Communications in Mathematics and Applications
Volume 2 (2011), Numbers 2-3, pp. 87–95

http://www.rgnpublications.com

Existence of Large Solutions for Quasilinear Elliptic Equation

Xiao Li and Zuodong Yang

Abstract. In this paper, we consider the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda u - b(x)h(u), & x \in \Omega \\ u = +\infty, & \text{on } \partial\Omega. \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R^N}$. The weight function $b(x)$ is a non-negative continuous function in the domain, $h(u)$ is locally Lipschitz continuous, $h(u)/u^{p-1}$ is increasing on $(0,\infty)$ and $h(u) \sim Hu^{m(p-1)}$ for sufficiently large u with $H > 0$ and $m > 1$. We establish conditions sufficient to ensure the existence of positive large solutions of the equation.

1. Introduction

In this paper, we are concerned with the existence of positive solution of quasilinear elliptic equations with singular boundary value condition in the following form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda u - b(x)h(u), & x \in \Omega \\ u = +\infty, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbf{R^N}$ is a smooth bounded domain, $p > 2$. The boundary condition in (1.1) is understood as $u(x) \to \infty$ when $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0^+$. The non-negative solutions of (1.1) are called large (or blow-up).

For the following singular boundary value problem

$$\begin{cases} \triangle u(x) = f(u(x)) & x \in \Omega, \\ u|_{\partial\Omega} = \infty \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N (N \geq 1)$ arises naturally from a number of different areas and has a long history, see [1, 8, 9, 15]. A problem with

2010 *Mathematics Subject Classification.* 35J25, 35B50, 35J40.
Key words and phrases. Blow up; Large solution; Quasilinear elliptic equation; Upper-lower solution; Existence.
Project supported by the National Natural Science Foundation of China (Grant No.11171092). Project Supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB110005).

$f(u) = e^u$ and $N = 2$ was first considered by Bieberbach [1] in 1916. Rademacher [7], using the idea of Bieberbach, extended the above result to a smooth bounded domain in $\mathbf{R}^3$. In this case the problem plays an important role, when $N = 2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N = 3$, in the study of the electric potential in a glowing hollow metal body. Lazer and McKenna [6] extended the results for a bounded domain Ω in $\mathbf{R}^N (N \geq 1)$ satisfying a uniform external sphere condition and the non-linearity $f = f(x,u) = p(x)e^u$, where $p(x)$ is continuous and strictly positive on $\overline{\Omega}$. Lazer and McKenna [15] obtained similar results when $\triangle$ is replaced by the Monge-Ampere operator and Ω is a smooth, strictly convex, bounded domain. Similar results were also obtained for $f = p(x)u^a$ with $a > 1$.

For the following singular boundary value problem

$$\begin{cases} -\triangle u = \lambda u - b(x)h(u), & x \in \Omega \\ u = +\infty, & \text{on } \partial\Omega \end{cases}$$

have been extensively studied, for example, see [13, 16, 17, 18] and the references therein.

Quasilinear elliptic problems with boundary blow-up

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) = f(u(x)), & x \in \Omega \\ u|_{\partial\Omega} = \infty. \end{cases} \tag{1.3}$$

have been studied, see [10, 11, 12] and the references therein. Diaz and Letelier proved the existence and uniqueness of large solutions to the problem (1.3) both for $f(u) = u^\gamma$, $\gamma > m - 1$ (super-linear case) and $\partial\Omega$ being of the class C^2. Recently, Lu, Yang and E.H. Twizell [10] proved the existence of Large solutions to the problem (1.3) both for $f(u) = u^\gamma$, $\gamma > m - 1$, $\Omega = \mathbf{R}^N$ or Ω being a bounded domain (super-linear case) and $\gamma \leq m - 1$, $\Omega = \mathbf{R}^N$ (sub-linear case) respectively.

