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On the Stabilization of A Flexible Cable with Boundary Feedback

Tour K. Augustin, Mensah E. Patrice, and Taha M. Mathurin

Abstract. In this paper we study the stability of a flexible cable that is clamped at one end and free at the other. To stabilize this system we apply a control force in position and velocity at the free end of the cable. We prove that the closed-loop system is well-posed and is exponentially stable. We then analyze the spectrum of the system. Using a method due to Shkalikov we prove that the spectrum determines the exponential decay rate of the energy under certain conditions.

1. Introduction

We study the stability of a flexible cable that is clamped at one end and is submitted to boundary control force in position and velocity at the free end. The equations of motion for this system are given by

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t \geq 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad t \geq 0, \tag{1.2}$$

$$u_x(1,t) = -\alpha u(1,t) - \beta u_t(1,t), \quad t \geq 0, \tag{1.3}$$

where α, β are two given positive constants; $u(x,t)$ stands for transversal deviation at the position x and time t; a subscript letter denotes the partial derivation with respect to that variable. For simplicity, and without loss of generality, the length of the cable, the mass per unit length, and the flexural rigidity of the cable are chosen to be unity. Many authors have studied exponential stabilization of this simplified model with boundary control feedback in velocity only. They then have got the exact locus of the spectrum and proved that for the considered case the spectrum determines the exponential decay rate for $\beta \neq 1$. To obtain this result they proved that the above system verifies the Riesz basis property. We recall that the Riesz basis property means that the generalized

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eigenvectors of the system form an unconditional basis for the state Hilbert space. In fact Riesz basis is a powerful method in the study of controllability of hyperbolic systems [11]. One of the classical methods used to prove the Riesz basis property for such evolutive system is an application of Bari's classical theorem [6]. This method seems to be very difficult to apply when the spectral parameter appears in the boundary conditions of the spectral problem, this is the case here for our system. The goal of this work is to establish conditions on the feedback parameters α and β, to get the Riesz basis property for the evolutive system (1.1)-(1.3). To obtain this result we used a method due to Shkalikov [14]. The content of this paper is as follows. In the next section we recall the well-posedness and prove the uniform stability of (1.1)-(1.3). Then we study the spectrum of the system and prove that for the case considered the spectrum determines the exponential decay rate for a suitable choice of α and β.

2. **Stability results**

Let us introduce the following spaces:

$$\mathscr{V} = \{u : [0,1] \to \mathbb{R} \,/\, u \in H^1(0,1),\ u_x(0) = 0\}, \tag{2.1}$$

$$\mathscr{H} = \{(u,v)^T / u \in \mathscr{V},\ v \in L^2(0,1)\} = \mathscr{V} \times L^2(0,1), \tag{2.2}$$

$$\mathscr{D}(0,1) = \text{the space of smooth functions with compact support}, \tag{2.3}$$

$$\mathscr{D}'(0,1) = \text{the space of continuous linear functionals } T : \mathscr{D}(0,1) \to \mathbb{C}, \tag{2.4}$$

where the superscript T stands for the transpose, the spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as:

$$L^2(0,1) = \left\{ y : [0,1] \to \mathbb{R} \,\Big/ \int_0^1 y^2 dx < \infty \right\}, \tag{2.5}$$

$$H^k(0,1) = \{y : [0,1] \to \mathbb{R} \,/\, y, y^{(1)}, \dots, y^{(k)} \in L^2(0,1)\}. \tag{2.6}$$

In $\mathscr{H}$ we define the following inner-product:

$$\langle y, \widetilde{y} \rangle_{\mathscr{H}} = \int_0^1 (u_x \widetilde{u}_x + v\widetilde{v})dx + \alpha u(1)\widetilde{u}(1), \tag{2.7}$$

where $y = (u,v)^T \in \mathscr{H}$, $\widetilde{y} = (\widetilde{u}, \widetilde{v})^T \in \mathscr{H}$. Next we define the unbounded operator $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ as follows:

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \end{pmatrix}, \tag{2.8}$$

where the domain $D(A)$ of the operator A is defined as

$$D(A) = \{(u,v)^T \,/\, u \in H^2(0,1) \cap \mathscr{V},\ v \in H^1(0,1),\ u_x(1) = -\alpha u(1) - \beta v(1)\}. \tag{2.9}$$

With the previous notation, the set of equations (1.1)-(1.3) can be formally written as

$$\dot{y} = Ay, \quad y(0) \in \mathscr{H}, \tag{2.10}$$

where $y = (u, v)^T$ and $v = u_t$.

Theorem 2.1. *The operator A, defined by* (2.8) *and* (2.9)*, generates a $\mathscr{C}_0$ semigroup of contractions on $\mathscr{H}$.* (*For the terminology on the semigroup theory, the reader is referred to* [8]).

Proof. We apply the Lumer-Phillips theorem (see, e.g. [8, p. 14]). First we show that the operator A is dissipative. For any $y = (u, v)^T \in D(A)$,

$$\langle Ay, y\rangle_{\mathscr{H}} = \int_0^1 (v_x u_x + u_{xx} v) + \alpha v(1)u(1) = -\beta v^2(1), \tag{2.11}$$

where to derive the last equation we integrated by parts and used (1.2)-(1.3). It follows from (2.11) that the operator A is dissipative.

