Communications in Mathematics and Applications
Vol. 13, No. 3, pp. 1037–1046, 2022
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v13i3.1369

http://www.rgnpublications.com

Research Article

2-Vertex Self Switching of Trees

C. Jayasekaran*, J. Christabel Sudha and M. Ashwin Shijo

Department of Mathematics, Pioneer Kumaraswamy College (Manonmaniam Sundaranar University), Nagercoil 629003, Tamil Nadu, India
***Corresponding author:** jaya_pkc@yahoo.com

Received: March 3, 2020 **Accepted:** May 21, 2022

Abstract. For a finite undirected graph $G(V,E)$ and a non empty subset $\sigma \subseteq V$, the *switching* of G by σ is defined as the graph $G^{\sigma}(V,E')$ which is obtained from G by removing all edges between σ and its complement V-σ and adding as edges all non-edges between σ and V-σ. For $\sigma = \{v\}$, we write G^{v} instead of $G^{\{v\}}$ and the corresponding switching is called as *vertex switching*. We also call it as $|\sigma|$-vertex switching. When $|\sigma| = 2$, it is termed as 2-vertex switching. If $G \cong G^{\sigma}$, then it is called *self vertex switching.* A subgraph B of G which contains $G[\sigma]$ is called a *joint* at σ in G if $B-\sigma$ is connected and maximal. If B is connected, then we call B as a *c-joint* and otherwise a *d-joint*. A graph with no cycles is called an acyclic graph. A connected acyclic graph is called a tree. In this paper, we give necessary and sufficient conditions for a graph G, for which G^{σ} at $\sigma = \{u,v\}$ to be connected and acyclic when $uv \in E(G)$ and $uv \notin E(G)$. Using this, we characterize trees with a 2-vertex self switching.

Keywords. Switching, 2-vertex self switching, $SS_2(G)$, $ss_2(G)$

Mathematics Subject Classification (2020). 05C60

1. Introduction

For a finite undirected simple graph $G(V,E)$ with $|V(G)| = p$ and a non-empty set $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V,E')$ which is obtained from G by removing all edges between σ and its complement, $V-\sigma$ and adding as edges all non-edges between σ and $V-\sigma$. *Switching* has been defined by Seidel [1,4] and is also referred to as Seidel switching. We also call it as $|\sigma|$-*vertex switching*. When $|\sigma| = 2$, we call it as 2-vertex switching [5]. Two graphs are said to be *switching equivalent* if they belong to the same switching class [2]. A graph G

is said to be a connected graph if every pair of vertices are joined by a path in G. A maximal connected subgraph of G is called a connected component or simply a component of G. A graph G is called disconnected if it is not connected. Clearly, a graph G is disconnected if and only if G has more than one component. The number of components of a graph G is represented by $k(G)$. A graph which contains no cycles is called an acyclic graph. A connected acyclic graph is called a tree. Any graph without cycles is a forest. Thus the components of a forest are trees.

In [6] the concept of branches and joints in graphs were introduced. A subgraph B of G which contains $G[\sigma]$ is called a *joint* at σ in G if $B-\sigma$ is connected and maximal. If B is connected, then we call B as a *c-joint* and otherwise a *d-joint*. B is called a *total joint* if B is the join of σ and $B-\sigma$, that is $B=\sigma+(B-\sigma)$ [3,6].

For the graph G given in Figure 1.1, G^σ is given in Figure 1.2, $G[\sigma]$ is given in Figure 1.3 and $G-\sigma$ is given in Figure 1.4, where $\sigma=\{u,v\}$. The c-joint, d-joint and the total joint are given in Figures 1.5, 1.6 and 1.7, respectively.

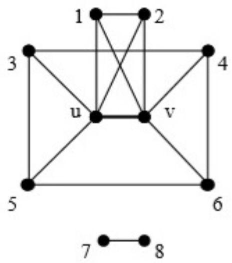

Figure 1.1. G

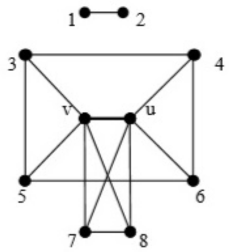

Figure 1.2. G^σ

Figure 1.3. $G[\sigma]$

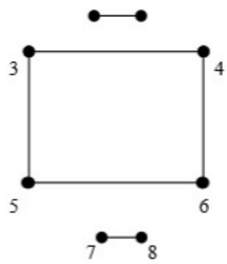

Figure 1.4. $G-\sigma$

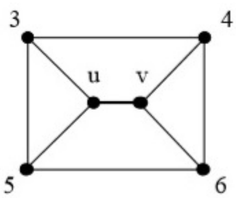

Figure 1.5. c-joint

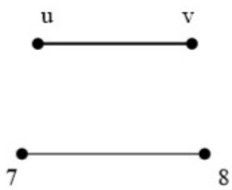

Figure 1.6. d-joint

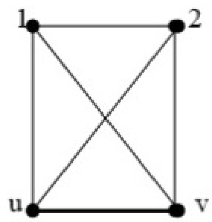

Figure 1.7. Total joint

2-Vertex Switching of Acyclic joints in Graphs

Now, consider the following results, which are required in the subsequent sections.