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory ([2]-[3]), non-Newtonian filtration ([4]) and the turbulent flow of a gas in porous medium ([5]). In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

Motivated by the results of the above cited papers, we shall attempt to treat such equation (1.1), the results of the semilinear equations are extended to the quasilinear ones. We can find the related existence results for $p = 2$ in [13].

2. Preliminaries

Consider the singular boundary value problem (1.1), where $\lambda \in \mathbf{R}^+$, Ω is a smooth bounded domain in $\mathbf{R}^\mathbf{N}$, and the weight function $b(x) > 0$ in Ω, the nonlinear function $h(u)$ satisfies:

($\mathscr{A}$) $h(u) \geq 0$ is locally Lipschitz continuous on $[0,\infty)$ and $h(u)/u^{p-1}$ is increasing on $(0,\infty)$; and, for some $m > 1$,

$$H = \lim_{u\to\infty} \frac{h(u)}{u^{m(p-1)}} > 0\,. \tag{2.1}$$

Note that (2.1) implies that h satisfies Keller-Osserman condition. Indeed, according to (2.1), there exists $\zeta > 0$ such that

$$h(u) > \frac{H}{2} u^{m(p-1)}, \quad u \geq \zeta\,.$$

Therefore

$$\begin{aligned}
&\int_1^\infty [g(t)]^{\frac{-1}{p}} dt \\
&= \int_1^\infty \left[\int_0^t h(u)du\right]^{\frac{-1}{p}} dt \\
&= \int_1^\zeta \left[\int_0^t h(u)du\right]^{\frac{-1}{p}} dt + \int_\zeta^\infty \left[\int_0^t h(u)du\right]^{\frac{-1}{p}} dt \\
&= \int_1^\zeta \left[\int_0^t h(u)du\right]^{\frac{-1}{p}} dt + \int_\zeta^\infty \left[\int_0^\zeta h(u)du + \int_\zeta^t h(u)du\right]^{\frac{-1}{p}} dt \\
&\leq M^{\frac{-1}{p}}(\zeta - 1) + \int_\zeta^\infty \left[\frac{H}{2}\frac{u^{m(p-1)+1}}{m(p-1)} - \frac{H}{2}\frac{\xi^{m(p-1)+1}}{m(p-1)}\right]^{\frac{-1}{p}} dt < \infty
\end{aligned}$$

because $m(p-1) > 1$ and $0 < M = \int_0^1 h(u)du < \int_0^t h(u)du$ for $t \geq 1$, so the existence of large solutions of (1.1) is guaranteed.

Consider the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda u - b(x)h(u), & x \in \Omega \\ u = \phi, & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain, $\phi \in C(\partial\Omega)$, h satisfies ($\mathscr{A}$) and $b \in C(\Omega, \mathbf{R}^+)$.

Lemma 2.1 ([13]). *Let $\underline{u}, \bar{u} \in W^{1,p}(\bar{\Omega})$ such that*

$$\begin{cases} -\mathrm{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \leq \lambda \underline{u} - b(x)h(\underline{u}), & x \in \Omega \\ -\mathrm{div}(|\nabla \bar{u}|^{p-2}\nabla \bar{u}) \geq \lambda \bar{u} - b(x)h(\bar{u}), & x \in \Omega\,. \end{cases}$$

If $\underline{u} \leq \phi \leq \bar{u}$ on $\partial\Omega$, then $\underline{u}(x) \leq \bar{u}(x)$ on $\bar{\Omega}$.

Lemma 2.2 ([14]). *Let $\phi = \infty$ in (2.1), let $\underline{u}$, $\bar{u} \in W^{1,p}(\bar{\Omega})$ such that $\bar{u} = \infty$ on $\partial\Omega$ and $\underline{u} = \infty$ on $\partial\Omega$. If*

$$-\text{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \le \lambda \underline{u} - b(x)h(\underline{u}), \quad x \in \Omega$$
$$-\text{div}(|\nabla \bar{u}|^{p-2}\nabla \bar{u}) \ge \lambda \bar{u} - b(x)h(\bar{u}), \quad x \in \Omega$$

and $\underline{u} \le \bar{u}$ in Ω, then there exists at least one solution u such that $\underline{u} \le u \le \bar{u}$ and $u(x) \to \infty$ as $x \to \partial\Omega$.

