lambda_2) + \widehat{P}_1 \det(\Lambda_3)) \\
&= \frac{1}{\det(\Lambda)}(14\det(\Lambda_1) + 4\det(\Lambda_2) + \det(\Lambda_3)) \\
&= \left(14 \begin{vmatrix} \theta_1^{n-1} & \theta_1 & 1 \\ \theta_2^{n-1} & \theta_2 & 1 \\ \theta_3^{n-1} & \theta_3 & 1 \end{vmatrix} + 4 \begin{vmatrix} \theta_1^2 & \theta_1^{n-1} & 1 \\ \theta_2^2 & \theta_2^{n-1} & 1 \\ \theta_3^2 & \theta_3^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \theta_1^2 & \theta_1 & \theta_1^{n-1} \\ \theta_2^2 & \theta_2 & \theta_2^{n-1} \\ \theta_3^2 & \theta_3 & \theta_3^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \theta_1^2 & \theta_1 & 1 \\ \theta_2^2 & \theta_2 & 1 \\ \theta_3^2 & \theta_3 & 1 \end{vmatrix} \\
&= \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}.
\end{aligned}$$

Similarly, we obtain the Binet formulas for binomial transforms of third-order Pell-Lucas and modified third-order Pell numbers as

$$\begin{aligned}
\widehat{Q}_n &= \frac{1}{\Lambda}(\widehat{Q}_3 \det(\Lambda_1) + \widehat{Q}_2 \det(\Lambda_2) + \widehat{Q}_1 \det(\Lambda_3)) \\
&= \left(44 \begin{vmatrix} \theta_1^{n-1} & \theta_1 & 1 \\ \theta_2^{n-1} & \theta_2 & 1 \\ \theta_3^{n-1} & \theta_3 & 1 \end{vmatrix} + 13 \begin{vmatrix} \theta_1^2 & \theta_1^{n-1} & 1 \\ \theta_2^2 & \theta_2^{n-1} & 1 \\ \theta_3^2 & \theta_3^{n-1} & 1 \end{vmatrix} + 5 \begin{vmatrix} \theta_1^2 & \theta_1 & \theta_1^{n-1} \\ \theta_2^2 & \theta_2 & \theta_2^{n-1} \\ \theta_3^2 & \theta_3 & \theta_3^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \theta_1^2 & \theta_1 & 1 \\ \theta_2^2 & \theta_2 & 1 \\ \theta_3^2 & \theta_3 & 1 \end{vmatrix} \\
&= \theta_1^n + \theta_2^n + \theta_3^n
\end{aligned}$$

and

$$\begin{aligned}
\widehat{E}_n &= \frac{1}{\Lambda}(\widehat{E}_3 \det(\Lambda_1) + \widehat{E}_2 \det(\Lambda_2) + \widehat{E}_1 \det(\Lambda_3)) \\
&= \left(9 \begin{vmatrix} \theta_1^{n-1} & \theta_1 & 1 \\ \theta_2^{n-1} & \theta_2 & 1 \\ \theta_3^{n-1} & \theta_3 & 1 \end{vmatrix} + 3 \begin{vmatrix} \theta_1^2 & \theta_1^{n-1} & 1 \\ \theta_2^2 & \theta_2^{n-1} & 1 \\ \theta_3^2 & \theta_3^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \theta_1^2 & \theta_1 & \theta_1^{n-1} \\ \theta_2^2 & \theta_2 & \theta_2^{n-1} \\ \theta_3^2 & \theta_3 & \theta_3^{n-1} \end{vmatrix} \right) \Big/ \begin{vmatrix} \theta_1^2 & \theta_1 & 1 \\ \theta_2^2 & \theta_2 & 1 \\ \theta_3^2 & \theta_3 & 1 \end{vmatrix}
\end{aligned}$$

$$= \frac{(-2+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{(-2+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{(-2+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}, \quad \text{respectively.}$$

5. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Tribonacci sequence $\{W_n\}$.

Theorem 5.1 (Simson Formula of Generalized Tribonacci Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{5.1}$$

Proof. Eq. (5.1) is given in Soykan [19]. □

Taking $\{W_n\} = \{b_n\}$ in the above theorem and considering $b_{n+3} = 5b_{n+2} - 6b_{n+1} + 3b_n$, $r = 5$, $s = -6$, $t = 3$, we have the following proposition.

Proposition 5.2. *For all integers n, Simson formula of binomial transforms of generalized third-order Pell numbers is given as*

$$\begin{vmatrix} b_{n+2} & b_{n+1} & b_n \\ b_{n+1} & b_n & b_{n-1} \\ b_n & b_{n-1} & b_{n-2} \end{vmatrix} = 3^n \begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}.$$

The previous Proposition gives the following results as particular examples.