Next we show that it is m-dissipative. It suffice to prove that the range of the operator $(I - A) : D(A) \subset \mathscr{H} \to \mathscr{H}$ is onto; that is for any given $(f, g)^T \in \mathscr{H}$, we have to find $y = (u, v)^T \in D(A)$ so that

$$(I - A)y = z, \tag{2.12}$$

which is equivalent to the following set of equations:

$$-u_{xx} + u = l, \tag{2.13}$$

$$u_x(0) = 0, \tag{2.14}$$

$$u_x(1) = -\alpha u(1) - \beta v(1), \tag{2.15}$$

where v and l are given by

$$v = u - f, \qquad l = f + g \in L^2(0, 1). \tag{2.16}$$

Multiplying (2.13) by $\varphi \in \mathscr{V}$ and integrate on $[0, 1]$, we obtain

$$\int_0^1 u\varphi\, dx - \int_0^1 u_{xx}\varphi\, dx = \int_0^1 l\varphi\, dx. \tag{2.17}$$

Using integration by parts, (2.14) and (2.15) we obtain

$$\int_0^1 u_x\varphi_x\, dx + \int_0^1 u\varphi\, dx + (\alpha + \beta)u(1)\varphi(1) = \int_0^1 l\varphi\, dx + \beta f(1)\varphi(1). \tag{2.18}$$

This is the weak formulation of (2.13)-(2.15). The left-hand side of (2.18) is a continuous coercive bilinear form of φ and u, which will be noted by a. Moreover the right-hand side of (2.18) is a continuous linear form on $\mathscr{V}$ noted L. Using the well-known Lax-Milgram theorem (see, e.g. [16, p. 26]), there exists an unique $u \in \mathscr{V}$ so that:

$$a(u,\varphi) = L(\varphi) \quad \text{for each } \varphi \in \mathscr{V}, \tag{2.19}$$

Regularity of the solution. Let $\varphi \in \mathscr{D}(0,1)$, (2.18) becomes

$$\int_0^1 u_{xx}\varphi dx + \int_0^1 u\varphi = \int_0^1 l\varphi dx. \tag{2.20}$$

Then we have

$$\int_0^1 (u_{xx} + u - l)\varphi dx = 0.$$

Which leads to

$$u_{xx} + u = l \ \text{ in } \ \mathscr{D}'(0,1), \tag{2.21}$$

the same equality holds in $L^2(0,1)$ because $u_{xx} = l - u \in L^2(0,1)$. By using particular φ in $\mathscr{V}$, one recovers the boundary conditions in u. Hence we have found a unique solution $u \in H^2(0,1) \cap \mathscr{V}$ of (2.13)-(2.15). This shows that the operator $I - A$ is onto and the proof of theorem 2.1 follows from the Lumer-Phillips theorem [8]. □

Remark 2.2. It follows from Theorem 2.1 that for $(u_0, v_0)^T \in D(A)$, the problem (2.10) has a unique solution $(u,v)^T \in \mathscr{C}^1(\mathbb{R}_+, \mathscr{H}) \cap \mathscr{C}^0(\mathbb{R}_+, D(A))$, where $v = u_t$. Thus we have $u \in \mathscr{C}^2(\mathbb{R}_+, L^2(0,1)) \cap \mathscr{C}^1(\mathbb{R}_+, \mathscr{V}) \cap \mathscr{C}^0(\mathbb{R}_+, H^2(0,1) \cap \mathscr{V})$. We also have

$$\begin{aligned} D(A) &\longrightarrow \mathscr{H}, \\ (u_0, v_0)^T &\longmapsto (u,v)^T, \end{aligned} \tag{2.22}$$

can be extended in a contraction $S(t)$ on $\mathscr{H}$ so that $(S(t))_{t\geq 0}$ is strongly continuous and for each $(u_0, v_0)^T \in \mathscr{V}$, the weak solution of (2.18) is defined by

$$(u,v)^T = S(t)(u_0, v_0)^T, \quad t \geq 0, \tag{2.23}$$

with $(u,v)^T \in \mathscr{C}^0(\mathbb{R}_+, \mathscr{H})$. Finally $u \in \mathscr{C}^1(\mathbb{R}_+, L^2(0,1)) \cap \mathscr{C}^0(\mathbb{R}_+, \mathscr{V})$.

Next we prove that the semigroup generated by A decays exponentially to zero. Let us define the energy function of the system (1.1)-(1.3):

$$E(t) = \frac{1}{2}\|(u,u_t)^T\|_H^2 = \frac{1}{2}\int_0^1 (u_x^2 + u_t^2)dx + \frac{\alpha}{2}u^2(1), \quad t \geq 0. \tag{2.24}$$

we also define the auxiliary function

$$p(t) = 2\int_0^1 u_t x u_x dx + 2\alpha u(1)\int_0^1 u_t dx, \quad t \geq 0. \tag{2.25}$$

To obtain the desired result we need the following lemma.

Lemma 2.3. *With the previous notations we have:*

$$|p(t)| \leq (4+2\alpha)E(t), \quad t \geq 0, \tag{2.26}$$

$$p'(t) \leq -\gamma E(t) + (\beta^2 + 2\alpha^2 + 1)u_t^2(1), \tag{2.27}$$

where $\gamma = \min(1, 2\alpha)$.

Proof. First let us prove (2.26). For $t \geq 0$ we have:

$$|p(t)| \leq \int_0^1 2|u_t||u_x|dx + 2\alpha|u(1)|\int_0^1 |u_t|dx. \tag{2.28}$$

Next by using Cauchy-Schwartz and Young's inequalities, we obtain:

$$|p(t)| \leq \int_0^1 (|u_t|^2 + |u_x|^2)dx + \alpha^2 u^2(1) + \int_0^1 |u_t|^2 dx. \tag{2.29}$$

In view of (2.24) we have:

$$|p(t)| \leq (4+2\alpha)E(t), \quad t \geq 0.$$

This completely proves (2.26).