Definition. A function $\bar{u} \in W^{1,p}$ is a subsolution to problem (1.1) if $\underline{u} = +\infty$ on $\partial\Omega$ and

$$-\text{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \le \lambda \underline{u} - b(x)h(\underline{u}), \quad x \in \Omega.$$

Similarly, $\bar{u}$ is a supersolution to problem (1.1) if $\bar{u} = +\infty$ on $\partial\Omega$ and

$$-\text{div}(|\nabla \bar{u}|^{p-2}\nabla \bar{u}) \ge \lambda \bar{u} - b(x)h(\bar{u}), \quad x \in \Omega.$$

Lemma 2.3. *Let $b(r) : [0,R] \mapsto [0,\infty)$ be continuous function such that $b(r) > 0$ for $r \in [0,R)$. Define $B(r) = \int_r^R b(s)ds$, $b^*(r) = \int_r^R B(s)ds$. If $g(r) = \frac{B(r)}{b(r)}$ is differentiable in $[0,R]$ and $\lim_{r\to R} g(r) = 0$, $\lim_{r\to R} g'(r) \le 0$, then we have*

$$\lim_{r\to R} \frac{B^\mu(r)}{b(r)} = 0, \text{ for all } \mu \ge 1, \quad \lim_{r\to R} \frac{b^*(r)}{B(r)} = 0 \text{ and } \lim_{r\to R} \frac{(B(r))^2}{b^*(r)b(r)} = c_0 \ge 1.$$

Proof. Since $\frac{B^\mu(r)}{b(r)} = \frac{B(r)}{b(r)}(B(r))^{\mu-1}$, $\lim_{r\to R} \frac{B(r)}{b(r)} = 0$ for all $\mu \ge 1$ follows easily. By the *L'Hospital* rule, we have

$$\begin{aligned}\lim_{r\to R} b^*(r)/B(r) &= \lim_{r\to R}(b^*(r)/B(r))(B(r)/B(r)) \\ &= \lim_{r\to R} B(r)/b(r) \lim_{r\to R} B(r)/B(r) = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to R} \frac{(B(r))^2}{b(r)b^*(r)} &= \lim_{r\to R} \frac{(B(r)/b(r))B(r)}{b^*(r)} \\ &= \lim_{r\to R} \frac{(B(r)/b(r))'B(r) - B(r)}{-B(r)} \\ &= 1 - \lim_{r\to R}(B(r)/b(r))' \\ &= c_0 \ge 1.\end{aligned}$$

□

3. Main results

Theorem 3.1. *Suppose that $\Omega = B_R(x_0)$ is a ball in $\mathbf{R}^\mathbf{N}$ of radius R centered at x_0 and $h(u)$ satisfies $(\mathscr{A})$, $m(p-1)$ is odd, $\lambda \in R^+$ and $b(x) = b(\|x - x_0\|)$ is a radially symmetric function on the ball, $b \in C([0,R];[0,\infty))$ satisfies $b > 0$ in $[0,R)$, $\lim_{r\to R} b(r) = 0$, $\lim_{r\to R} B(r)/b(r) = 0$,* *and* $c_0 = \lim_{r\to R}(B(r))^2/(b^*(r)b(r)) \geq 1$, *where* $B(r) = \int_r^R b(s)ds$ *and* $b^*(r) = \int_r^R B(s)ds$. *Then the problem* (1.1) *has a positive solution* u.