Theorem 1.1 ([6]). *If $B_1, B_2, \dots, B_k$ are the distinct joints at σ in G such that $G=\bigcup_{i=1}^{k} B_i$ where $k\ge 2$, then $G^\sigma=\bigcup_{i=1}^{k} B_i^\sigma$.*

Theorem 1.2 ([5]). *Let G be a graph of order $p\ge 3$ and let $\sigma=\{u,v\}$ be a subset of $V(G)$ such that $uv\in E(G)$. Let B be a c-joint at σ in G. Then B^σ is a c-joint and acyclic at σ in G^σ if and only if $B-\sigma$ is connected, acyclic and $\{d_B(u), d_B(v)\}=\{|V(B)|-1, |V(B)|-2\}$.*

Theorem 1.3 ([5]). *Let G be a graph of order $p \geq 4$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a c-joint at σ in G. Then B^σ is a c-joint and acyclic if and only if $B-\sigma$ is connected, acyclic, $|V(B)| \geq 4$ and $d_B(u) = d_B(v) = |V(B)|-3$.*

Theorem 1.4 ([5]). *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a c-joint at σ in G. Then B^σ is a d-joint and acyclic if and only if $B-\sigma$ is connected, acyclic and either $d_B(u) = d_B(v) = |V(B)|-2$ or $\{d_B(u), d_B(v)\} = \{|V(B)|-2, |V(B)|-3\}$.*

Theorem 1.5 ([5]). *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a d-joint at σ in G. Then B^σ is a c-joint and acyclic at σ in G^σ if and only if $B = 3K_1$.*

Theorem 1.6 ([5]). *Let G be a graph of order $p \geq 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let B be a d-joint at σ in G. Then B^σ is a d-joint and acyclic at σ in G^σ if and only if $B = K_1 \cup K_2$, where K_1 is either u or v.*

Theorem 1.7 ([5]). *If $\sigma = \{u,v\} \subseteq V$ is a 2-vertex self switching of a graph G, then*

$$d_G(u) + d_G(v) = \begin{cases} p & \text{if } uv \in E(G) \\ p-2 & \text{if } uv \notin E(G). \end{cases}$$

2. Main Results

2-Vertex Self Switching of Trees

Observation 2.1. *If G is a connected graph and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. If $B_1, B_2, B_3, \ldots, B_k$ are the k joints at σ in G, then each joint B_i at σ in G is a c-joint, $1 \leq i \leq k$.*

Consider the graph G given in Figure 2.1. The graph $G-\sigma$ is the union of three components K_1, P_2 and P_3 which is given in Figure 2.2. The three joints B_1, B_2 and B_3 are given in Figures 2.3, 2.4 and 2.5, respectively. Clearly, B_1, B_2 and B_3 are c-joints.

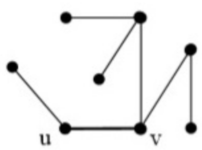

Figure 2.1. G

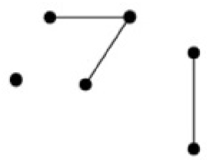

Figure 2.2. $G-\sigma$

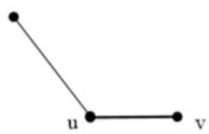

Figure 2.3. B_1

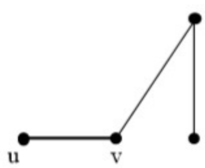

Figure 2.4. B_2

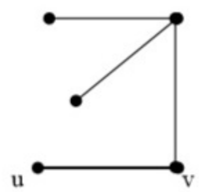

Figure 2.5. B_3

Theorem 2.2. *Let G be a connected graph of order $p \geq 3$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Then G^σ is connected and acyclic if and only if $B-\sigma$ is connected, acyclic and $\{d_B(u), d_B(v)\} = \{|V(B)|-2, |V(B)|-1\}$ for all joints B at σ in G.*