Corollary 5.3. *For all integers n, Simson formula of binomial transforms of third-order Pell, Pell-Lucas and modified Pell numbers are given as*

$$\begin{vmatrix} \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n \\ \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} \\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} \end{vmatrix} = -3^{n-2},$$

$$\begin{vmatrix} \widehat{Q}_{n+2} & \widehat{Q}_{n+1} & \widehat{Q}_n \\ \widehat{Q}_{n+1} & \widehat{Q}_n & \widehat{Q}_{n-1} \\ \widehat{Q}_n & \widehat{Q}_{n-1} & \widehat{Q}_{n-2} \end{vmatrix} = -29 \times 3^{n-1}$$

and

$$\begin{vmatrix} \widehat{E}_{n+2} & \widehat{E}_{n+1} & \widehat{E}_n \\ \widehat{E}_{n+1} & \widehat{E}_n & \widehat{E}_{n-1} \\ \widehat{E}_n & \widehat{E}_{n-1} & \widehat{E}_{n-2} \end{vmatrix} = -3^{n-1}$$

respectively.

6. Some Identities

In this section, we obtain some identities of binomial transforms of third order Pell, third order Pell-Lucas and modified third order Pell numbers. First, we can give a few basic relations between $\{\widehat{P}_n\}$ and $\{\widehat{Q}_n\}$.

Lemma 6.1. *The following equalities are true:*

$$3\widehat{Q}_n = -11\widehat{P}_{n+4} + 46\widehat{P}_{n+3} - 24\widehat{P}_{n+2}, \tag{6.1}$$
$$\widehat{Q}_n = -3\widehat{P}_{n+3} + 14\widehat{P}_{n+2} - 11\widehat{P}_{n+1}, \tag{6.2}$$
$$\widehat{Q}_n = -\widehat{P}_{n+2} + 7\widehat{P}_{n+1} - 9\widehat{P}_n, \tag{6.3}$$
$$\widehat{Q}_n = 2\widehat{P}_{n+1} - 3\widehat{P}_n - 3\widehat{P}_{n-1}, \tag{6.4}$$
$$\widehat{Q}_n = 7\widehat{P}_n - 15\widehat{P}_{n-1} + 6\widehat{P}_{n-2} \tag{6.5}$$

and

$$261\widehat{P}_n = -16\widehat{Q}_{n+4} + 89\widehat{Q}_{n+3} - 108\widehat{Q}_{n+2}, \tag{6.6}$$
$$87\widehat{P}_n = 3\widehat{Q}_{n+3} - 4\widehat{Q}_{n+2} - 16\widehat{Q}_{n+1}, \tag{6.7}$$
$$87\widehat{P}_n = 11\widehat{Q}_{n+2} - 34\widehat{Q}_{n+1} + 9\widehat{Q}_n, \tag{6.8}$$
$$29\widehat{P}_n = 7\widehat{Q}_{n+1} - 19\widehat{Q}_n + 11\widehat{Q}_{n-1}, \tag{6.9}$$
$$29\widehat{P}_n = 16\widehat{Q}_n - 31\widehat{Q}_{n-1} + 21\widehat{Q}_{n-2}. \tag{6.10}$$

Proof. Note that all the identities hold for all integers n. We prove (6.1). To show (6.1), writing

$$\widehat{Q}_n = a \times \widehat{P}_{n+4} + b \times \widehat{P}_{n+3} + c \times \widehat{P}_{n+2}$$

and solving the system of equations

$$\widehat{Q}_0 = a \times \widehat{P}_4 + b \times \widehat{P}_3 + c \times \widehat{P}_2,$$
$$\widehat{Q}_1 = a \times \widehat{P}_5 + b \times \widehat{P}_4 + c \times \widehat{P}_3,$$
$$\widehat{Q}_2 = a \times \widehat{P}_6 + b \times \widehat{P}_5 + c \times \widehat{P}_4,$$

we find that $a = -\frac{11}{3}$, $b = \frac{46}{3}$, $c = -8$. The other equalities can be proved similarly. □

Note that all the identities in the above lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{\widehat{P}_n\}$ and $\{\widehat{E}_n\}$.

Lemma 6.2. *The following equalities are true:*

$$3\widehat{E}_n = -\widehat{P}_{n+3} + 2\widehat{P}_{n+2} + 6\widehat{P}_{n+1},$$
$$\widehat{E}_n = -\widehat{P}_{n+2} + 4\widehat{P}_{n+1} - \widehat{P}_n,$$
$$\widehat{E}_n = -\widehat{P}_{n+1} + 5\widehat{P}_n - 3\widehat{P}_{n-1}$$

and

$$3\widehat{P}_n = \widehat{E}_{n+3} - 4\widehat{E}_{n+2} + 3\widehat{E}_{n+1},$$
$$3\widehat{P}_n = \widehat{E}_{n+2} - 3\widehat{E}_{n+1} + 3\widehat{E}_n,$$
$$3\widehat{P}_n = 2\widehat{E}_{n+1} - 3\widehat{E}_n + 3\widehat{E}_{n-1}.$$

Thirdly, we give a few basic relations between $\{\widehat{Q}_n\}$ and $\{\widehat{E}_n\}$.

Lemma 6.3. *The following equalities are true:*

$$3\widehat{Q}_n = -4\widehat{E}_{n+3} + 18\widehat{E}_{n+2} - 9\widehat{E}_{n+1},$$
$$3\widehat{Q}_n = -2\widehat{E}_{n+2} + 15\widehat{E}_{n+1} - 12\widehat{E}_n,$$
$$3\widehat{Q}_n = 5\widehat{E}_{n+1} - 6\widehat{E}_{n-1},$$

and

$$87\widehat{E}_n = 20\widehat{Q}_{n+3} - 75\widehat{Q}_{n+2} + 19\widehat{Q}_{n+1},$$
$$87\widehat{E}_n = 25\widehat{Q}_{n+2} - 101\widehat{Q}_{n+1} + 60\widehat{Q}_n,$$
$$29\widehat{E}_n = 8\widehat{Q}_{n+1} - 30\widehat{Q}_n + 25\widehat{Q}_{n-1}.$$

We now present a few special identities for the binomial transform of the third order Pell-Lucas sequence $\{\widehat{Q}_n\}$.