Next we prove (2.27). By differentiating (2.25) with respect to time we get:

$$\begin{aligned} p'(t) = 2\int_0^1 u_{tt}xu_x dx + 2\int_0^1 u_t xu_{xt}dx + 2\alpha u(1)\int_0^1 u_{tt}dx \\ + 2\alpha u_t(1)\int_0^1 u_t dx. \end{aligned} \tag{2.30}$$

Using (1.1) we obtain

$$\int_0^1 u_{tt}xu_x dx = \int_0^1 u_{xx}xu_x dx. \tag{2.31}$$

Next using integration by parts we have

$$\int_0^1 u_{xx}xu_x dx = [u_x xu_x]_0^1 - \int_0^1 u_x u_x dx - \int_0^1 u_x xu_{xx}dx, \tag{2.32}$$

then we obtain

$$2\int_0^1 u_{xx}xu_x = u_x^2(1) - \int_0^1 u_x^2 dx. \tag{2.33}$$

Using again integration by parts we get

$$\int_0^1 u_{tx}xu_t dx = [u_t xu_t]_0^1 - \int_0^1 u_t u_t dx - \int_0^1 u_{tx}xu_t dx. \tag{2.34}$$

Thus we obtain

$$2\int_0^1 u_{tx}xu_t dx = u_t^2(1) - \int_0^1 u_t^2 dx. \tag{2.35}$$

Using (1.3), (2.33) and (2.35) we get

$$2\left[\int_0^1 u_{tt}xu_x dx + \int_0^1 u_t x u_{tx} dx\right]$$
$$= -\int_0^1 u_x^2 dx - \int_0^1 u_t^2 dx + \alpha^2 u^2(1) + (\beta^2+1)u_t^2(1) + 2\alpha\beta u_t(1)u(1). \tag{2.36}$$

Using (1.1) and (1.2) we get

$$2\alpha u_t(1)\int_0^1 u_t dx + 2\alpha u(1)\int_0^1 u_{tt} dx$$
$$= 2\alpha u_t(1)\int_0^1 u_t dx + 2\alpha u(1)\int_0^1 u_{xx} dx$$
$$= 2\alpha u_t(1)\int_0^1 u_t dx + 2\alpha u(1)u_x(1). \tag{2.37}$$

Using Young's inequality we get:

$$\int_0^1 (2\alpha u_t(1))u_t dx + 2\alpha u(1)u_x(1)$$
$$\leq 2\alpha^2 u_t^2(1) + \frac{1}{2}\int_0^1 u_t^2 dx + 2\alpha u(1)u_x(1). \tag{2.38}$$

Finally using (1.3) we get

$$2\alpha u_t(1)\int_0^1 u_t dx + 2\alpha u(1)\int_0^1 u_{tt} dx$$
$$\leq 2\alpha^2 u_t^2(1) + \frac{1}{2}\int_0^1 u_t^2 dx - 2\alpha^2 u^2(1) - 2\alpha\beta u(1)u_t(1). \tag{2.39}$$

Using (2.36) and (2.39) we have

$$p'(t) \leq -\frac{1}{2}\int_0^1 u_x^2 dx - \frac{1}{2}\int_0^1 u_t^2 dx - \alpha^2 u^2(1) + (\beta^2 + 1 + 2\alpha^2)u_t^2(1). \tag{2.40}$$

By choosing $\gamma = \min(1, 2\alpha)$ we finally get

$$p'(t) \leq -\gamma E(t) + (\beta^2 + 2\alpha^2 + 1)u_t^2(1).$$

and (2.27) is completely proved. □

Theorem 2.4. *For the system* (1.1)-(1.3) *there exist two constants* $M > 1$ *and* $\omega > 0$ *such that the following holds:*

$$E(t) \leq ME(0)e^{-\omega t}, \quad \textit{for all} \ \ t \geq 0 \tag{2.41}$$

Proof. Let $\varepsilon > 0$. We define the following function:

$$E_\varepsilon(t) = E(t) + \varepsilon p(t), \quad t \geq 0. \tag{2.42}$$

We have:

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon |p(t)|. \tag{2.43}$$

Using the Lemma 2.3 get:

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon(4 + 2\alpha)E(t), \tag{2.44}$$

which implies

$$(1 - \varepsilon(4 + 2\alpha))E(t) \leq E_\varepsilon(t) \leq (1 + \varepsilon(4 + 2\alpha))E(t), \quad t \geq 0. \tag{2.45}$$

By differentiating (2.42) with respect to t we get

$$E'_\varepsilon(t) = E'(t) + \varepsilon p'(t). \tag{2.46}$$

In view of (2.24), we have:

$$E'(t) = -\beta u_t^2(1) \leq 0. \tag{2.47}$$

By using (2.46), (2.47) and the Lemma 2.3 we get

$$E'_\varepsilon(t) \leq [(\beta^2 + 2\alpha^2 + 1)\varepsilon - \beta]u_t^2(1) - \varepsilon\gamma E(t). \tag{2.48}$$

By choosing ε so that

$$0 < \varepsilon < \frac{\beta}{\beta^2 + 2\alpha^2 + 1} = \varepsilon_1. \tag{2.49}$$

We then obtain

$$E'_\varepsilon(t) \leq -\varepsilon\gamma E(t). \tag{2.50}$$

By choosing $\varepsilon > 0$ so that

$$0 < \varepsilon \leq \min\left(\varepsilon_1, \frac{1}{4 + 2\alpha}\right), \tag{2.51}$$

the following holds $(1 - \varepsilon(4 + 2\alpha)) > 0$.