Proof. Let G be a connected graph and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Let $B_1, B_2, B_3, \dots, B_k$ be the k joints at σ in G. Then $G = \bigcup_{i=1}^{k} B_i$ and $G^\sigma = \bigcup_{i=1}^{k} B_i^\sigma$. Since G is connected, by Observation 2.1, each B_i is connected and hence a c-joint for $1 \leq i \leq k$. Suppose G^σ is connected and acyclic. Then each B_i^σ is connected and hence a c-joint and acyclic for $1 \leq i \leq k$. By Theorem 1.2, each $B_i - \sigma$ is connected, acyclic and $\{d_{B_i}(u), d_{B_i}(v)\} = \{|V(B_i)|-1, |V(B_i)|-2\}$ for $1 \leq i \leq k$.

Conversely, let $B-\sigma$ be connected, acyclic and $\{d_B(u), d_B(v)\} = \{|V(B)|-1, |V(B)|-2\}$ for all joints B at σ in G. By Theorem 1.2, each B^σ is a c-joint and acyclic. Since $G^\sigma = \cup B^\sigma$ and $uv \in E(G^\sigma)$, G^σ is connected and acyclic. □

Observation 2.3. *Let G be a connected graph and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. If $B_1, B_2, \dots, B_k$ are the k joints at σ in G, then each B_i is either a c-joint or a d-joint for $1 \leq i \leq k$.*

Consider the graph G given in Figure 2.6. The graph $G-\sigma$ is the union of P_2 and P_3 which is given in Figure 2.7. The joints B_1 and B_2 at σ are given in Figure 2.8 and Figure 2.9, respectively. Here B_1 is a d-joint and B_2 is a c-joint at σ in G.

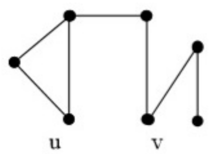

Figure 2.6. G

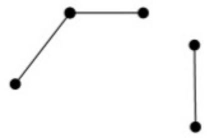

Figure 2.7. $G-\sigma$

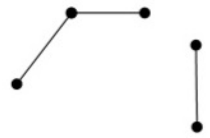

Figure 2.8. B_1

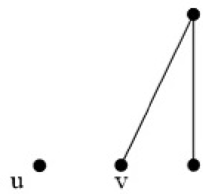

Figure 2.9. B_2

Theorem 2.4. *Let G be a connected graph of order $p \geq 4$ and let $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. Let $k \geq 1$ be the number of joints at σ in G. Then G^σ is connected and acyclic if and only if there exists at least one c-joint at σ in G, $B-\sigma$ is connected and acyclic for each joint B at σ in G, $d_B(u) = d_B(v) = |V(B)|-3$ and $|V(B)| \geq 4$ for exactly one c-joint $B = B^*$, $\{d_B(u), d_B(v)\} = \{|V(B)|-2, |V(B)|-3\}$ for all c-joints $B \neq B^*$ and $B = K_1 \cup K_2$ for all d-joints B, if exists, where K_1 is either u or v.*

Proof. Let G be a connected graph and $\sigma = \{u,v\}$ be a subset of $V(G)$ such that $uv \notin E(G)$. By Observation 2.3, G is connected implies that each joint at σ in G is either a c-joint or a d-joint. Let $B_{C_1}, B_{C_2}, \dots, B_{C_m}$ be the m c-joints and $B_{d_1}, B_{d_2}, \dots, B_{d_n}$ be the n d-joints at σ in G so that $m+n = k$. Then by Theorem 1.1, $G = \left(\bigcup_{i=1}^{m} B_{c_i}\right) \cup \left(\bigcup_{j=1}^{n} B_{d_j}\right)$ and $G^\sigma = \left(\bigcup_{i=1}^{m} B_{c_i}^\sigma\right) \cup \left(\bigcup_{j=1}^{n} B_{d_j}^\sigma\right)$. By Observation 2.3, each $B_{c_i}^\sigma$ is either a c-joint or a d-joint at σ in G^σ for $1 \leq i \leq m$, and each $B^{\sigma}_{d_j}$ is either a c-joint or a d-joint at σ in G for $1 \leq j \leq n$. Without loss of generality, let $B^{\sigma}_{c_1}, B^{\sigma}_{c_2}, \ldots, B^{\sigma}_{c_r}, B^{\sigma}_{d_1}, B^{\sigma}_{d_2}, \ldots, B^{\sigma}_{d_s}$ be the c-joints at σ in G^{σ} and $B^{\sigma}_{c_{r+1}}, B^{\sigma}_{c_{r+2}}, \ldots, B^{\sigma}_{c_m}, B^{\sigma}_{d_{s+1}}, B^{\sigma}_{d_{s+2}}, \ldots, B^{\sigma}_{d_n}$ be the d-joints at σ in G^{σ}.

