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On Tricomi and Hermite-Tricomi Matrix Functions of Complex Variable

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Abstract. In this paper, Tricomi and Hermite-Tricomi matrix functions are introduced starting from the Hermite matrix polynomials. The convergence, radius of regularity, integral form, generating matrix functions, matrix recurrence relations satisfied by these Tricomi matrix functions are derived. Finally, the generating matrix functions, matrix recurrence relations, addition theorems for the Hermite-Tricomi matrix functions are given and matrix differential equations satisfied by them are presented.

1. Introduction

Theory of special functions plays an important role in the formalism of mathematical physics. Hermite and Chebyshev polynomials are among the most important special functions, with very diverse applications to physics, engineering and mathematical physics ranging from abstract number theory to problems of physics and engineering. The hypergeometric matrix function has been introduced as a matrix power series and an integral representation and the hypergeometric matrix differential equation in [9, 12, 15, 17] and the explicit closed form general solution of it has been given in [10]. Recently, extension to the matrix framework of the classical families of Hermite-Hermite, Hermite, Laguerre, Bessel, Jacobi, Chebyshev and Gegenbauer matrix polynomials have been proposed and studied in a number of papers [1, 2, 3, 4, 7, 8, 11, 15, 16].

The primary goal of this paper is to consider a new system of matrix function, namely the Tricomi matrix functions and Hermite-Tricomi matrix functions. The structure of the paper is as follows: In Section 2 a definition of Tricomi matrix functions is given and the convergence properties, radius of convergence and integral form are given, the generating matrix functions and matrix recurrence relations are established and the matrix differential equation of three order

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satisfied by them is presented. Finally, we define the Hermite-Tricomi matrix functions and the matrix recurrence relations,addition theorems and matrix differential equations are investigated in Section 3.

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ is denoting the principle logarithm of z [5], then $z^{\frac{1}{2}}$ represents $\exp\left(\frac{1}{2}\log(z)\right)$. The set of all eigenvalues of A is denoted by $\sigma(A)$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset D_0$, then $A^{\frac{1}{2}}=\sqrt{A}=\exp\left(\frac{1}{2}\log(A)\right)$ denotes the image by $z^{\frac{1}{2}}$ of the matrix functional calculus acting on the matrix A. The two-norm of A is denoted by $\|A\|_2$ and it is defined by

$$\|A\|_2=\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2}$$

where for a vector y in $\mathbb{C}^N$, $\|y\|_2=(y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [5], it follows that

$$f(A)g(A)=g(A)f(A). \tag{1.1}$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B)\subset\Omega$ and also if $AB=BA$, then

$$f(A)g(B)=g(B)f(A). \tag{1.2}$$

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition [9, 10]

$$Re(z)>0, \quad \text{for all } z\in\sigma(A). \tag{1.3}$$

It has been seen by Defez and J dar [2] that if $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n\geq 0$, $k\geq 0$, it follows (in an analogous way to the proof of Lemma 11 of [13]) that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),\\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}A(k,n-2k).\end{aligned} \tag{1.4}$$

Similarly to (1.4), we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k),\\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}A(k,n)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k).\end{aligned} \tag{1.5}$$

The Hermite matrix polynomials $H_n(x,A)$ of single variable was defined by using the generating function [1, 6, 7] in the following form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,A) = \exp\left(xt\sqrt{2A} - t^2 I\right) \tag{1.6}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$. The Hermite matrix polynomials are explicitly expressed as follows

$$H_n(x,A) = n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{k!(n-2k)!} (x\sqrt{2A})^{n-2k}, \quad n \geq 0. \tag{1.7}$$

The Hermite matrix polynomials are defined through the operational rule [1] in the form

$$H_n(x,A) = \exp\left(-\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dx^2}\right)(x\sqrt{2A})^n. \tag{1.8}$$

In addition, the inverse of (1.8) allows concluding that

$$(x\sqrt{2A})^n = \exp\left(\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dx^2}\right) H_n(x,A). \tag{1.9}$$

In next section, we introduce to define and study of a new matrix function which represents of the Tricomi matrix functions as given by the relation and the convergence properties, radius of convergence and an integral form are given.