By using (2.51) we have:

$$E'_\varepsilon(t) \leq -\varepsilon\gamma \frac{1}{1 - \varepsilon(4 + 2\alpha)} E_\varepsilon(t). \tag{2.52}$$

By integrating we obtain:

$$E_\varepsilon(t) \leq E_\varepsilon(0) e^{-\frac{\varepsilon\gamma}{1-\varepsilon(4+2\alpha)}t}. \tag{2.53}$$

Using (2.45), (2.53) we deduce that

$$E(t) \leq \frac{1}{1 - \varepsilon(4 + 2\alpha)} E_\varepsilon(t) \leq \frac{1 + \varepsilon(4 + 2\alpha)}{1 - \varepsilon(4 + 2\alpha)} e^{-\frac{\varepsilon\gamma}{1-\varepsilon(4+2\alpha)}t} E(0). \tag{2.54}$$

To obtain the conclusion of the theorem it suffices to choose

$$M = \frac{1+\varepsilon(4+2\alpha)}{1-\varepsilon(4+2\alpha)} > 1, \tag{2.55}$$

$$\omega = \frac{\varepsilon\gamma}{1-\varepsilon(4+2\alpha)} > 0, \tag{2.56}$$

where ω does not depends on the choice of $E(0)$.

Thus the system (1.1)-(1.3) is uniformly and exponentially stable. □

3. **Spectral analysis**

In this section, we prove that the spectrum determines the optimal exponential decay rate w given by (2.41). Our approach consists in proving that there is a sequence of generalized eigenvectors of the operator A which forms a Riesz basis of the energy space. The study of the spectral problem associated to the evolutive system (1.1)-(1.3) reveals that the spectral parameter appears in boundary conditions. For this kind of problems the classical theorem of Bari seems very difficult to apply [13]. Let us recall that the basic idea of Bari's theorem is that if $\{\phi_n\}_1^\infty$ is a Riesz basis for a Hilbert space $\mathscr{H}$ and another ω-linearly independent sequence basis $\{\psi_n\}_1^\infty$ of $\mathscr{H}$ satisfying $\sum\limits_{n=1}^{\infty} \|\psi_n - \varphi_n\|^2 < \infty$, then $\{\psi_n\}_1^\infty$ also forms a Riesz basis itself. Here we use a method due to Shkalikov [14]. The basic idea of this method is to build with the operator A a new operator called the Shkalikov's linearized operator which verifies the Riesz basis property and then deduce the same property for the operator A. Here we have to work in complexified Hilbert spaces of $\mathscr{V}$, $L^2(0,1)$ and $\mathscr{H}$. For the convenience we do not change the notation for these spaces. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A and let $y = (u,v)^T \in D(A)$ be a corresponding eigenvector. To find y we have to solve the equation $Ay = \lambda y$ and hence the following set of equations

$$\lambda u - v = 0, \tag{3.1}$$

$$\lambda v - u_{xx} = 0, \tag{3.2}$$

$$u_x(0) = 0, \tag{3.3}$$

$$u_x(1) = -\alpha u(1) - \beta v(1). \tag{3.4}$$

By eliminating v, we get

$$u_{xx} - \lambda^2 u = 0, \tag{3.5}$$

$$u_x(0) = 0, \tag{3.6}$$

$$u_x(1) = -(\alpha + \beta\lambda)u(1). \tag{3.7}$$

The orders of the boundary conditions are respectively $k_1 = 1$, $k_2 = 1$, the global order is then $k_1 + k_2 = 2$.

When λ is a nonzero eigenvalue, the Shkalikov's characteristic polynomial (see [14]) associated to (3.5) is

$$\omega^2 - 1 = 0. \tag{3.8}$$

Which zeros are

$$\omega_1 = 1 \text{ and } \omega_2 = -1. \tag{3.9}$$

The solutions of (3.5) can be found as

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}. \tag{3.10}$$

Upon substituting (3.10) in the boundary conditions we obtain the following matrix equation

$$\begin{pmatrix} \lambda & -\lambda \\ ((1+\beta)\lambda + \alpha)e^{\lambda} & ((\beta-1)\lambda + \alpha)e^{-\lambda} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.11}$$

A necessary and sufficient condition for this matrix equation to have nontrivial solutions for c_1 and c_2 is that the following characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} \lambda & -\lambda \\ ((1+\beta)\lambda + \alpha)e^{\lambda} & ((\beta-1)\lambda + \alpha)e^{-\lambda} \end{vmatrix}, \tag{3.12}$$

vanishes; in the other words

$$\Delta(\lambda) = \lambda^2 \{[1 + \beta + \alpha\lambda^{-1}]e^{\lambda} + [-1 + \beta + \alpha\lambda^{-1}]e^{-\lambda}\} = 0. \tag{3.13}$$

Hence for the eigenvalues of large modulus $|\lambda|$ the dominant terms of the expression in bracket are $(1 + \beta)$ and $(\beta - 1)$ which are nonzero if $\beta \geq 0$ and $\beta \neq 1$.

In this case, according to the theory of Shkalikov, we say that the boundary conditions of (3.6)-(3.7) are regular.

Our next task is to prove that the eigenvalues of the operator A, with sufficiently large modulus are algebraically simple and isolated.

These properties are very important in order to use the fundamental Theorem 3.1 of Shkalikov (see [14]) in the perspective of finding a Riesz basis property for the operator A in the energy space $\mathscr{H}$.

Since the operator A is m-dissipative the eigenvalues λ of A are all in the left half complex plane, and hence verify $\Re e(\lambda) \leq 0$.