Theorem 6.4 (Catalan's identity)**.** *For all integers n and m, the following identity holds*

$$\widehat{Q}_{n+m}\widehat{Q}_{n-m} - \widehat{Q}_n^2 = \frac{1}{9}(5\widehat{E}_{n+m+1} - 6\widehat{E}_{n+m-1})(5\widehat{E}_{n-m+1} - 6\widehat{E}_{n-m-1}) - \frac{1}{3}(5\widehat{E}_{n+1} - 6\widehat{E}_{n-1})^2.$$

Proof. We use the identity

$$\widehat{Q}_n = \frac{1}{3}(5\widehat{E}_{n+1} - 6\widehat{E}_{n-1}).$$

□

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the binomial transform of the third order Pell-Lucas sequnce

Corollary 6.5 (Cassini's identity)**.** *For all integers n and m, the following identity holds*

$$\widehat{Q}_{n+1}\widehat{Q}_{n-1} - \widehat{Q}_n^2 = \frac{1}{9}(5\widehat{E}_{n+2} - 6\widehat{E}_n)(5\widehat{E}_n - 6\widehat{E}_{n-2}) - \frac{1}{3}(5\widehat{E}_{n+1} - 6\widehat{E}_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $\widehat{Q}_n = \frac{1}{3}(5\widehat{E}_{n+1} - 6\widehat{E}_{n-1})$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of the binomial transform of third order Pell-Lucas sequence $\{\widehat{Q}_n\}$.

Theorem 6.6. *Let n and m be any integers. Then the following identities are true:*

(a) (*d'Ocagne's identity*)

$$\widehat{Q}_{m+1}\widehat{Q}_n - \widehat{Q}_m\widehat{Q}_{n+1} = \frac{1}{9}((5\widehat{E}_{m+2} - 6\widehat{E}_m)(5\widehat{E}_{n+1} - 6\widehat{E}_{n-1}) - (5\widehat{E}_{m+1} - 6\widehat{E}_{m-1})(5\widehat{E}_{n+2} - 6\widehat{E}_n)).$$

(b) (*Gelin-Cesàro's identity*)

$$\widehat{Q}_{n+2}\widehat{Q}_{n+1}\widehat{Q}_{n-1}\widehat{Q}_{n-2} - \widehat{Q}_n^4 = \frac{1}{81}((5\widehat{E}_{n+3} - 6\widehat{E}_{n+1})(5\widehat{E}_{n+2} - 6\widehat{E}_n)(5\widehat{E}_n - 6\widehat{E}_{n-2})(5\widehat{E}_{n-1} - 6\widehat{E}_{n-3}) - (5\widehat{E}_{n+1} - 6\widehat{E}_{n-1})^4).$$

(c) (*Melham's identity*)

$$\widehat{Q}_{n+1}\widehat{Q}_{n+2}\widehat{Q}_{n+6}-\widehat{Q}^3_{n+3}=\frac{1}{27}((5\widehat{E}_{n+2}-6\widehat{E}_n)(5\widehat{E}_{n+3}-6\widehat{E}_{n+1})(5\widehat{E}_{n+7}-6\widehat{E}_{n+5})-(5\widehat{E}_{n+1}-6\widehat{E}_{n-1})^3).$$

Proof. Use the identity $\widehat{Q}_n=\frac{1}{3}(5\widehat{E}_{n+1}-6\widehat{E}_{n-1})$. □

7. Linear Sums

The following theorem presents some linear summing formulas of generalized Tribonacci numbers with positive subscripts.

Theorem 7.1. *For $n\geq 0$, we have the following formulas:*

(a) (*Sum of the generalized Tribonacci numbers*) *If $r+s+t-1\neq 0$, then*

$$\sum_{k=0}^{n}W_k=\frac{W_{n+3}+(1-r)W_{n+2}+(1-r-s)W_{n+1}-W_2+(r-1)W_1+(r+s-1)W_0}{r+s+t-1}.$$

(b) *If $2s+2rt+r^2-s^2+t^2-1=(r+s+t-1)(r-s+t+1)\neq 0$ then*

$$\sum_{k=0}^{n}W_{2k}=\frac{\begin{pmatrix}(-s+1)W_{2n+2}+(t+rs)W_{2n+1}+(t^2+rt)W_{2n}+(-1+s)W_2\\+(-t-rs)W_1+(-1+r^2-s^2+rt+2s)W_0\end{pmatrix}}{(r+s+t-1)(r-s+t+1)}$$

and

$$\sum_{k=0}^{n}W_{2k+1}=\frac{\begin{pmatrix}(r+t)W_{2n+2}+(s-s^2+t^2+rt)W_{2n+1}+(t-st)W_{2n}+(-r-t)W_2\\+(-1+s+r^2+rt)W_1+(-t+st)W_0\end{pmatrix}}{(r-s+t+1)(r+s+t-1)}.$$

Proof. This is given in Soykan [22]. □

Taking $r=5$, $s=-6$, $t=3$, $b_n=W_n$ in Theorem 7.1(a,b), we obtain the following proposition which gives sum formulas for the binomial transform of the generalized third order Pell sequence

Proposition 7.2. *For $n\geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n}b_k=b_{n+3}-4b_{n+2}+2b_{n+1}-b_2+4b_1-2b_0$.