Proof. We consider the corresponding singular problem (1.1) in radial form

$$\begin{cases} -(|\psi'|^{p-2}\psi')' - \frac{N-1}{r}|\psi'|^{p-2}\psi' = \lambda\psi - b(r)h(\psi) \ \text{ in } (0,R) \\ \lim_{r\to R}\psi(r) = \infty \\ \psi'(0) = 0 \end{cases} \tag{3.1}$$

We claim that for each $\varepsilon > 0$, the problem (3.1) possesses a large solution ψ_ε where we denoted

$$\beta = \frac{1}{p-1}, \quad b^*(r) = \int_r^R \int_s^R b(t)dt\, ds.$$

Therefore, for each $x_0 \in \mathbf{R^N}$, the function

$$u_\varepsilon(x) = \psi_\varepsilon(r), \quad r = \|x - x_0\|$$

provides us with a radially symmetric positive large solution of (1.1) with the assumption in Theorem 3.1.

To prove the claim, we first construct a supersolution of (3.1) for each $\varepsilon > 0$. Let

$$\overline{\psi}_\varepsilon(r) = A + B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta}$$

where $A > 0$ and $B_+ > 0$. Then

$$\begin{aligned} \overline{\psi}'_\varepsilon(r) &= 2B_+\frac{r}{R^2} b^*(r))^{-\beta} - \beta B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))', \\ \overline{\psi}''_\varepsilon(r) &= 2B_+\frac{1}{R^2}(b^*(r))^{-\beta} - 4\beta B_+\frac{r}{R^2}(b^*(r))^{-\beta-1}(b^*(r))' \\ &\quad + \beta(\beta+1)B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-2}[(b^*(r))']^2 \\ &\quad - \beta B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))'' \end{aligned}$$

$\overline{\psi}_\varepsilon(r) \to \infty$ as $r \to R$ because $b^*(r) \to 0$ as $r \to R$ and $\beta > 0$. Also $\overline{\psi}'_\varepsilon(r) \to 0$ as $r \to 0$. Then $\overline{\psi}_\varepsilon(r)$ is a supersolution if

$$\begin{aligned} &-(|\overline{\psi}'_\varepsilon(r)|^{p-2}\overline{\psi}'_\varepsilon(r))' - \frac{N-1}{r}(|\overline{\psi}'_\varepsilon(r)|^{p-2}\overline{\psi}'_\varepsilon(r)) \\ &\quad \geq \lambda\overline{\psi}_\varepsilon(r) - b(r)h(\overline{\psi}_\varepsilon(r)). \end{aligned} \tag{3.2}$$

By the assumption $(\mathscr{A})$ on h, it is easy to see that for the same $\varepsilon > 0$

$$(1-\varepsilon)H\overline{\psi}_{\varepsilon}^{m(p-1)}(r) \le h(\overline{\psi}_{\varepsilon}(r)) \le (1+\varepsilon)H\overline{\psi}_{\varepsilon}^{m(p-1)}(r)$$

for all $r \in [0,R)$, by choosing A sufficiently large, say $A \ge A_0$. The inequality (3.2) holds if

$$\begin{aligned} &-(|\overline{\psi}_{\varepsilon}'(r)|^{p-2}\overline{\psi}_{\varepsilon}'(r))' - \frac{N-1}{r}(|\overline{\psi}_{\varepsilon}'(r)|^{p-2}\overline{\psi}_{\varepsilon}'(r)) \\ &\qquad \ge \lambda\overline{\psi}_{\varepsilon}(r) - b(r)(1-\varepsilon)H\overline{\psi}_{\varepsilon}^{m(p-1)}(r) \end{aligned}$$

That is

$$\begin{aligned} &-(p-1)\left[2B_+\frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}b^*(r))'\right]^{p-2} \\ &\times\left\{2B_+\frac{1}{R^2}(b^*(r))^{-\beta} - 4\beta B_+\frac{r}{R^2}(b^*(r))^{-\beta-1}(b^*(r))'\right. \\ &+\beta(\beta+1)B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-2}[(b^*(r))']^2\Big] \\ &\left.-\beta B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}(b^*(r))''\right\} \\ &-\frac{N-1}{r}\left[2B_+\frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}b^*(r))'\right]^{p-1} \\ &\quad \ge \lambda(b^*(r))^{-\beta}\left[A(b^*(r))^{\beta} + B_+\left(\frac{r}{R}\right)^2\right] \\ &\qquad - b(r)(1-\varepsilon)H\left[A+B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}\right]^{m(p-1)}. \end{aligned}$$