Lemma 3.1 (Control feedback in velocity only)**.** *Consider the system given by* (2.10) *where* $\alpha = 0$ *and* $\beta > 0$*. Then the following holds.*

(1) *Zero is a geometrically simple eigenvalue of A with algebraic multiplicity of two.*
(2) *All nonzero eigenvalues of A are algebraically simple.*
(3) *The eigenvalues of A are countable and isolated.*

Proof. (1) Let $\lambda = 0$, the solutions of the Cauchy's problem (3.5)-(3.7) are nonzero complex constant functions. Hence $\lambda = 0$ is an eigenvalue of A and the corresponding eigenvectors subspace is generated by the vector $(1,0)^T$ of $\mathscr{H}$. Thus the geometrical multiplicity of $\lambda = 0$ is 1. Let us now prove that its algebraic multiplicity is 2. The algebraic multiplicity of $\lambda = 0$ is greater than 1 if and only if there exists $W = (u,v)^T \in \ker A^2 \setminus \ker A$, which is equivalent to:

$$A(AW) = 0 \quad \text{and} \quad AW \neq 0, \tag{3.14}$$

Hence the vector AW belongs to $KerA$, by normalizing W, we may suppose that

$$AW = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.15}$$

The later is equivalent to the following set of equations:

$$v = 1, \tag{3.16}$$

$$u_{xx} = 0, \tag{3.17}$$

$$u_x(1) = 0, \tag{3.18}$$

$$u_x(0) = 0, \tag{3.19}$$

and the solutions of the problem (3.17)-(3.19) are nonzero complex constant functions. Hence W belongs to the vector subspace of $\mathscr{H}$ generated by the vectors $(1,0)^T$ and $(0,1)^T$.

Next we prove that $\ker A^3 \setminus \ker A^2 = \{0\}$. Consider $W = (u,v)^T \in \ker A^3 \setminus \ker A^2$. Then we have

$$A(A^2W) = 0 \quad \text{with} \quad A^2W \neq 0. \tag{3.20}$$

Hence $A^2W \in \ker A = \text{Vect}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and by normalizing W we can suppose that $A^2W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Since

$$A^2W = \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}, \tag{3.21}$$

the function u verifies the following set of equations

$$u_{xx} = 1, \tag{3.22}$$

$$u_x(0) = 0, \tag{3.23}$$

$$u_x(1) = 0. \tag{3.24}$$

The solutions of the problem (3.22)-(3.24) take the form

$$u(x) = \frac{1}{2}x^2 + bx + c,$$

where $(b, c) \in \mathbb{R}^2$. Next using (3.23) we obtain $b = 0$, which leads to contradiction since (3.24) becomes $(u_x(1) = 2 = 0)$. Hence the algebraic multiplicity of $\lambda = 0$ is 2.

(2) Using (3.12), we deduce by a straightforward computation that the nonzero eigenvalues of A are exactly the roots of the following equation

$$e^{2\lambda} = \frac{1-\beta}{1+\beta}. \tag{3.25}$$

If λ is a nonzero eigenvalue of A with algebraic multiplicity greater than 1, by deriving (3.25), we obtain

$$2\lambda e^{2\lambda} = 0 \text{ which leads to a contradiction.} \tag{3.26}$$

Hence all the nonzero eigenvalues of A are algebraically simple.

(3) the roots of (3.25) take the form

$$\lambda_n = \begin{cases} \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right) + in\pi, & n \in \mathbb{Z}, \text{ if } \beta < 1, \\ \frac{1}{2}\ln\left(\frac{\beta-1}{1+\beta}\right) + i\left(n + \frac{1}{2}\right)\pi, & n \in \mathbb{Z}, \text{ if } \beta > 1. \end{cases} \tag{3.27}$$

Hence the eigenvalues of A are countable and isolated. □

Remark 3.2. We observe that if there is no feedback control in position ($\alpha = 0$), the nonzero eigenvalues can be explicitly found by solving the characteristic equation (3.13) and they take the form (3.27).

Unfortunately, when an additional feedback control in position is applied ($\alpha > 0$), this characteristic equation can not be explicitly solved. One needs asymptotic methods to investigate the behavior of the eigenvalues. For these methods the reader can be referred to [7] or [14].

Here we recall the following important result due to R.E. Langer [7], which can be also found in [3].

Proposition 3.3. *Let $f(s)$ a function of the form*

$$f(s) = \sum_{k=0}^{n} a_k(1 + o(1))e^{\alpha_k s}, \quad |s| \to +\infty, \; s = \sigma + i\tau,$$

where a_k are nonzero complex numbers and α_k being real numbers such that $\alpha_0 < \alpha_1 < \ldots < \alpha_n$.

Then the zeros of $f(s)$ all lie in a strip $a \leq \sigma \leq b$ and can asymptotically represented by those of the following comparison function $f^*(s) = \sum_{k=0}^{n} a_k e^{\alpha_k s}$. In particular, there exists $R > 0$, such that the number $N(T_1, T_2)$ of the zeros of $f(s)$

in any rectangle $a < \sigma \leq b$, $T_1 < \sigma < T_2$ with $T_1 > R$ is limited by the following relation

$$\left| N(T_1, T_2) - \frac{T_2 - T_1}{2\pi}(\alpha_n - \alpha_0) \right| \leq n.$$

Note that $h = o(1)$ when $|s| \to +\infty$ means here that $h(s) \to 0$ when $|s| \to +\infty$.

Remark 3.4. In particular the multiplicity of any zero of $f(s)$ is not greater than n.