2. **Tricomi matrix functions**

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3). The Tricomi matrix functions $C_n(z,A)$ is defined by the series

$$C_n(z,A) = \sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{2^k k!(n+k)!}. \tag{2.1}$$

Using (1.4), (1.5) and (2.1), we arrange the series

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} t^n\, C_n(z,A) &= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{2^k k!(n+k)!} t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{2^k k! n!} t^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} t^n I \sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{2^k k!} t^k \\
&= e^{tI - \frac{z\sqrt{2A}}{2t}}.
\end{aligned}$$

We obtain an explicit representation for the Tricomi matrix functions by the generating matrix function in the form

$$F(z,t,A)=\sum_{n=-\infty}^{\infty} t^n\, C_n(z,A)=e^{tI-\frac{z\sqrt{2A}}{2t}}. \tag{2.2}$$

Now, we will investigate the convergence of the series (2.1) and by using the ratio test, one gets

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{\|U_{k+1}(z)\|}{\|U_k(z)\|}\right| &= \lim_{k\to\infty}\left\|\left\|\frac{(-1)^{k+1}(z\sqrt{2A})^{k+1}}{2^{k+1}(k+1)!(n+k+1)!}\,\frac{2^k k!(n+k)!}{(-1)^k(z\sqrt{2A})^k}\right\|\right\| \\ &= \lim_{k\to\infty}\left\|\left\|\frac{-z\sqrt{2A}}{2(k+1)(n+k+1)}\right\|\right\| \\ &= \lim_{k\to\infty}\left\|\frac{z\sqrt{2A}}{2(k+1)(n+k+1)}\right\|=0\end{aligned}$$

where

$$U_k(z)=\frac{(-1)^k(z\sqrt{2A})^k}{2^k k!(n+k)!}.$$

Now, we begin the study of this function by calculating its radius of convergence R. For this purpose, we recall relation of [14], then

$$\begin{aligned}\frac{1}{R} &= \limsup_{k\to\infty}(\|U_k\|)^{\frac{1}{k}} \\ &= \lim_{k\to\infty}\sup\left(\left\|\frac{(-1)^k(\sqrt{2A})^k}{2^k k!(n+k)!}\right\|\right)^{\frac{1}{k}} \\ &= \lim_{k\to\infty}\sup\left(\left\|\frac{(-1)^k(\sqrt{2A})^k}{2^k\sqrt{2\pi\, k}(\frac{k}{e})^k\sqrt{2\pi\,(n+k)}(\frac{n+k}{e})^{n+k}}\right\|\right)^{\frac{1}{k}}=0.\end{aligned}$$

Then, the Tricomi matrix functions is an entire function. The integral form of Tricomi matrix functions is provided by the following theorem:

Theorem 2.1. *Suppose that A is a matrix in $C^{N\times N}$ satisfying* (1.3)*, then*

$$C_n(z,A)=\frac{1}{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{n-\frac{1}{2}}\cos(wt)\mathrm{dt} \tag{2.3}$$

where $|\arg(z)|<\pi$ *and* $w^2=2z\sqrt{2A}$.

Proof. By Lemma 2 of [9], we can state that

$$\begin{aligned}\frac{1}{(n+k)!} &= \frac{1}{\Gamma(n+k+1)} \\ &= \frac{1}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}\int_{-1}^{1}t^{2h}(1-t^2)^{n-\frac{1}{2}}dt.\end{aligned} \tag{2.4}$$

From (2.1) and (2.4), we get

$$\begin{aligned}
C_n(z,A) &= \sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{2^k k!} \frac{1}{\Gamma(n+\frac{1}{2})\Gamma(k+\frac{1}{2})} \int_{-1}^{1} t^{2h}(1-t^2)^{n-\frac{1}{2}} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{\Gamma(n+\frac{1}{2})} \frac{1}{2^k k!\Gamma(k+\frac{1}{2})} \int_{-1}^{1} t^{2h}(1-t^2)^{n-\frac{1}{2}} dt \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (z\sqrt{2A})^k}{\Gamma(n+\frac{1}{2})} \frac{1}{2^{-k}\sqrt{\pi}(2k)!} \int_{-1}^{1} t^{2h}(1-t^2)^{n-\frac{1}{2}} dt \\
&= \frac{1}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k (2z\sqrt{2A})^k t^{2k}}{(2k)!} dt \\
&= \frac{1}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} \cos(wt) dt
\end{aligned}$$

where $|\arg(z)| < \pi$ and $w^2 = 2z\sqrt{2A}$, the result is established. □

2.1. *Matrix recurrence relations*

Some matrix recurrence relations will be established for the Tricomi matrix functions. First, we obtain

Theorem 2.2. *The Tricomi matrix functions $C_n(x,A)$ satisfy the relations*

$$\frac{d^r}{dz^r} C_n(z,A) = (-1)^r \frac{(\sqrt{2A})^r}{2^r} C_{n+r}(z,A). \tag{2.5}$$