Case 1. B is a c-joint at σ in G and B^{σ} is a c-joint at σ in G^{σ}.
Then $B = B_{c_i}$, $1 \leq i \leq r$. By Theorem 1.3, $B - \sigma$ is connected, acyclic, $|V(B)| \geq 4$ and $d_B(u) = d_B(v) = |V(B)| - 3$. If $r > 1$, then there exist c-joints B_1 and B_2 at σ in G such that $d_{B_1}(u) = d_{B_1}(v) = |V(B_1)| - 3$ and $d_{B_2}(u) = d_{B_2}(v) = |V(B_2)| - 3$. $d_{B_1}(u) = |V(B_1)| - 3$ implies that u is non-adjacent to only one vertex, say a, of $V(B_1) - \sigma$ in B_1 and hence u is adjacent to the unique vertex a in B^{σ}_1. In a similar argument, v is adjacent to the unique vertex, say b, in B^{σ}_1, u is adjacent to the unique vertex, say c, in B^{σ}_2 and v is adjacent to the unique vertex, say d, in B^{σ}_2. Since $B_1 - \sigma$ and $B_2 - \sigma$ are connected, there exist paths $a - b$ and $c - d$ in $B_1 - \sigma$ and $B_2 - \sigma$, respectively and hence in B^{σ}_1 and B^{σ}_2, respectively. Now, the edge ua, the path $a - b$, the edge bv is a $u - v$ path P in B^{σ}_1 and hence in G^{σ}Ȧlso, the edge uc, the path $c - d$ and the edge dv form a $u - v$ path P' in B^{σ}_2 and hence in G^{σ}. Thus P and P' are two distinct $u - v$ paths in G^{σ} and hence G^{σ} contains a cycle, which is a contradiction to G^{σ} is acyclic. Therefore, $r = 1$ and hence $d_B(u) = d_B(v) = |V(B)| - 3$ and $|V(B)| \geq 4$ for exactly one c-joint $B = B^*$ at σ in G.

Case 2. B is a c-joint at σ in G and B^{σ} is a d-joint at σ in G^{σ}
Here $B = B_{c_i}$, $2 \leq i \leq m$. By Theorem 1.4, B $-$ σ is connected, acyclic and either $d_B(u) = d_B(v) = |V(B)| - 2$ or $\{d_B(u), d_B(v)\} = \{|V(B)| - 2, |V(B)| - 3\}$. If $d_B(u) = d_B(v) = |V(B)| - 2$, then both u and v are isolated vertices in B^{σ} since $uv \notin E(G)$. This implies that $B - \sigma$ is a component of G^{σ} which is a contradiction to G^{σ} is connected. Hence $\{d_B(u), d_B(v)\} = \{|V(B)| - 2, |V(B)| - 3\}$.

Case 3. B is a d-joint at σ in G and B^{σ} is a c-joint at σ in G^{σ}.
In this case, $B = B^{\sigma}_{d_j}$, $1 \leq j \leq s$. By Theorem 1.5, $B = 3K_1$. This implies that K_1 is a component of G which is a contradiction to G is connected. Hence there do not exist any joint B at σ in G.

Case 4. B is a d-joint at σ in G and B^{σ} is a d-joint at σ in G^{σ}.
Here $B = B_{d_j}$, $1 \leq j \leq n$. By Theorem 1.6, $B = K_1 \cup K_2$, where K_1 is either u or v.
From Cases 1, 2, 3 and 4, we see that $B - \sigma$ is connected and acyclic for all joints B at σ in G, $d_B(u) = d_B(v) = |V(B)| - 3$ and $|V(B^*)| \geq 4$ for exactly one c-joint $B = B^*$, $\{d_B(u), d_B(v)\} = \{|V(B)| - 2, |V(B)| - 3\}$ for all c-joints $B \neq B^*$ and $B = K_1 \cup K_2$ for all d-joints B, if exists, where K_1 is either u or v.
Conversely, let $B - \sigma$ be connected and acyclic for each joint B at σ in G, $d^*_B(u) = d_{\boldsymbol{B}}(v) = |V(B)| - 3$ and $|V(B)| \geq 4$ for exactly one c-joint $B = B^*$, $\{d_B(u), d_B(v)\} = \{|V(B)| - 2, |V(B)| - 3\}$ for all c-joints $B \neq B^*$ and $B = K_1 \cup K_2$ for all d-joints B, if exists, where K_1 is either u or v. By Theorem 1.3, $B^{*^{\sigma}}$ is an acyclic c-joint at σ in G^{σ}. Hence there exists a $u - v$ path in B^*. Let $B \neq B^*$ be a c-joint at σ in G. Then $\{d_B(u), d_B(v)\} = \{|V(B)| - 2, |V(B)| - 3\}$. By Theorem 1.4, B^{σ} is a d-joint and acyclic at σ in G^{σ}. This implies that either $d_{B^{\sigma}}(u) = 0$ or $d_{B^{\sigma}}(v) = 0$. If B is a d-joint, then $B = K_1 \cup K_2$ where K_1 is either u or v. Now, $B^{\sigma} = K_1 \cup K_2$, where K_1 is either u or v and hence a d-joint at σ in G^{σ}.