Lemma 3.5 (Control feedback in position and velocity). *Consider the system given by* (2.10) *where $\alpha > 0$ and $\beta > 0$. Then the following holds.*

(1) *Each eigenvalue λ of A is nonzero and is such that $\Re e(\lambda) < 0$.*

(2) *The eigenvalues of A with sufficiently large modulus are isolated and have the following asymptotic expansion:*
for $\beta < 1$,

$$\lambda'_m = \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right) + i\left[m\pi + \frac{\alpha}{m(1-\beta^2)\pi}\right] + O\left(\frac{1}{m^2}\right) \text{ as } m \to \infty, \tag{3.28}$$

for $\beta > 1$,

$$\lambda'_m = \frac{1}{2}\ln\left(\frac{\beta-1}{\beta+1}\right) + i\left[(m+\frac{1}{2})\pi + \frac{\alpha}{m(1-\beta^2)\pi}\right] + O\left(\frac{1}{m^2}\right) \text{ as } m \to \infty. \tag{3.29}$$

(3) *The eigenvalues of A with sufficiently large modulus are algebraically simple.*

Proof. (1) Consider λ an eigenvalue of the operator A and $y = (u, v)^T \in D(A)$ the corresponding eigenvector. Since A is a dissipative operator, $\Re e(\lambda) \leq 0$. Multiplying both sides of (3.5) by the conjugate function $\bar{u}$ of u and integrating by parts with respect to x we get

$$\int_0^1 u_{xx}\bar{u}dx - \int_0^1 \lambda^2 u\bar{u}dx = \left[u_x\bar{u}\right]_0^1 - \int_0^1 u_x\bar{u}_x dx - \lambda^2\int_0^1 u\bar{u}dx = 0. \tag{3.30}$$

Now using the boundary conditions (3.6) and (3.7) we have

$$(\alpha + \beta\lambda)|u(1)|^2 + \int_0^1 |u_x|^2 dx + \lambda^2 \int_0^1 |u|^2 dx = 0. \tag{3.31}$$

If $\Re e(\lambda) = 0$ then $\lambda^2 \leq 0$ and we deduce from (3.31) that the complex number $\beta\lambda|u(1)|^2$ is both real and imaginary. Hence we have $\beta\lambda|u(1)|^2 = 0$ and using the fact $\beta > 0$ we obtain $\lambda|u(1)|^2 = 0$.

If $\lambda = 0$, the solutions of (3.5) are the complex functions of the form $u(x) = ax + b$ where $a, b \in \mathbb{C}$. From (3.6) and (3.7) we obtain

$$u_x(1) = -\alpha u(1) = -\alpha b = 0. \tag{3.32}$$

Since $\alpha \neq 0$ we obtain $b = 0$ and $u = 0$. Hence $y = (u, v)^T = 0$ which leads to a contradiction.

If $u(1) = 0$, from (3.7) we deduce that $u_x(1) = 0$. Hence u is a solution of the following Cauchy's problem

$$\begin{cases} u_{xx} - \lambda^2 u = 0, \\ u_x(0) = 0, \\ u_x(1) = 0, \end{cases} \tag{3.33}$$

which unique solution is $u = 0$. Hence we obtain $y = (u, v)^T = 0$ which leads to a contradiction.

Therefore we get $\Re e(\lambda) < 0$.

(2) Let λ be an eigenvalue of A. Equation (3.12) shows that λ verifies the following equation

$$[(-1+\beta)\lambda + \alpha] + [(1+\beta)\lambda + \alpha]e^{2\lambda} = 0. \tag{3.34}$$

In this equation, the left hand expression is called an exponential sum, see [7] for more details. Next using the proposition 3.3 we deduce that for $\alpha > 0$ and $\beta \in \mathbb{R}_+^* - \{1\}$, the zeros of the above equation are asymptotically those of (3.25). we consider now the following function of the complex variable λ:

$$P(\lambda) = \left[(-1+\beta)\lambda + \alpha\right] + \left[(1+\beta)\lambda + \alpha\right]e^{2\lambda}.$$

We search an asymptotic expansion of the zeros of P in the following form

$$\lambda'_m = \lambda_m + \frac{x}{m} + O\left(\frac{1}{m^2}\right), \quad m \in \mathbb{N},\ m \to \infty, \tag{3.35}$$

where x is constant complex number to be determined later and λ_m verifies (3.27). It suffices now to insert the above expression of λ'_m in that of $P(\lambda'_m)$. After a direct computation we obtain

for $\beta < 1$

$$P(\lambda'_m) = -2\frac{i\pi x(\beta^2-1) - \alpha}{\beta+1} + O\left(\frac{1}{m}\right), \quad m \in \mathbb{N},\ m \to \infty. \tag{3.36}$$

We choose the complex number x in such a way to have a zero constant in the above expansion. We get

$$x = \frac{i\alpha}{(1-\beta^2)\pi}. \tag{3.37}$$

Finally we have

$$\lambda'_m = \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right) + i\left[m\pi + \frac{\alpha}{m(1-\beta^2)\pi}\right] + O\left(\frac{1}{m^2}\right), \quad m \in \mathbb{N},\ m \to \infty. \tag{3.38}$$

For $\beta > 1$

$$P(\lambda'_m) = -2\frac{i\pi x(\beta^2-1)-\alpha}{\beta+1} + O\left(\frac{1}{m}\right), \quad m \in \mathbb{N},\ m \to \infty. \tag{3.39}$$

Choosing the complex number x in such a way to have a zero constant in the above expansion, gives

$$x = \frac{i\alpha}{(1-\beta^2)\pi}. \tag{3.40}$$

Hence we have

$$\lambda'_m = \frac{1}{2}\ln\left(\frac{\beta-1}{\beta+1}\right) + i\left[(m+\frac{1}{2})\pi + \frac{\alpha}{m(1-\beta^2)\pi}\right] + O\left(\frac{1}{m^2}\right),$$
$$m \in \mathbb{N},\ m \to \infty. \tag{3.41}$$

From (3.38) and (3.41), we deduce that the eigenvalues of A with sufficiently large modulus are isolated.