Proof. Differentiating the identity (2.2) with respect to z yields

$$\sum_{n=-\infty}^{\infty} t^n \frac{d}{dz} C_n(z,A) = -\frac{\sqrt{2A}}{2t} e^{tI - \frac{z\sqrt{2A}}{2t}}. \tag{2.6}$$

From (2.6) and (2.2), we have

$$\sum_{n=-\infty}^{\infty} t^n \frac{d}{dz} C_n(z,A) = -\frac{\sqrt{2A}}{2} \sum_{n=-\infty}^{\infty} t^{n-1}\, C_n(z,A). \tag{2.7}$$

Hence, identifying the coefficients of t^n, we obtain

$$\frac{d}{dz} C_n(z,A) = -\frac{\sqrt{2A}}{2}\, C_{n+1}(z,A). \tag{2.8}$$

Iteration (2.8) for $0 \le r \le n$ implies (2.5). Therefore, the expression (2.5) is established and the proof of Theorem 2.2 is completed. □

The above three-terms matrix recurrence relation will be used in the following theorem.

Theorem 2.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.3). *Then, we have*

$$z\sqrt{2A} C_{n+1}(z,A) - 2n C_n(z,A) + 2C_{n-1}(z,A) = 0. \tag{2.9}$$

Proof. Differentiating (2.2) with respect to z and t, we find respectively

$$\begin{aligned}\frac{\partial F(z,t,A)}{\partial z} &= \sum_{n=-\infty}^{\infty} t^n \frac{d}{dz} C_n(z,A)\\ &= -\frac{\sqrt{2A}}{2t} e^{tI-\frac{z\sqrt{2A}}{2t}}\\ &= -\frac{\sqrt{2A}}{2t} \sum_{n=-\infty}^{\infty} t^n C_n(z,A)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F(z,t,A)}{\partial t} &= \sum_{n=-\infty}^{\infty} nt^{n-1} C_n(z,A)\\ &= \left(I + \frac{\sqrt{2A}z}{2t^2}\right) e^{tI-\frac{z\sqrt{2A}}{2t}}\\ &= \left(I + \frac{\sqrt{2A}z}{2t^2}\right) \sum_{n=-\infty}^{\infty} t^n C_n(z,A)\\ &= \sum_{n=-\infty}^{\infty} t^n C_n(z,A) + \frac{\sqrt{2A}z}{2} \sum_{n=-\infty}^{\infty} t^{n-2} C_n(z,A).\end{aligned}$$

Hence, identifying the coefficients of t^{n-1}, we obtain

$$nC_n(z,A) = C_{n-1}(z,A) + \frac{\sqrt{2A}x}{2} C_{n+1}(z,A).$$

Therefore, $F(z,t,A)$ satisfies the partial matrix differential equation

$$\left(I + \frac{\sqrt{2A}z}{2t^2}\right)\frac{\partial F(z,t,A)}{\partial z} + \frac{\sqrt{2A}}{2t}\frac{\partial F(z,t,A)}{\partial t} = 0$$

or

$$(2t^2 I + \sqrt{2A}z)\frac{\partial F(z,t,A)}{\partial z} + \sqrt{2A}t\frac{\partial F(z,t,A)}{\partial t} = 0$$

which, by virtue of (2.2), becomes

$$(2t^2 I + \sqrt{2A}z) \sum_{n=-\infty}^{\infty} t^n \frac{d}{dz} C_n(z,A) + \sqrt{2A}t \sum_{n=-\infty}^{\infty} nt^{n-1} C_n(z,A) = 0.$$

It follows that

$$2\frac{d}{dz} C_{n-2}(z,A) + \sqrt{2A}z\frac{d}{dz} C_n(z,A) + n\sqrt{2A}C_n(z,A) = 0. \tag{2.10}$$

Using (2.8) and (2.10), we get (2.9). The proof of Theorem 2.3 is completed. □

The above recurrence properties can be derived either from (2.8) or from (2.9). It is easy to prove that

$$\begin{aligned} &\frac{d}{dz}C_n(z,A) = -\frac{\sqrt{2A}}{2}\, C_{n+1}(z,A), \\ &C_{n-1}(z,A) = \left[n + z\frac{d}{dz}\right] C_n(z,A). \end{aligned} \tag{2.11}$$