(b) $\sum_{k=0}^{n}b_{2k}=\frac{1}{15}(7b_{2n+2}-27b_{2n+1}+24b_{2n}-7b_2+27b_1-9b_0)$.

(c) $\sum_{k=0}^{n}b_{2k+1}=\frac{1}{15}(8b_{2n+2}-18b_{2n+1}+21b_{2n}-8b_2+33b_1-21b_0)$.

From the above proposition, we have the following corollary which gives linear sum formulas of the binomial transform of the third order Pell numbers (take $b_n=\widehat{P}_n$ with $\widehat{P}_0=0$, $\widehat{P}_1=1$, $\widehat{P}_2=4$).

Corollary 7.3. *For $n\geq 0$, binomial transform of the third order Pell numbers have the following properties.*

(a) $\sum_{k=0}^{n}\widehat{P}_k=\widehat{P}_{n+3}-4\widehat{P}_{n+2}+2\widehat{P}_{n+1}$.

(b) $\sum_{k=0}^{n} \widehat{P}_{2k} = \frac{1}{15}(7\widehat{P}_{2n+2} - 27\widehat{P}_{2n+1} + 24\widehat{P}_{2n} - 1)$.

(c) $\sum_{k=0}^{n} \widehat{P}_{2k+1} = \frac{1}{15}(8\widehat{P}_{2n+2} - 18\widehat{P}_{2n+1} + 21\widehat{P}_{2n} + 1)$.

Taking $b_n = \widehat{Q}_n$ with $\widehat{Q}_0 = 3$, $\widehat{Q}_1 = 5$, $\widehat{Q}_2 = 13$ in the above proposition, we have the following corollary which presents linear sum formulas of the binomial transform of the third order Pell-Lucas numbers.

Corollary 7.4. *For $n \geq 0$, binomial transform of the third order Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} \widehat{Q}_k = \widehat{Q}_{n+3} - 4\widehat{Q}_{n+2} + 2\widehat{Q}_{n+1} + 1$.

(b) $\sum_{k=0}^{n} \widehat{Q}_{2k} = \frac{1}{15}(7\widehat{Q}_{2n+2} - 27\widehat{Q}_{2n+1} + 24\widehat{Q}_{2n} + 17)$.

(c) $\sum_{k=0}^{n} \widehat{Q}_{2k+1} = \frac{1}{15}(8\widehat{Q}_{2n+2} - 18\widehat{Q}_{2n+1} + 21\widehat{Q}_{2n} - 2)$.

From the above Proposition, we have the following corollary which gives linear sum formulas of the binomial transform of the third order modified Pell numbers (take $b_n = \widehat{E}_n$ with $\widehat{E}_0 = 0$, $\widehat{E}_1 = 1$, $\widehat{E}_2 = 3$).

Corollary 7.5. *For $n \geq 0$, binomial transform of the third order modified Pell numbers have the following properties.*

(a) $\sum_{k=0}^{n} \widehat{E}_k = \widehat{E}_{n+3} - 4\widehat{E}_{n+2} + 2\widehat{E}_{n+1} + 1$.

(b) $\sum_{k=0}^{n} \widehat{E}_{2k} = \frac{1}{15}(7\widehat{E}_{2n+2} - 27\widehat{E}_{2n+1} + 24\widehat{E}_{2n} + 6)$.

(c) $\sum_{k=0}^{n} \widehat{E}_{2k+1} = \frac{1}{15}(8\widehat{E}_{2n+2} - 18\widehat{E}_{2n+1} + 21\widehat{E}_{2n} + 9)$.

The following Theorem presents some linear summing formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 7.6. *For $n \geq 1$, we have the following formulas:*

(a) (*Sum of the generalized Tribonacci numbers with negative indices*) *If $r + s + t - 1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{-(r+s+t)W_{-n-1} - (s+t)W_{-n-2} - tW_{-n-3} + W_2 + (1-r)W_1 + (1-r-s)W_0}{r+s+t-1}.$$

(b) *If $(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\begin{pmatrix} -(r+t)W_{-2n+1} + (r^2+rt+s-1)W_{-2n} + (st-t)W_{-2n-1} \\ +(1-s)W_2 + (t+rs)W_1 + (1-rt-2s-r^2+s^2)W_0 \end{pmatrix}}{(r+s+t-1)(r-s+t+1)}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{\begin{pmatrix} (s-1)W_{-2n+1} - (t+rs)W_{-2n} - (t^2+rt)W_{-2n-1} \\ +(r+t)W_2 + (1-r^2-rt-s)W_1 + (t-st)W_0 \end{pmatrix}}{(r+s+t-1)(r-s+t+1)}.$$

Proof. This is given in Soykan [22]. □

Taking $r = 5$, $s = -6$, $t = 3$, $b_n = W_n$ in Theorem 7.6(a,b), we obtain the following proposition which gives sum formulas for the binomial transform of the generalized third order Pell sequence with negative subscripts.