(3) We use Remark 3.4 to conclude that the eigenvalues of A are asymptotically and algebraically simple. □

Remark 3.6. From (2) of the above lemma we deduce that

$$\Re e(\lambda) \longrightarrow \frac{1}{2}\ln\left|\frac{1-\beta}{1+\beta}\right| \quad \text{as } |\lambda| \to +\infty. \tag{3.42}$$

Lemma 3.7 (Control feedback in position only). *Consider the system given by* (2.10) *where* $\alpha > 0$ *and* $\beta = 0$. *Then the following holds.*

(1) *The eigenvalues of A are all imaginary.*

(2) *The eigenvalues of A with sufficiently large modulus are isolated and algebraically simple.*

Proof. (1) Consider λ an eigenvalue of the operator A. Since $\beta = 0$ the relation (3.31) becomes

$$\alpha|u(1)|^2 + \int_0^1 |u_x|^2 dx + \lambda^2 \int_0^1 |u|^2 dx = 0. \tag{3.43}$$

Hence $\lambda^2 \leq 0$. This proves the first assertion of the lemma.

(2) Using the relation (3.34) and Proposition 3.3 we deduce that the eigenvalues of the operator A are asymptotically of the form $im\pi$, $m \in \mathbb{N}$ and are asymptotically and algebraically simple and isolated. □

Finally we can conclude that for $\alpha \geq 0$ and $\beta \in \mathbb{R}_+^* - \{1\}$, the eigenvalues of the operator A are asymptotically and algebraically simple and isolated. This property is essential for applying Theorem 3.1 of Shkalikov [14], in order to obtain the Riesz basis property for the operator A in the energy space $\mathscr{H}$.

Theorem 3.8. *Consider the system given by* (2.10) *where* $\alpha > 0$ *and* $\beta \in \mathbb{R}_+^* - \{1\}$ *then there exists a fundamental system of generalized eigenvectors of the operator* A *which forms a Riesz basis in* $\mathscr{H} = \mathscr{V} \times L^2(0,1)$.

Proof. Following the notations of Shkalikov in [14], for integer $r \geq 0$ we set

$$\mathscr{W}_2^r = W_2^{1+r}(0,1) \oplus W_2^r(0,1), \tag{3.44}$$

where $W_2^k(0,1)$ is the Sobolev space of smooth functions on the segment $[0,1]$, having $(k-1)$ absolutely continuous derivatives and $(k-th)$ derivative from $L^2(0,1)$ with the norm $\|f\|_{W_2^k} = \|f^{(k)}\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}$.

We rewrite (3.5) in the form

$$l(u,\lambda) = -u'' + \lambda^2 u = 0. \tag{3.45}$$

Now we consider the operator H defined as follows

$$\widetilde{v} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in \mathscr{W}_2^r \longmapsto H\widetilde{v} = \begin{bmatrix} v_1 \\ v_0'' \end{bmatrix} \in \mathscr{W}_2^r, \tag{3.46}$$

where $v_0 = u$, $v_1 = \lambda v_0$. We also define $H^i(\tilde{v}) \in \mathscr{W}_2^{r-i}$ where H^i is the power of H at order i. We now normalize the boundary conditions (3.6) and (3.7) according to Shkalikov's method [14]. First we rewrite them as follows

$$U_1(u,\lambda) = u'(0) = 0, \tag{3.47}$$

$$U_2(u,\lambda) = u'(1) + \beta\lambda u(1) + \alpha u(1) = 0, \tag{3.48}$$

and we do the following transformations:

$$\lambda^i u^{(k)}(x) = (H^i\tilde{v})_0^{(k)}(x) \quad \text{if } \; i+k < n+r,$$

$$\lambda^i u^{(k)}(x) = \lambda^{i+k-n-r+1}(H^{n+r-k-1}\tilde{v})_0^{(k)}(x) \quad \text{if } \; i+k \geq n+r,$$

where $x = 0$ or 1, n is the number of boundary conditions and the subscript index means that one takes the first component of the associated vector. In our case we have $n = 2$. We rewrite the above boundary conditions as follows:

$$\widetilde{U}_i(\tilde{v},\lambda) = \sum_{k=0}^{v_i(r)} \lambda^k U_i^k(\tilde{v}), \quad 1 \leq i \leq n,$$

where for any $1 \leq i \leq n$, $U_i^k(\tilde{v})$ is a linear form of the variable $\tilde{v}$ which does not depends on λ. We set

$$N_r = v_1(r) + v_2(r) + \cdots + v_q(r),$$

where the numbers v_i are those who appear above. If they are all zero, then $N_r = 0$.

Here we take $r = 0$, and replace the term $\lambda u(1)$ by $v_1(1)$ in (3.48). The other conditions remain unchanged. We get

$$\widetilde{U}_1(\widetilde{u},\lambda) = v_0'(0) = 0, \tag{3.49}$$

$$\widetilde{U}_2(\widetilde{v},\lambda) = v_0'(1) + \beta v(1) + \alpha v_0(1) = 0. \tag{3.50}$$

We denote by $\mathscr{W}^r_{2,U}$ the Shkalikov space defined as follows

$$\mathscr{W}^r_{2,U} = \{\widetilde{v} \in \mathscr{W}^r_2, \widetilde{U}_j(H^k(\widetilde{v})) = 0, \quad 1 \le j \le n \text{ for } 0 \le k \le n+r-2$$
$$\text{and all the others orders of boundary conditions verifying}$$
$$\le n+r-k-2\}. \tag{3.51}$$

Following the theory of Shkalikov $\mathscr{W}^r_{2,U}$ is a closed subspace of finite codimension in $\mathscr{W}^r_2$. In our case, since $n=2$ for $r=0$ we have

$$\mathscr{W}^0_{2,U} = \left\{ \widetilde{v} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in W^1_2(0,1) \oplus W^0_2(0,1),\ v_0'(0) = 0, \right.$$
$$\left. v_0'(1) + \beta v_1(1) + \alpha v_0(1) = 0 \right\}. \tag{3.52}$$

We define the Shkalikov's operator as follows

$$H_0 \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = H \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_0'' \end{bmatrix}. \tag{3.53}$$

From Corollary 3.2 of Theorem 3.1 of Shkalikov [14], there is a set of generalized eigenvectors of the operator H_0 which forms a Riesz basis of the Hilbert space $\mathscr{W}^0_{2,U}$.