Each B^σ is acyclic, exactly B^{*^σ} is a c-joint at σ in G^σ and all other joints at σ in G^σ are d-joints implies that G^σ is acyclic. B^{*^σ} is a c-joint at σ in G^σ implies that there exists a $u-v$ path in G^σ. To prove G^σ is connected. Let x and y be any two vertices in G^σ. We consider the following three cases.

Case 1. $\{x,y\} \neq \{u,v\}$.

Subcase 1.1. x and y are in different joints at σ in G^σ.
Let B_1^σ and B_2^σ be two joints at σ in G^σ such that $x \in V(B_1^\sigma)$ and $y \in V(B_2^\sigma)$. Since B^{*^σ} is the only c-joint at σ in G^σ, we have the following possibilities:

Subcase 1.1.a. B_1^σ is a c-joint and B_2^σ is an d-joint at σ in G^σ.
Then $B_1^\sigma = B^{*^\sigma}$. The paths $x-u$ and $u-v$ in B^{*^σ} and either the $v-y$ path in B_2^σ if $d_{B_2^\sigma}(u)=0$ or the $u-y$ path in B_2^σ if $d_{B_2^\sigma}(v)=0$ form $ax-y$ walk in G^σ and hence there is a $x-y$ path in G^σ.

Subcase 1.1.b. B_1^σ and B_2^σ are d-joints at σ in G^σ
If $d_{B_1^\sigma}(u)=0$ and $d_{B_2^\sigma}(u)=0$, then $x-v$ and $v-y$ form a $x-y$ path in G^σ.
If $d_{B_1^\sigma}(v)=0$ and $d_{B_2^\sigma}(u)=0$, then the $x-u$ path in B_1^σ, $u-v$ path in B^{*^σ} and the $v-y$ path in B_2^σ form a $x-y$ path in G^σ.

Subcase 1.2. x and y are in the same joint at σ in G^σ
Let $x,y \in V(B_i^\sigma)$, $1 \le i \le k$. Clearly $x,y \in V(B_i^\sigma)-\sigma$. Since $B_i^\sigma-\sigma$ is connected, there is a $x-y$ path in $B_i^\sigma-\sigma$ and hence in B_i^σ.

Case 2. $\{x,y\} = \{u,v\}$.
Then $x,y \in V(B^{*^\sigma})$. Since B^{*^σ} is connected, there is a $x-y$ path in B^{*^σ} and hence in G^σ.

Case 3. $x=u$ and $y \neq v$.
Then $x \in V(B^{*^\sigma})$. Since B^{*^σ} is connected, there is a $x-v$ path in B^{*^σ} and hence in G^σ. Since $y \neq v$, $y \in V(B)$ such that B may be a c-joint or a d-joint.

Subcase 3.a. B is a c-joint
Then there exist a $v-y$ path in B^σ and hence a $x-y$ path in G^σ.

Subcase 3.b. B is a d-joint
Here $B = K_1 \cup K_2$ where K_1 is either u or v. If $K_1 = u = x$, then K_2 is the edge vy. Now $B^\sigma = K_1 \cup K_2$ where K_1 is v and K_2 is the edge uy which is same as xy and hence there is $ax-y$ path in G^σ.
If $K_1 = v$, then K_2 is the edge xy. This implies that $B^\sigma = K_1 \cup K_2$ where K_1 is $x=u$ and K_2 is vy. Now $x-v$ path in B^{*^σ} and vy edge form a $x-y$ path in G^σ.
From Cases 1, 2 and 3, G^σ is connected. Hence the theorem is proved. □