Proposition 7.7. *For $n \geq 1$, we have the following formulas:*

(a) $\sum_{k=1}^{n} b_{-k} = -2b_{-n-1} + 3b_{-n-2} - 3b_{-n-3} + b_2 - 4b_1 + 2b_0$.

(b) $\sum_{k=1}^{n} b_{-2k} = \frac{1}{15}(-8b_{-2n+1} + 33b_{-2n} - 21b_{-2n-1} + 7b_2 - 27b_1 + 9b_0)$.

(c) $\sum_{k=1}^{n} b_{-2k+1} = \frac{1}{15}(-7b_{-2n+1} + 27b_{-2n} - 24b_{-2n-1} + 8b_2 - 33b_1 + 21b_0)$.

Taking $b_n = \widehat{P}_n$ with $\widehat{P}_0 = 0, \widehat{P}_1 = 1, \widehat{P}_2 = 4$ in the above proposition, we have the following corollary which gives linear sum formulas of the binomial transform of third order Pell sequence with negative subscripts.

Corollary 7.8. *For $n \geq 1$, binomial transform of the third order Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{P}_{-k} = -2\widehat{P}_{-n-1} + 3\widehat{P}_{-n-2} - 3\widehat{P}_{-n-3}$.

(b) $\sum_{k=1}^{n} \widehat{P}_{-2k} = \frac{1}{15}(-8\widehat{P}_{-2n+1} + 33\widehat{P}_{-2n} - 21\widehat{P}_{-2n-1} + 1)$.

(c) $\sum_{k=1}^{n} \widehat{P}_{-2k+1} = \frac{1}{15}(-7\widehat{P}_{-2n+1} + 27\widehat{P}_{-2n} - 24\widehat{P}_{-2n-1} - 1)$.

From the last Proposition, we have the following corollary which gives linear sum formulas of the binomial transform of the third order Pell-Lucas numbers (take $b_n = \widehat{Q}_n$ with $\widehat{Q}_0 = 3$, $\widehat{Q}_1 = 5$, $\widehat{Q}_2 = 13$).

Corollary 7.9. *For $n \geq 1$, binomial transform of the third order Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{Q}_{-k} = -2\widehat{Q}_{-n-1} + 3\widehat{Q}_{-n-2} - 3\widehat{Q}_{-n-3} - 1$.

(b) $\sum_{k=1}^{n} \widehat{Q}_{-2k} = \frac{1}{15}(-8\widehat{Q}_{-2n+1} + 33\widehat{Q}_{-2n} - 21\widehat{Q}_{-2n-1} - 17)$.

(c) $\sum_{k=1}^{n} \widehat{Q}_{-2k+1} = \frac{1}{15}(-7\widehat{Q}_{-2n+1} + 27\widehat{Q}_{-2n} - 24\widehat{Q}_{-2n-1} + 2)$.

Taking $b_n = \widehat{E}_n$ with $\widehat{E}_0 = 0$, $\widehat{E}_1 = 1\widehat{E}_2 = 3$ in the above proposition, we have the following corollary which gives linear sum formulas of the binomial transform of third order modified Pell sequence with negative subscripts.

Corollary 7.10. *For $n \geq 1$, binomial transform of the third order modified Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{E}_{-k} = -2\widehat{E}_{-n-1} + 3\widehat{E}_{-n-2} - 3\widehat{E}_{-n-3} - 1.$

(b) $\sum_{k=1}^{n} \widehat{E}_{-2k} = \frac{1}{15}(-8\widehat{E}_{-2n+1} + 33\widehat{E}_{-2n} - 21\widehat{E}_{-2n-1} - 6).$

(c) $\sum_{k=1}^{n} \widehat{E}_{-2k+1} = \frac{1}{15}(-7\widehat{E}_{-2n+1} + 27\widehat{E}_{-2n} - 24\widehat{E}_{-2n-1} - 9).$

8. Matrices Related with Binomial Transform of Generalized Third-Order Pell numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{8.1}$$

For matrix formulation (8.1), see [10]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 5 & -6 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 3$. From (2.1) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 5 & -6 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n+1} \\ b_n \\ b_{n-1} \end{pmatrix} \tag{8.2}$$

and from (8.1) (or using (8.2) and induction), we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 5 & -6 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}.$$

If we take $b_n = \widehat{P}_n$ in (8.2), we have

$$\begin{pmatrix} \widehat{P}_{n+2} \\ \widehat{P}_{n+1} \\ \widehat{P}_n \end{pmatrix} = \begin{pmatrix} 5 & -6 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{P}_{n+1} \\ \widehat{P}_n \\ \widehat{P}_{n-1} \end{pmatrix}. \tag{8.3}$$