The matrix differential equation satisfied by $C_n(z,A)$ can be straightforwardly inferred by introducing the shift operators

$$\begin{aligned} &\widehat{P} = -2\frac{d}{dz}, \\ &\widehat{M} = n + z\frac{d}{dz} \end{aligned} \tag{2.12}$$

which act on $C_n(z,A)$ according to the rules

$$\widehat{P}\, C_n(z,A) = \sqrt{2A}\, C_{n+1}(z,A), \qquad \widehat{M}\, C_n(z,A) = C_{n-1}(z,A). \tag{2.13}$$

Using the identity

$$\widehat{P}\widehat{M}\, C_n(z,A) = \sqrt{2A}\, C_n(z,A) \tag{2.14}$$

from (2.12), we find that $C_n(z,A)$ satisfies the following ordinary matrix differential equation of second order

$$\left[2z\frac{d^2}{dz^2} + 2(n+1)\frac{d}{dz} + \sqrt{2A}\right] C_n(z,A) = 0. \tag{2.15}$$

In the next result, the Tricomi matrix functions appear as finite series solutions of the second order matrix differential equation.

Corollary 2.1. *The Tricomi matrix functions are solutions of the matrix differential equation of the second order*

$$\left[2z\frac{d^2}{dz^2} + 2(n+1)\frac{d}{dz} + \sqrt{2A}\right] C_n(z,A) = 0, \quad n \geq 0. \tag{2.16}$$

Proof. From (2.8) and using (2.9), becomes

$$C_{n-1}(z,A) - z\frac{d}{dz}\, C_n(x,A) - nC_n(z,A) = 0.$$

Differentiating the identity (2.2) with respect to z yields

$$\frac{d}{dz}C_{n-1}(z,A) - \frac{d}{dz}\, C_n(z,A) - z\frac{d^2}{dz^2}\, C_n(z,A)) - n\frac{d}{dz}C_n(z,A) = 0. \tag{2.17}$$

Substituting from (2.8) into (2.17), we obtain (2.16). Thus the proof of Corollary 2.1 is completed. □

Corollary 2.2. *The Tricomi matrix functions satisfy the following relations*

$$C_n(z \pm w, A) = \sum_{k=0}^{\infty} \frac{(\mp 1)^k (w\sqrt{2A})^k}{2^k\, k!} C_{n+k}(z,A). \tag{2.18}$$

Proof. Using (2.2), we get directly the equation (2.18). The proof of Corollary 2.2 is completed. □

3. Hermite-Tricomi Matrix Functions

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.3). We define the new generating matrix function which represents the Hermite-Tricomi matrix functions in the form

$$\sum_{n=-\infty}^{\infty} t^n \, {}_HC_n(z,A) = \exp\left(tI - \frac{z\sqrt{2A}}{t} - \frac{1}{t^2}I\right) \tag{3.1}$$

and by the series expansion

$${}_HC_n(z,A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} H_k(z,A). \tag{3.2}$$

It is clear that

$$\begin{aligned}
{}_HC_{-1}(z,A) &= 0,\\
{}_HC_0(z,A) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} H_k(z,A),\\
{}_HC_1(z,A) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} H_k(z,A).
\end{aligned}$$

By exploiting the same argument of the previous section, it is evident that Hermite-Tricomi matrix functions ${}_HC_n(z,A)$ satisfies the properties. In the following theorem, we obtain another representation for the Hermite-Tricomi matrix functions ${}_HC_n(z,A)$ as follows:

Theorem 3.1. *The Hermite-Tricomi matrix functions have the following representation*

$${}_HC_n(z,A) = \exp\left(-\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dz^2}\right) C_n(2z,A). \tag{3.3}$$

Proof. By using (1.8) and (3.2), we consider the series

$$\begin{aligned}
{}_HC_n(z,A) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} H_k(z,A)\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \exp\left(-\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dz^2}\right)\left(z\sqrt{2A}\right)^k\\
&= \exp\left(-\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dz^2}\right)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(z\sqrt{2A}\right)^k\\
&= \exp\left(-\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dz^2}\right) C_n(2z,A).
\end{aligned}$$

The proof of Theorem 3.1 is completed. Furthermore, in view of (3.3), we can write

$$C_n(2z,A) = \exp\left(\frac{1}{(\sqrt{2A})^2}\frac{d^2}{dz^2}\right) {}_HC_n(z,A). \qquad \square \tag{3.4}$$

The new properties of the Hermite-Tricomi matrix functions generated by (3.1) yields as given in the following theorem.