Notation 2.5. *Let G be a connected graph and let $\{v_1,v_2,\dots,v_n\} \subseteq V(G)$ such that $G[\{v_1,v_2,\dots,v_n\}] = P_n$ and each edge of P_n is a bridge in G. Without loss of generality, let P_n be $v_1v_2\dots v_n$. Let $B_{i_1}, B_{i_2}, \dots, B_{i_{r_i}}$ be the r_i (>0) branches at v_i in G, $1 \le i \le n$. We denote the* *graph G by $P_{n(v_1-v_n)}\left(\bigcup_{i=1}^{r_1} B_{1_i}, \bigcup_{i=1}^{r_2} B_{2_i}, \dots, \bigcup_{i=1}^{r_n} B_{n_i}\right)$. If there is no branch at v_j, then we put 0 in the place $\bigcup_{i=1}^{r_1} B_{j_i}$.*

Example 2.6. Consider the graph G given in Figure 2.10. It can be denoted by $P_{6(u-v)}(2P_2, C_3, P_2, C_4 \cup P_2, 2P_2 \cup P_3)$ or $P_{6(u-w)}(2P_2, C_3, P_2, C_4 \cup P_2, 2P_2, P_2)$ or $P_{7(u-x)}(2P_2, C_3, P_2, C_4 \cup P_2, 2P_2, 0, 0)$.

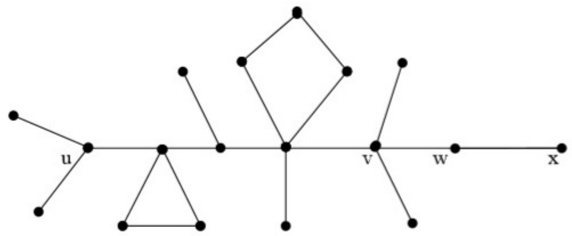

Figure 2.10. G

Theorem 2.7. *Let G be a connected acyclic graph of order $p \geq 4$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$. Then G has a 2-vertex self switching at σ in G if and only if one of the following holds:*

(i) *$G = B_{m,n}$ where $m + n = p - 2$ is the number of c-joints at σ in G, uv is an edge in G and u and v are the central vertices of G.*

(ii) *G is either $P_{4(u-v)}(mP_2, 0, 0, nP_2)$ or $P_{3(u-v)}(mP_2, P_2, nP_2)$ where $p-4 = m+n$ is the number of d-joints at σ in G, u and v are end vertices of both P_3 and P_4 and uv is not an edge of G.*

Proof. Let G be a connected acyclic graph and $\sigma = \{u, v\}$ be a subset of $V(G)$. Let $B_1, B_2, \dots, B_k$ be the k joints at σ in G. Then by Theorem 1.1, $G = \bigcup_{i=1}^{k} B_i$ and $G^\sigma = \bigcup_{i=1}^{k} B_i^\sigma$. Let σ be a 2-vertex self switching of G. Then $G \cong G^\sigma$. We consider the following two cases.

Case 1. $uv \in E(G)$.
Since G is connected and $uv \in E(G)$, by Observation 2.1, each B_i is a c-joint, $1 \leq i \leq k$. Since G is acyclic, each B_i is acyclic, $1 \leq i \leq k$. By Theorem 1.2, $B_i - \sigma$ is connected, acyclic and $\{d_{B_i}(u), d_{B_i}(v)\} = \{|V(B_i)| - 1, |V(B_i)| - 2\}$, $1 \leq i \leq k$. Without loss of generality, let B_1 be a joint at σ in G such that $d_{B_1}(u) = |V(B_1)| - 1$ and $d_{B_1}(v) = |V(B_1)| - 2$.
If $|V(B_1)| \geq 4$, then $d_{B_1}(u) \geq 3$ and $d_{B_1}(v) \geq 2$. $uv \in E(G)$ and $d_{B_1}(u) \geq 3$ implies that there exist at least two vertices, say a and b, in $V(B_1) - \sigma$ such that u is adjacent to a and b in B_1. Since $B_1 - \sigma$ is connected, there is an $a - b$ path in $B_1 - \sigma$ and hence in B_1. Now the edge ua, the path $a - b$ and the edge bu form a cycle in B, which is a contradiction to G is acyclic. Therefore, $|V(B_1)| = 3$. This implies that $d_{B_1}(u) = 2$ and $d_{B_1}(v) = 1$ and hence $B_1 = P_3$.
Let B_2 be a joint at σ in G such that $d_{B_2}(u) = |V(B_2)| - 2$ and $d_{B_2}(v) = |V(B_2)| - 1$. Then as before $B_2 = P_3$ where $d_{B_2}(u) = 2$ and $d_{B_2}(v) = 1$. Hence, each $B_i = P_3$ in which $\{d_{B_i}(u), d_{B_i}(v)\} = \{1, 2\}$, $1 \leq i \leq k$ and hence either u or v is an end vertex of P_3. Therefore, $G = \bigcup_{i=1}^{k} P_3 = B_{m,n}$, where