For $n \geq 0$, we also define

$$B_n = \begin{pmatrix} \sum\limits_{k=0}^{n+1} \widehat{P}_k & -3\left(2\sum\limits_{k=0}^{n} \widehat{P}_k - \sum\limits_{k=0}^{n-1} \widehat{P}_k\right) & 3\sum\limits_{k=0}^{n} \widehat{P}_k \\ \sum\limits_{k=0}^{n} \widehat{P}_k & -3\left(2\sum\limits_{k=0}^{n-1} \widehat{P}_k - \sum\limits_{k=0}^{n-2} \widehat{P}_k\right) & 3\sum\limits_{k=0}^{n-1} \widehat{P}_k \\ \sum\limits_{k=0}^{n-1} \widehat{P}_k & -3\left(2\sum\limits_{k=0}^{n-2} \widehat{P}_k - \sum\limits_{k=0}^{n-3} \widehat{P}_k\right) & 3\sum\limits_{k=0}^{n-2} \widehat{P}_k \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} b_{n+1} & -6b_n + 3b_{n-1} & 3b_n \\ b_n & -6b_{n-1} + 3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -6b_{n-2} + 3b_{n-3} & 3b_{n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \widehat{P}_k = 0, \quad \sum_{k=0}^{-2} \widehat{P}_k = \frac{1}{3}, \quad \sum_{k=0}^{-3} \widehat{P}_k = \frac{2}{3}.$$

Theorem 8.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$,

(b) $C_1 A^n = A^n C_1$,

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. (a) Proof can be done by mathematical induction on n.

(b) After matrix multiplication, (b) follows.

(c) We have

$$AC_{n-1} = \begin{pmatrix} 5 & -6 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -6b_{n-1} + 3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -6b_{n-2} + 3b_{n-3} & 3b_{n-2} \\ b_{n-2} & -6b_{n-3} + 3b_{n-4} & 3b_{n-3} \end{pmatrix}$$
$$= \begin{pmatrix} b_{n+1} & -6b_n + 3b_{n-1} & 3b_n \\ b_n & -6b_{n-1} + 3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -6b_{n-2} + 3b_{n-3} & 3b_{n-2} \end{pmatrix} = C_n.$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$ □

Some properties of matrix A^n can be given as

$$A^n = 5A^{n-1} - 6A^{n-2} + 3A^{n-3} = 2A^{n+1} - \frac{5}{3}A^{n+2} + \frac{1}{3}A^{n+3},$$
$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 3^n$$

for all integer $m, n \geq 0$.

Theorem 8.2. *For $m,n \geq 0$, we have*

$$b_{n+m} = b_n \sum_{k=0}^{m+1} \widehat{P}_k + b_{n-1}\left(-6\sum_{k=0}^{m} \widehat{P}_k + 3\sum_{k=0}^{m-1} \widehat{P}_k\right) + 3b_{n-2}\sum_{k=0}^{m} \widehat{P}_k \tag{8.4}$$

$$= \left(\sum_{k=0}^{m} \widehat{P}_k\right)(3b_{n-2} - 6b_{n-1}) + 3\left(\sum_{k=0}^{m-1} \widehat{P}_k\right) b_{n-1} + \left(\sum_{k=0}^{m+1} \widehat{P}_k\right) b_n . \tag{8.5}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Corollary 8.3. *For $m,n \geq 0$, we have*

$$\widehat{P}_{n+m} = \widehat{P}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{P}_{n-1}\left(-6\sum_{k=0}^{m} \widehat{P}_k + 3\sum_{k=0}^{m-1} \widehat{P}_k\right) + 3\widehat{P}_{n-2}\sum_{k=0}^{m} \widehat{P}_k ,$$

$$\widehat{Q}_{n+m} = \widehat{Q}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{Q}_{n-1}\left(-6\sum_{k=0}^{m} \widehat{P}_k + 3\sum_{k=0}^{m-1} \widehat{P}_k\right) + 3\widehat{Q}_{n-2}\sum_{k=0}^{m} \widehat{P}_k ,$$

$$\widehat{E}_{n+m} = \widehat{E}_n \sum_{k=0}^{m+1} \widehat{P}_k + \widehat{E}_{n-1}\left(-6\sum_{k=0}^{m} \widehat{P}_k + 3\sum_{k=0}^{m-1} \widehat{P}_k\right) + 3\widehat{E}_{n-2}\sum_{k=0}^{m} \widehat{P}_k .$$

From Corollary 7.3, we know that for $n \geq 0$,

$$\sum_{k=0}^{n} \widehat{P}_k = \widehat{P}_{n+3} - 4\widehat{P}_{n+2} + 2\widehat{P}_{n+1} .$$

Thus Theorem 8.2 and Corollary 8.3 can be written in the following forms:

Theorem 8.4. *For $m,n \geq 0$, we have*

$$\begin{aligned} b_{n+m} &= b_n(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 3\widehat{P}_{m+1}) + b_{n-1}(-4\widehat{P}_{m+3} + 17\widehat{P}_{m+2} - 12\widehat{P}_{m+1}) \\ &\quad + 3b_{n-2}(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 2\widehat{P}_{m+1}). \end{aligned}$$