Theorem 3.2. *The Hermite-Tricomi matrix functions satisfy the following relations*

$$ {}_HC_n(z+w,A) = \sum_{k=0}^{\infty} \frac{(-w\sqrt{2A})^k}{k!}\, {}_HC_{n+k}(z,A). \tag{3.5}$$

Proof. Using (3.1), the series can be given in the form

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} {}_HC_n(z+w,A)t^n &= \exp\left(t - \frac{(z+w)\sqrt{2A}}{t} - \frac{1}{t^2}\right)\\
&= \exp\left(-\frac{w\sqrt{2A}}{t}\right)\exp\left(tI - \frac{z\sqrt{2A}}{t} - \frac{1}{t^2}I\right)\\
&= \sum_{n=-\infty}^{\infty} {}_HC_n(z,A)t^n \sum_{k=0}^{\infty}\frac{(-w\sqrt{2A})^k t^{-k}}{k!}\\
&= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-w\sqrt{2A})^k}{k!}\, {}_HC_n(z,A)t^{n-k}.
\end{aligned}$$

Comparing the coefficients of t^n, we get (3.5) and the proof is established. $\square$

In the following corollary, we obtain the properties of Hermite-Tricomi functions as follows:

Corollary 3.1. *The addition corollary is easily derived from* (3.5) *which yields*

$$ {}_HC_n(z\pm w,A) = \sum_{k=0}^{\infty} \frac{(\mp w\sqrt{2A})^k}{k!}\, {}_HC_{n+k}(z,A). \tag{3.6}$$

Proof. By exploiting the addition formula

$$ {}_HC_n(z-w,A) = \sum_{k=0}^{\infty} \frac{(w\sqrt{2A})^k}{k!}\, {}_HC_{n+k}(z,A).$$

Hence, the proof of Corollary 3.1 is established. $\square$

3.1. *Matrix recurrence relations*

Some matrix recurrence relation is carried out on the Hermite-Tricomi matrix functions. We obtain the following

Theorem 3.3. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.3). *The Hermite-Tricomi matrix functions satisfies the following relations*

$$\frac{d^k}{dz^k}\,{}_HC_n(z,A) = (-\sqrt{2A})^k\,{}_HC_{n+k}(z,A), \quad 0 \le k \le n. \tag{3.7}$$

Proof. Differentiating the identity (3.1) with respect to z yields

$$\sum_{n=-\infty}^{\infty} {}_HC'_n(z,A)t^n = \frac{-\sqrt{2A}}{t}\exp\left(tI - \frac{z\sqrt{2A}}{t} - \frac{1}{t^2}I\right). \tag{3.8}$$

From (3.1) and (3.8), we have

$$\sum_{n=-\infty}^{\infty} {}_HC'_n(z,A)t^n = -\sqrt{2A}\sum_{n=-\infty}^{\infty} {}_HC_n(z,A)t^{n-1}. \tag{3.9}$$

Hence, from identifying coefficients in t^n, it follows that

$${}_HC'_n(z,A) = -\sqrt{2A}\,{}_HC_{n+1}(z,A). \tag{3.10}$$

Iteration (3.10), for $0 \le k \le n$, implies (3.7). Hence for particular values of k and n, (3.7) yield

$${}_HC_n(z,A) = (-\sqrt{2A})^{k-n}\frac{d^{n-k}}{dz^{n-k}}\,{}_HC_k(z,A). \tag{3.11}$$

Therefore, the expression (3.7) is established and the proof of Theorem 3.3 is completed. Differentiating the identity (3.1) with respect to t yields

$$\sum_{n=-\infty}^{\infty} n\,{}_HC_n(z,A)t^{n-1} = (t^3I + zt\sqrt{2A} + 2I)\sum_{n=-\infty}^{\infty} {}_HC_n(z,A)t^{n-3}$$

from which by comparing the coefficients of t^n on both sides of the identity, we obtain both the pure matrix recurrence relation

$$(n+1)\,{}_HC_{n+1}(z,A) = {}_HC_n(z,A) + z\sqrt{2A}\,{}_HC_{n+2}(z,A) + 2\,{}_HC_{n+3}(z,A). \quad \square \tag{3.12}$$

The above matrix recurrence relation will be used in the following theorem.