$m+n=k$ and u and v are central vertices of G. Clearly, $d_G(u)+d_G(v)=m+n+2$. By Theorem 1.7, $d_G(u)+d_G(v)=p$. This implies that $m+n=p-2$, which is the number of c-joints at σ in G. Thus (i) is proved.

Figure 2.11. $G=B_{4,5}$

Case 2. $uv \notin E(G)$.

Let $B_1,B_2,\dots,B_r$ the r c-joints and $B_{r+1},B_{r+2},\dots,B_k$ be the $(k-r)$ d-joints at σ in G. Since $G \cong G^\sigma$ and G^σ is connected and acyclic, G is also connected. By Theorem 2.4, there exist at least one c-joint at σ in G, $B-\sigma$ is connected and acyclic for each joint B at σ in G, $d_{\boldsymbol{B}}(u)=d_{\boldsymbol{B}}(v)=|V(B)|-3$ and $|V(B)|\geq 4$ for exactly one c-joint $B=B^*$, $\{d_B(u),d_B(v)\}=\{|V(B)|-2,|V(B)|-3\}$ for all c-joints $B\neq B^*$ and $B=K_1\cup K_2$ for all d-joints B, if exists, where K_1 is either u or v. Let $B_1=B^*$. If $|V(B*)|\geq 5$, then $d_{B^*}(u)=d_{B^*}(v)\geq 2$. This implies that there exists at least two vertices, say a and b, in $V(B^*)-\sigma$ which are adjacent to u in B^* and at least two vertices, say c and d, in $V(B^*)-\sigma$ which are adjacent to v in B^*. Since $B^*-\sigma$ is connected, there exist paths $a-b$ and $c-d$ in $B^*-\sigma$ and hence in B^*. Then the edge ua, the path $a-b$ and the edge bu form a cycle in B^* and hence in G, which is a contradiction to G is acyclic. Hence $|V(B^*)|=4$. This implies that $d_{B^*}(u)=d_{B^*}(v)=1$. All possible connected acyclic graphs on four vertices are given in Figure 2.12 and Figure 2.13. Clearly, $P_4^\sigma \cong P_4$ and $K_{1,3}^\sigma \cong K_{1,3}$ where u and v are end vertices and hence P_4 and $K_{1,3}$ are self switching joints at σ.

Figure 2.12. $B^*=P_4$

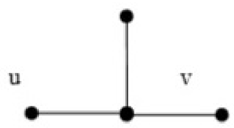

Figure 2.13. $B^*=K_{1,3}$

Now, $\{d_{B_i}(u),d_{B_i}(v)\}=\{|V(B_i)|-2,|V(B_i)|-3\}$, $2\leq i\leq r$. If $|V(B_i)|\geq 4$, then either $d_{B_i}(u)\geq 2$ and $d_{B_i}(v)\geq 1$ or $d_{B_i}(u)\geq 1$ and $d_{B_i}(v)\geq 2$. If either $d_{B_i}(u)\geq 2$ or $d_{B_i}(v)\geq 2$ then as before we get a cycle in G, which is a contradiction to G is acyclic. If $|V(B_i)|=3$, then either $d_{B_i}(u)=1$ and $d_{B_i}(v)=0$ or $d_{B_i}(u)=0$ and $d_{B_i}(v)=1$. This implies that $B=K_1\cup K_2$, where K_1 is either u or v and hence B_i is a d-joint which is a contradiction to B_i is a c-joint, $2\leq i\leq r$.

Thus there exists exactly one c-joint B^* at σ in G which is either P_4 or $K_{1,3}$, where u and v are the end vertices and each of the remaining $(k-1)$ d-joints is $K_1\cup K_2$ where K_1 is either u or v. Let m be the number of d-joints with K_1 as u and n be the number of d-joints with K_1 as v so that $m+n=k-1$. Clearly, $d_G(u)=m+1$ and $d_G(v)=n+1$ and hence $d_G(u)+d_G(v)=m+n+2$. By Theorem 1.7, $d_G(u)+d_G(v)=p-2$ which implies that $m+n=p-4$, the number of d-joints. Therefore, G is either $P_{4(u-v)}(mP_2,0,0,nP_2)$ or $P_{3(u-v)}(mP_2,P_2,nP_2)$ where $m+n=p-4$ is the number of d-joints at σ in G and u and v are end vertices of P_3 and P_4.