Multiplying both sides of this inequality by $\frac{(b^*(r))^{m\beta(p-1)}}{b(r)}$ and taking into consideration that $m\beta = \beta + 1$,

$$\begin{aligned} 0 \ge{}& \lambda\frac{b^*(r)}{b(r)}(b^*(r))^{m\beta(p-2)}\left[A(b^*(r))^{\beta} + B_+\left(\frac{r}{R}\right)^2\right] \\ &-(1-\varepsilon)H\left[A+B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}\right]^{m(p-1)}. \end{aligned}$$

Since when $r \to R$, $\frac{b^*(r)}{b(r)} \to 0$, $\frac{(b^*(r))'}{b(r)} \to 0$, $\frac{[(b^*(r))']^2}{b^*(r)b(r)} \to c_0 \ge 1$ and $\frac{(b^*(r))''}{b(r)} \to 1$, by Lemma 2.3, then the above inequality becomes into

$$0 \ge -(1-\varepsilon)H(B_+)^{m(p-1)} \quad \text{as } r \to R.$$

So, $\overline{\psi}_{\varepsilon}(r) = A + B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}$ is a supersolution of (3.1).

Next, we construct a subsolution of (3.1). Due to the assumption ($\mathscr{A}$) on h, for $u \geq A_0$ large, $(1-\varepsilon)Hu^{m(p-1)} \leq h(u) \leq (1+\varepsilon)Hu^{m(p-1)}$.

For each $A_0 > 0$, and $0 < R_0 < R$, we consider the auxiliary problem

$$\begin{cases} -(|\psi'|^{p-2}\psi')' - \frac{N-1}{r}|\psi'|^{p-2}\psi' = \lambda\psi - b(r)h(\psi), & \text{in } (0,R_0) \\ \psi(R_0) = A_0, \\ \psi'(0) = 0. \end{cases} \tag{3.3}$$

By the assumption on b and h, we have

$$\min_{r\in[0,R_0]} b(r) > 0, \;\; h(0) = 0, \;\; \text{and} \;\; \frac{h(u)}{u} \to \infty, \;\; \text{as } u \to \infty$$

Then it is easy to know that

$$\underline{\psi}_{A_0} = 0, \quad \overline{\psi}_{A_0} = A_0$$

provides us with an ordered sub-supersolution pair of (3.3). Thus (3.3) possesses a solution ψ_{A_0} such that $\psi_{A_0}(r) \in [0, A_0]$ for all $r \in [0, R_0]$. For each $\varepsilon > 0$ sufficiently small, we claim that there exists $0 < A_0 < C$ for which the function

$$\psi_\varepsilon(r) = \begin{cases} \psi_{A_0}(r) & r \in [0,R_0] \\ \max\{A_0, C + B_-(r/R)^2(b^*(r))^{-\beta}\} & r \in (R_0, R] \end{cases}$$

provides a subsolution, where $B_- < 0$.

In fact, denoting $f_c(r) = C + B_-(r/R)^2(b^*(r))^{-\beta}$ we have

$$\begin{aligned} f_c'(r) &= 2B_-\frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}(b^*(r))' \\ &= 2B_-\frac{r}{R^2}(b^*(r))^{-\beta} + \beta B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}\int_r^R b(s)ds \end{aligned}$$

which is strictly smaller than zero in $(0,R)$. It follows that $f_c(r)$ is decreasing and