Theorem 3.4. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.3). *Then Hermite-Tricomi matrix functions, we have*

$$2\,{}_HC_{n+1}(z,A) + z\sqrt{2A}\,{}_HC_n(z,A) - (n-1)\,{}_HC_{n-1}(x) + {}_HC_{n-2}(x) = 0. \tag{3.13}$$

Proof. We define the new generating matrix function which represents of the Hermite-Tricomi matrix functions by

$$W(z,t,A) = \sum_{n=-\infty}^{\infty} {}_HC_n(z,A)t^n = \exp\left(tI - \frac{z\sqrt{2A}}{t} - \frac{1}{t^2}I\right). \tag{3.14}$$

Differentiating (3.14) with respect to z and t, we find respectively

$$\frac{\partial W(z,t,A)}{\partial z} = \sum_{n=-\infty}^{\infty} {}_HC'_n(z,A)t^n = -\sqrt{2A}\sum_{n=-\infty}^{\infty} {}_HC_n(z,A)t^{n-1}$$

and

$$\frac{\partial W(z,t,A)}{\partial t} = \sum_{n=-\infty}^{\infty} n\, {}_HC_n(z,A)t^{n-1}$$
$$= (t^3I + xt\sqrt{2A} + 2I)\sum_{n=-\infty}^{\infty} {}_HC_n(z,A)t^{n-3}.$$

Therefore, $W(z,t,A)$ satisfies the partial matrix differential equation

$$(t^3I + zt\sqrt{2A} + 2I)\frac{\partial W(z,t,A)}{\partial z} + t^2\sqrt{2A}\frac{\partial W(z,t,A)}{\partial t} = 0$$

which, by using (3.1), becomes

$$(t^3I + zt\sqrt{2A} + 2I)\sum_{n=-\infty}^{\infty} {}_HC'_n(z,A)t^n + t^2\sqrt{2A}\sum_{n=-\infty}^{\infty} n\, {}_HC_n(z,A)t^{n-1} = 0$$

it follows

$$ {}_HC'_{n-3}(x) + z\sqrt{2A}\, {}_HC'_{n-1}(x) + 2\, {}_HC'_n(x) + (n-1)\sqrt{2A}\, {}_HC_{n-1}(x) = 0. \quad (3.15)$$

Using (3.10) and (3.15), we get (3.13) and the proof of Theorem 3.4 is completed. □

The following result, the Hermite-Tricomi matrix functions appear as finite series solutions of the three order matrix differential equation.

Corollary 3.2. *The Hermite-Tricomi matrix functions are solutions of the matrix differential equation of the three order in the form*

$$\left[2\frac{d^3}{dz^3} - z(\sqrt{2A})^2\frac{d^2}{dz^2} - (n+1)(\sqrt{2A})^2\frac{d}{dz} - (\sqrt{2A})^3\right] {}_HC_n(z,A) = 0. \quad (3.16)$$

Proof. Using (3.10) gives

$$\begin{aligned}
\frac{d}{dz}\, {}_HC_n(z,A) &= -\sqrt{2A}\, {}_HC_{n+1}(z,A),\\
\frac{d^2}{dz^2}\, {}_HC_n(z,A) &= -\sqrt{2A}\frac{d}{dz}\, {}_HC_{n+1}(z,A) = (-\sqrt{2A})^2\, {}_HC_{n+2}(z,A),\\
\frac{d^3}{dz^3}\, {}_HC_n(z,A) &= \frac{d^2}{dz^2}(-\sqrt{2A}\, {}_HC_{n+1}(z,A))\\
&= (-\sqrt{2A})^2\frac{d}{dz}\, {}_HC_{n+2}(z,A) = (-\sqrt{2A})^3\, {}_HC_{n+3}(z,A).
\end{aligned} \quad (3.17)$$

Substituting from (3.17) into (3.12) to obtain (3.16). Thus the proof of Corollary 3.2 is completed. □

Finally, the extension to forms of the type in (2.1) and (3.2) are straightforward to give definitions of this family matrix functions and will be reported here.

It is the purpose of this end section to introduce a new matrix function of complex variable which can be stated in the following form:

$$J_n(z,A) = \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{2A})^k}{2^k k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k} \tag{3.18}$$

where A is a matrix in $\mathbb{C}^{N\times N}$satisfying (1.3), this is known as Bessel matrix functions and is characterized by the following link with Tricomi matrix functions of complex variable:

$$J_n(z,A) = \left(\frac{z}{2}\right)^n C_n\left(\frac{z^2}{4},A\right). \tag{3.19}$$

The Hermite-Bessel matrix polynomials is defined by the series expansion

$${}_HJ_n(z,A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} H_{n+2k}(z,A). \tag{3.20}$$

Further examples proving the usefulness of the present methods can be easily worked out, but are not reported here for conciseness. These last identities indicate that the method described in this paper can go beyond the specific problem addressed here and can be exploited in a wider context.

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