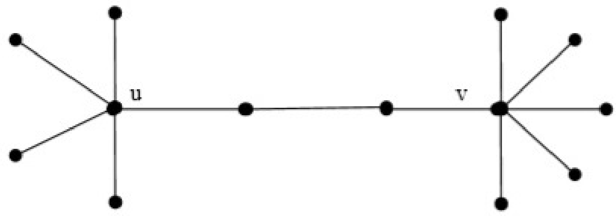

Figure 2.14. $P_{4(u-v)}(4P_2,0,0,5P_2)$

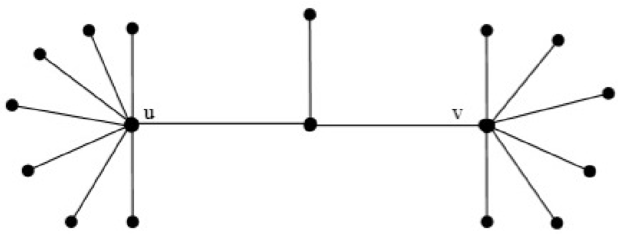

Figure 2.15. $P_{3(u-v)}(7P_2,P_2,6P_2)$

Thus from Cases 1 and 2, if $uv\in E(G)$, then $G=B_{m,n}$ where $n+m=p-2$ is the number of c-joints at σ in G and u and v are central vertices of G and if $uv\notin E(G)$, then G is either $P_{4(u-v)}(nP_2,0,0,mP_2)$ or $P_{3(u-v)}(mP_2,P_2,nP_2)$, where $n+m=p-4$ is the number of d-joints at σ in G and u and v are end vertices of both P_3 and P_4.
Conversely, let $G=B_{m,n}$ where $n+m=p-2$ is the number of c-joints at $\sigma=\{u,v\}$ in G, where u and v are central vertices and $uv\in E(G)$ or G is either $P_{4(u-v)}(nP_2,0,0,mP_2)$ or $P_{3(u-v)}(mP_2,P_2,nP_2)$, where $n+m=p-4$ is the number of d-joints at $\sigma=\{u,v\}$ in G, where u and v are end vertices of both P_4 and P_3 and $uv\notin E(G)$. Then each case leads to G has a 2-vertex self switching at σ in G. □

3. Conclusion

In this paper, we have given necessary and sufficient conditions for a graph G, for which G^σ at $\sigma=\{u,v\}$ to be connected and acyclic when $uv\in E(G)$ and $uv\notin E(G)$. Finally, we characterized trees with a 2-vertex self switching.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. G. Corneil and R. A. Mathon (editors), *Geometry and Combinatorics Selected Works of J. J. Seidel*, Academic Press, Boston (1991).

[2] J. Hage and T. Harju, Acyclicity of switching classes, *European Journal of Combinatorics* **19** (1998), 321 – 327, DOI: 10.1006/eujc.1997.0191.

[3] V. V. Kamalappan, J. P. Joseph and C. Jayasekaran, Self vertex switchings of trees, *Ars Combinatoria* **127** (2016), 33 – 43.

[4] J. J. Seidel, A survey of two-graphs, *Geometry and Combinatorics* **1991** (1991), 146 – 176, DOI: 10.1016/B978-0-12-189420-7.50018-9.

[5] C. Jayasekaran, J. C. Sudha and M. A. Shijo, Some results on 2-vertex switching in joints, *Communications in Mathematics and Applications* **12**(1) (2021), 59 – 69, DOI: 10.26713/cma.v12i1.1426.

[6] C. Jayasekaran, J. C. Sudha and M. A. Shijo, 2-Vertex self switching in acyclic joints in graph, *AIP Conference Proceedings* **2516** (2022), 210017, DOI: 10.1063/5.0108449.

[7] C. Jayasekaran, J. C. Sudha and M. A. Shijo, 2-Vertex self switching of forests, *Nonlinear Studies* **28**(3) (2021), 749 – 759, URL: http://www.nonlinearstudies.com/index.php/nonlinear/article/view/2627.

[8] V. Vilfred, J. P. Joseph and C. Jayasekaran, Branches and joints in the study of self switching of graphs, *The Journal of Combinatorial Mathematics and Combinatorial Computing* **67** (2008), 111 – 122.