Corollary 8.5. *For $m,n \geq 0$, we have*

$$\begin{aligned} \widehat{P}_{n+m} &= \widehat{P}_n(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 3\widehat{P}_{m+1}) + \widehat{P}_{n-1}(-4\widehat{P}_{m+3} + 17\widehat{P}_{m+2} - 12\widehat{P}_{m+1}) \\ &\quad + 3\widehat{P}_{n-2}(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 2\widehat{P}_{m+1}), \\ \widehat{Q}_{n+m} &= \widehat{Q}_n(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 3\widehat{P}_{m+1}) + \widehat{Q}_{n-1}(-4\widehat{P}_{m+3} + 17\widehat{P}_{m+2} - 12\widehat{P}_{m+1}) \\ &\quad + 3\widehat{Q}_{n-2}(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 2\widehat{P}_{m+1}), \\ \widehat{E}_{n+m} &= \widehat{E}_n(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 3\widehat{P}_{m+1}) + \widehat{E}_{n-1}(-4\widehat{P}_{m+3} + 17\widehat{P}_{m+2} - 12\widehat{P}_{m+1}) \\ &\quad + 3\widehat{E}_{n-2}(\widehat{P}_{m+3} - 4\widehat{P}_{m+2} + 2\widehat{P}_{m+1}). \end{aligned}$$

Now, we consider non-positive subscript cases. For $n \geq 0$, we define

$$B_{-n} = \begin{pmatrix} -\sum_{k=0}^{n-2} \widehat{P}_{-k} & 3(2\sum_{k=0}^{n-1} \widehat{P}_{-k} - \sum_{k=0}^{n} \widehat{P}_{-k}) & -3\sum_{k=0}^{n-1} \widehat{P}_{-k} \\ -\sum_{k=0}^{n-1} \widehat{P}_{-k} & 3(2\sum_{k=0}^{n} \widehat{P}_{-k} - \sum_{k=0}^{n+1} \widehat{P}_{-k}) & -3\sum_{k=0}^{n} \widehat{P}_{-k} \\ -\sum_{k=0}^{n} \widehat{P}_{-k} & 3(2\sum_{k=0}^{n+1} \widehat{P}_{-k} - \sum_{k=0}^{n+2} \widehat{P}_{-k}) & -3\sum_{k=0}^{n+1} \widehat{P}_{-k} \end{pmatrix}$$

and

$$C_{-n} = \begin{pmatrix} b_{-n+1} & -6b_{-n}+3b_{-n-1} & 3b_{-n} \\ b_{-n} & -6b_{-n-1}+3b_{-n-2} & 3b_{-n-1} \\ b_{-n-1} & -6b_{-n-2}+3b_{-n-3} & 3b_{-n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \widehat{P}_{-k} = 0, \quad \sum_{k=0}^{-2} \widehat{P}_{-k} = -1.$$

Theorem 8.6. *For all integer $m,n \geq 0$, we have*

(a) $B_{-n} = A^{-n}$,

(b) $C_{-1}A^{-n} = A^{-n}C_{-1}$,

(c) $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$.

Proof. (a) Proof can be done by mathematical induction on n.

(b) After matrix multiplication, (b) follows.

(c) We have

$$\begin{aligned} A^{-1}C_{-n-1} &= \begin{pmatrix} 5 & -6 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{-n} & -6b_{-n-1}+3b_{-n-2} & 3b_{-n-1} \\ b_{-n-1} & -6b_{-n-2}+3b_{-n-3} & 3b_{-n-2} \\ b_{-n-2} & -6b_{-n-3}+3b_{-n-4} & 3b_{-n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{-n+1} & -6b_{-n}+3b_{-n-1} & 3b_{-n} \\ b_{-n} & -6b_{-n-1}+3b_{-n-2} & 3b_{-n-1} \\ b_{-n-1} & -6b_{-n-2}+3b_{-n-3} & 3b_{-n-2} \end{pmatrix} = C_{-n}. \end{aligned}$$

i.e. $C_{-n} = A^{-1}C_{-n-1}$. From the last equation, using induction we obtain $C_{-n} = A^{-n-1}C_{-1}$. Now

$$C_{-n-m} = A^{-n-m-1}C_{-1} = A^{-n-1}A^{-m}C_{-1} = A^{-n-1}C_{-1}A^{-m} = C_{-n}B_{-m}$$

and similarly

$$C_{-n-m} = B_{-m}C_{-n}.$$ □

Some properties of matrix A^{-n} can be given as

$$A^{-n} = 5A^{-n-1} - 6A^{-n-2} + 3A^{-n-3} = 2A^{-n+1} - \frac{5}{3}A^{-n+2} + \frac{1}{3}A^{-n+3},$$

$$A^{-n-m} = A^{-n}A^{-m} = A^{-m}A^{-n}$$

and

$$\det(A^{-n}) = 3^{-n}$$

for all integer $m,n \geq 0$.

Theorem 8.7. *For $m,n \geq 0$, we have*

$$\begin{aligned} b_{-n-m} &= -b_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - b_{-n-1}\left(-6\sum_{k=0}^{m-1}\widehat{P}_{-k} + 3\sum_{k=0}^{m}\widehat{P}_{-k}\right) - 3b_{-n-2}\left(\sum_{k=0}^{m-1}\widehat{P}_{-k}\right) \\ &= -3\left(\sum_{k=0}^{m}\widehat{P}_{-k}\right)b_{-n-1} - b_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - (3b_{-n-2} - 6b_{-n-1})\sum_{k=0}^{m-1}\widehat{P}_{-k}. \end{aligned}$$

Proof. From the equation $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$ we see that an element of C_{-n-m} is the product of row C_{-n} and a column B_{-m}. From the last equation we say that an element of C_{-n-m} is the product of a row C_{-n} and column B_{-m}. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{-n-m} and $C_{-n}B_{-m}$. This completes the proof. □