$$\lim_{r\to R} f_c(r) = -\infty, \quad \lim_{r\to 0} f_c(r) = C > A_0$$

By the continuity of $f_c(r)$ and the intermediate-value theorem, there exists a unique $Z = Z(C) \in (0,R)$ such that

$$C + B_-(r/R)^2(b^*(r))^{-\beta} > A_0 \quad \text{when } r \in [0, Z(C))$$

$$C + B_-(r/R)^2(b^*(r))^{-\beta} \leq A_0 \quad \text{when } r \in [0, Z(C))$$

Let $R_0 = Z(C)$, from the definition of $\underline{\psi}_\varepsilon(r)$ and R_0, $\psi_\varepsilon(r) = \psi_{A_0}(r)$ in $[0, Z(C)]$, and then the inequality

$$-(|\underline{\psi}'_\varepsilon(r)|^{p-2}\underline{\psi}'_\varepsilon(r))' - \frac{N-1}{r}(|\underline{\psi}'_\varepsilon(r)|^{p-2}\underline{\psi}'_\varepsilon(r)) \le \lambda\underline{\psi}_\varepsilon(r) - b(r)h(\underline{\psi}_\varepsilon(r)) \tag{3.4}$$

holds in $[0, Z(C)]$. So $\psi_\varepsilon(r)$ is a subsolution if the (3.4) is satisfied in $[Z(C), R]$.

By direct computation and by using the fact $h(\underline{\psi}_\varepsilon(r)) \le (1+\varepsilon)H\underline{\psi}_\varepsilon{}^{m(p-1)}(r)$ in $[Z(C), R]$. (3.4) holds of

$$\begin{aligned}&-(|\underline{\psi}'_\varepsilon(r)|^{p-2}\underline{\psi}'_\varepsilon(r))' - \frac{N-1}{r}(|\underline{\psi}'_\varepsilon(r)|^{p-2}\underline{\psi}'_\varepsilon(r))\\ &\quad\le \lambda\underline{\psi}_\varepsilon(r) - b(r)(1-\varepsilon)H\underline{\psi}_\varepsilon{}^{m(p-1)}(r).\end{aligned}$$

That is

$$\begin{aligned}&-(-1)^p(p-1)\left[2B_-\frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}b^*(r))'\right]^{p-2}\\ &\times\left\{2B_-\frac{1}{R^2}(b^*(r))^{-\beta} - 4\beta B_-\frac{r}{R^2}(b^*(r))^{-\beta-1}(b^*(r))'\right.\\ &+\beta(\beta+1)B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-2}[(b^*(r))']^2]\\ &\left.-\beta B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}(b^*(r))''\right\}\\ &-(-1)^p\frac{N-1}{r}\left[2B_-\frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}b^*(r))'\right]^{p-1}\\ &\quad\le \lambda(b^*(r))^{-\beta}\left[A(b^*(r))^\beta + B_-\left(\frac{r}{R}\right)^2\right]\\ &\qquad - b(r)(1+\varepsilon)H(b^*(r))^{-m\beta(p-1)}\left[A(b^*(r))^\beta + B_-\left(\frac{r}{R}\right)^2\right]^{m(p-1)}.\end{aligned}$$

Multiply $\frac{(b^*(r))^{m\beta(p-1)}}{b(r)}$ by the inequality, we get

$$0 \le -(1+\varepsilon)H(B_-)^{m(p-1)} \quad \text{as } r \to R.$$

So, $\underline{\psi}_\varepsilon(r)$ is a supersolution of (3.1). By Lemma 2.2, that completes the theorem. □

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Xiao Li, *Zhejiang Water Conservancy and Hydropower College, Zhejiang Hangzhou 310018, China*.

Zuodong Yang, *Institute of Mathematics, School of Mathematics Science, Nanjing Normal University, Jiangsu Nanjing 210046, China; and*
College of Zhongbei, Nanjing Normal University, Jiangsu Nanjing 210046, China.
E-mail: zdyang_jin@263.net

Received August 11, 2011
Accepted December 26, 2011