We now build a Riesz basis for the operator A. We have

$$D(H_0) = W^1_{2,U} = \left\{ \begin{pmatrix} \omega \\ v \end{pmatrix} \in H^1(0,1) \oplus L^2(0,1) \,/\, \omega'(0) = 0 \right.$$
$$\left. \omega'(1) = -\alpha\omega(1) - \beta v(1) \right\}. \tag{3.54}$$

Next we prove that the spectral problem associated to the operator H_0 is equivalent to the one defined by A.

First, suppose that

$$H_0 U = \lambda U \quad \text{where } U = (v_0, v)^T \in D(H_0),$$

is an eigenvector associated to $\lambda \in Sp(H_0)$.
We obtain

$$v_1 = \lambda v_0, \tag{3.55}$$

$$v_0'' = \lambda v_1, \tag{3.56}$$

$$U \in D(H_0), \tag{3.57}$$

by substitution we have

$$v_0'' - \lambda^2 v_0 = 0, \tag{3.58}$$

$$v'(0) = 0,\ v_0'(1) = -\alpha v_0(1) - \beta v_1(1), \tag{3.59}$$

where $v_0 = u$.

Thus λ is an eigenvalue of A associated to the spectral problem (3.5)-(3.7).

Next we let λ be an eigenvalue of A associated to the spectral problem (3.5)-(3.7). We easily deduce that λ is an eigenvalue of H_0.

Since we know from the previous study of H_0 that there is a set of generalized eigenvectors of the operator H_0 which forms a Riesz basis of the Hilbert space $W^0_{2,U}$, we deduce that there is also a set of generalized eigenvectors of the operator A which forms a Riesz basis of the Hilbert space $\mathscr{H} = \mathscr{V} \oplus L^2(0,1)$.

The theorem is proved. □

Remark 3.9. The above result allows us to prove that the spectrum of the evolutive system defined by (1.1)-(1.3) determines the optimal decay rate of the energy.

Let ω denote the optimal exponential decay rate of the energy and μ the supremum of the real part of the eigenvalues of A. We have

$$\omega = \inf\{\varepsilon : \exists\, C(\varepsilon) > 0,\ E(t) \leq C(\varepsilon)E(0)e^{2\varepsilon t},\ \text{for all}\ \ t \geq 0\}, \tag{3.60}$$

$$\mu = \sup\{\Re e\lambda \,/\, \lambda \in \sigma(A)\}, \tag{3.61}$$

where $\sigma(A)$ is the spectrum of A.

Theorem 3.10. $\mu = \omega$.

Proof. Here we suppose that the eigenvalues of A are all simple.

If this is not the case, with little modifications the proof is the same since there is a finite number of eigenvalues which are not simple.

It is well known that $\omega \geq \mu$.

Consider now $(u_0, v_0)^T \in \mathscr{H}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ a Riesz basis of eigenvectors of the energy state Hilbert space $\mathscr{H}$. We have

$$(u_0, v_0)^T = \sum_{n=0}^{+\infty} \alpha_n \varphi_n, \tag{3.62}$$

hence we get

$$\begin{aligned} S(t)(u_0, v_0)^T &= \sum_{n=0}^{+\infty} \alpha_n S(t)\varphi_n \\ &= \sum_{n=0}^{+\infty} \alpha_n e^{\lambda_n t}\varphi_n. \end{aligned} \tag{3.63}$$

Which leads to

$$\|S(t)(u_0, v_0)^T\|^2_{\mathscr{H}} \leq C_1 \sum_{n=0}^{+\infty} |\alpha_n e^{\lambda_n t}|^2, \tag{3.64}$$

where C_1 a positive constant. Using the definition of μ we get

$$
\begin{aligned}
\|S(t)(u_0, v_0)^T\|^2_{\mathscr{H}} &\leq C_1 \sum_{n=0}^{+\infty} |\alpha_n e^{\lambda_n t}|^2 \\
&\leq C_1 e^{2\mu t} \sum_{n=0}^{+\infty} |\alpha_n|^2 \\
&\leq C e^{2\mu t} \|(u_0, v_0)^T\|^2_{\mathscr{H}}, \qquad (3.65)
\end{aligned}
$$

with $C = \frac{C_1}{B}$, where B is a positive constant such that $B \sum\limits_{n=0}^{+\infty} |\alpha_n|^2 \leq \|(u_0, v_0)^T\|^2_{\mathscr{H}}$

$$E(t) \leq CE(0)e^{2\mu t}. \qquad (3.66)$$

Hence

$$\omega \leq \mu. \qquad (3.67)$$

Finally, $\omega = \mu$. This complete the proof. $\square$

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Tour K. Augustin, *Institut National Polytechnique Houphout-Boigny de Yamoussoukro, BP 1093 Yamoussoukro, C te d'Ivoire*
E-mail: `latourci@yahoo.fr`

Mensah E. Patrice, *Institut National Polytechnique Houphout-Boigny de Yamoussoukro, BP 1093 Yamoussoukro, C te d'Ivoire*
E-mail: `pemensah@hotmail.com`

Taha M. Mathurin, *Institut National Polytechnique Houphout-Boigny de Yamoussoukro, BP 1093 Yamoussoukro, C te d'Ivoire*
E-mail: `malodja@yahoo.fr`

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