Corollary 8.8. *For $m,n \geq 0$, we have*

$$\begin{aligned}
\widehat{P}_{-n-m} &= -\widehat{P}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{P}_{-n-1}\left(-6\sum_{k=0}^{m-1}\widehat{P}_{-k} + 3\sum_{k=0}^{m}\widehat{P}_{-k}\right) - 3\widehat{P}_{-n-2}\left(\sum_{k=0}^{m-1}\widehat{P}_{-k}\right),\\
\widehat{Q}_{-n-m} &= -\widehat{Q}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{Q}_{-n-1}\left(-6\sum_{k=0}^{m-1}\widehat{P}_{-k} + 3\sum_{k=0}^{m}\widehat{P}_{-k}\right) - 3\widehat{Q}_{-n-2}\left(\sum_{k=0}^{m-1}\widehat{P}_{-k}\right),\\
\widehat{E}_{-n-m} &= -\widehat{E}_{-n}\sum_{k=0}^{m-2}\widehat{P}_{-k} - \widehat{E}_{-n-1}\left(-6\sum_{k=0}^{m-1}\widehat{P}_{-k} + 3\sum_{k=0}^{m}\widehat{P}_{-k}\right) - 3\widehat{E}_{-n-2}\left(\sum_{k=0}^{m-1}\widehat{P}_{-k}\right).
\end{aligned}$$

From Corollary 7.8, we know that for $n \geq 1$,

$$\sum_{k=1}^{n}\widehat{P}_{-k} = -2\widehat{P}_{-n-1} + 3\widehat{P}_{-n-2} - 3\widehat{P}_{-n-3}.$$

Since $\widehat{P}_0 = 0$, it follows that

$$\sum_{k=0}^{n}\widehat{P}_{-k} = -2\widehat{P}_{-n-1} + 3\widehat{P}_{-n-2} - 3\widehat{P}_{-n-3}.$$

Thus Theorem 8.7 and Corollary 8.8 can be written in the following forms.

Theorem 8.9. *For $m,n \geq 0$, we have*

$$\begin{aligned}
b_{-n-m} &= -b_{-n}(-7\widehat{P}_{-m} + 9\widehat{P}_{-m-1} - 6\widehat{P}_{-m-2}) - 9b_{-n-1}(\widehat{P}_{-m} - \widehat{P}_{-m-1} + \widehat{P}_{-m-2})\\
&\quad - 3b_{-n-2}(-2\widehat{P}_{-m} + 3\widehat{P}_{-m-1} - 3\widehat{P}_{-m-2}).
\end{aligned}$$

Corollary 8.10. *For $m,n \geq 0$, we have*

$$\begin{aligned}
\widehat{P}_{-n-m} &= -\widehat{P}_{-n}(-7\widehat{P}_{-m} + 9\widehat{P}_{-m-1} - 6\widehat{P}_{-m-2}) - 9\widehat{P}_{-n-1}(\widehat{P}_{-m} - \widehat{P}_{-m-1} + \widehat{P}_{-m-2})\\
&\quad - 3\widehat{P}_{-n-2}(-2\widehat{P}_{-m} + 3\widehat{P}_{-m-1} - 3\widehat{P}_{-m-2}),\\
\widehat{Q}_{-n-m} &= -\widehat{Q}_{-n}(-7\widehat{P}_{-m} + 9\widehat{P}_{-m-1} - 6\widehat{P}_{-m-2}) - 9\widehat{Q}_{-n-1}(\widehat{P}_{-m} - \widehat{P}_{-m-1} + \widehat{P}_{-m-2})\\
&\quad - 3\widehat{Q}_{-n-2}(-2\widehat{P}_{-m} + 3\widehat{P}_{-m-1} - 3\widehat{P}_{-m-2}),\\
\widehat{E}_{-n-m} &= -\widehat{E}_{-n}(-7\widehat{P}_{-m} + 9\widehat{P}_{-m-1} - 6\widehat{P}_{-m-2}) - 9\widehat{E}_{-n-1}(\widehat{P}_{-m} - \widehat{P}_{-m-1} + \widehat{P}_{-m-2})\\
&\quad - 3\widehat{E}_{-n-2}(-2\widehat{P}_{-m} + 3\widehat{P}_{-m-1} - 3\widehat{P}_{-m-2})).
\end{aligned}$$

9. Conclusion

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduced the binomial transform of the generalized third-order Pell sequence and as special cases, the binomial transform of the third-order Pell, third-order Pell-Lucas and modified third-order Pell sequences have been defined. In Section 1, we present some background about the generalized Tribonacci numbers and the generalized third-order Pell numbers. In Section 2, we defined the binomial transform of the generalized third-order Pell sequence. In Sections 3 and 4, we gave generating functions and Binet's formulas of the binomial transform of the generalized third-order Pell sequence. In Section 5, we present Simson formulas of the binomial transform of the generalized third-order Pell sequence. In Section 6, we obtained some identities of the binomial transform of the generalized third-order Pell sequence. In Section 7, we present sum formulas of the binomial transform of the generalized third-order Pell sequence. In Section 8, we gave some matrix formulation of the binomial transform of the generalized third-order Pell sequence.

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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