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Research Article

Classical Solution for the Boltzmann Equation with Absorption Term in Yang-Mills Field

David Dongo[1*], Norbert Noutchegueme[2] and Abel Kenfack Nguelemo[1]

[1] *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, POB: 67 Dschang, Cameroon*

[2] *Department of Mathematics, Faculty of Science, University of Yaoundé I, POB: 812 Yaoundé, Cameroon*

[*]**Corresponding author:** dongodavid@yahoo.fr

Abstract. We consider in this work the Boltzmann equation with absorption term in the presence of an external field which is of Yang-Mills type, on a Bianchi type 1 space-time. Such an equation governs the evolution with collisions of plasmas, for instance of quarks and gluons (quagma), where non-Abelian Yang-Mills field replaces the usual electromagnetic field. A local in time existence and uniqueness result for the classical solution is established, using a suitable combination of Faedo Galerkin method and the standard iteration method. We also prove the well-posedness of the solution.

Keywords. Boltzmann equation; Absorption term; Yang-Mills field; Classical solution; Bianchi type 1

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1. Introduction

The Boltzmann equation is one of the basic equations of the relativistic kinetic theory where particles are fast moving. It has proved fruitful, not only for the study of the classical gases that Boltzmann had in mind, but also properly generalized for the study of electron transport in solids and plasmas, neutron transport in nuclear reactors, phonon transport in superfluids and radiactive transfer in planetary and stellar atmospheres. Many authors have studied the Boltzmann equation in the relativistic situation, taking it alone or coupled with orther type of equations. Local existence was proved by Bancel and Choquet-Bruhat many years ago [5, 6]. Noutchegueme, Dongo and Takou studied the Boltzmann equation in some cosmological settings [17, 18] and they obtained milds solutions. Noutchegueme et al. improved the works done in [17, 18] to obtain classical solutions in [3, 4, 15, 16]. Lee in [12] has replaced the $\mu - N$ regularity assumption on the differential cross section of [18] by that of hard potentials. More recently, with Robertson-Walker spacetime as background, Bazow [7] solved the full nonlinear Boltzmann equation for an expanding massless gas. Takou and Ciake Ciake have also replaced the $\mu - N$ regularity on the differential cross section by that of "Israel particles" [20] or "hard potentials" [21, 22] and they have obtained global solutions using some space-time background. But we note that all this works has been done, mostly among them, in the case of uncharged particles. The few that takes into account the charged particles, consider the cas of Abelian chargesthat are governed by Maxwell's electromagnetic field.

In this work, we consider a more general plasma with particles having non-Abelian charges, for example the field of quarks and gluons that one meets in chromodynamics, the Maxwell electromagnetic field is replaced by the Yang-Mills field. The plasma obtained here, called "plasma quarks-gluons", is supposed to exist at very high temperature. In this case, unlike in the Abelian's one, the unknown of the Boltzmann equation depends, not only on the position $(x^\alpha) = (x^0, x^i)$ and the 4-momentum of particles denoted by $p = (p^\alpha) = (p^0, \bar{p})$, but also on the non-Abelian charge of particles denoted by $q = (q^I)$, $I = 1, 2, \dots, N$ (where N is the dimension of the Lie algebra $\mathcal{G}$ of a Lie group G). In the collisionless case where the Boltzmann equation is replaced by the Vlasov equation, many authors have already studied this kind of phenomenon: Choquet-Bruhat and Noutchegueme in [8] studied the Yang-Mills-Vlasov system using the characteristic method and obtained a local in time existence result. They also studied in [9] the Yang-Mills-Vlasov system only for the zero mass particles case and obtained a global existence theorem in Minkowski space-time for small initial data; Noutchegueme and Noundjeu in [19] proved a local in time existence theorem of solutions of the Cauchy problem for the Yang-Mills system in temporal gauge with current generated by a distribution function that satisfies a Vlasov equation; Ayissi et al. in [2] obtained the viscosity solutions for the one-Body Liouville equation in Yang-Mills charged Bianchi models with non-zero mass.

In the instantaneous, binary and elastic scheme due to Lichnerowicz [13], we consider that at a given position $x^\alpha = (t, x)$, two particles of momenta $p = (p^0, \bar{p})$, $p_* = (p^0_*, \bar{p}_*)$, and charges $q = (\widetilde{q}, q^N)$, $q_* = (\widetilde{q}_*, q^N_*)$ respectively, collide without destroying each other. The collision affecting their momenta and charges that change after the collision to become $p' = (p'^0, \bar{p}')$ and $p'_* = (p'^0_*, \bar{p}'_*)$ for momenta, $q' = (\widetilde{q}', q'^N)$ and $q'_* = (\widetilde{q}'_*, q'^N_*)$ for charges, respectively. They satisfy $p + p_* = p' + p'_*$ and $q + q_* = q' + q'_*$, which express the momentum conservation law and the charges conservation law, respectively. Using the line element (2), the collision operator $\mathcal{L} = \mathcal{L}(f, g)$ is defined by (13) below.

Unlike in the usual case, we consider in this work the Boltzmann equation in the more general case with absorption term writing in the form:

$$p \cdot \nabla_x f + P \cdot \nabla_p f + Q \cdot \nabla_q f + \varrho(t, p, q) f = \mathcal{L}(f, f), \tag{1}$$

where $\varrho(t,p,q) \geq 0$ is the absorption rate of the medium at the instant t, for particles of momentum p and charge q. The given function ϱ is a physical characteristic of the material. $P = (P^\alpha)$ and $Q = (Q^I)$ are defined by (10) below. Equation (1) can be seen as the generalized Boltzmann equation presented by Arlotti et al. in [1], taking in this case $\varrho = \nabla_{q^I} Q^I$, $Q = (Q^I)$ models the evolution of the internal state of particles. In this work, the gravitational effects are generated by the homogeneous type I Bianchi space-time.

Some of energy estimates used in this paper where made by authors in [11], where they considered the above equation (1), but in the place of the absorption term, they consider the term "af^2" which is not physically well motivated. In addition, the method that we use here to obtain the local existence and uniqueness of solution of equation (1) is as follows: First, by the standard iteration method, we linearize the equation by fixing for $n \in \mathbf{N}$, f_n in its non-linear part (which is the collision operator), we then use the Faedo-Galerking method [3, 16] to construct f_{n+1} which is the solution of the linearized equation (see Theorem 1). Second, we show that, the sequence $(f_n)_{n\in\mathbf{N}}$ is bounded in the reflexive Hilbert space; thanks to Banach-Alaoglu theorem, we then extract a sub-sequence which converges weakly to the solution of the equation (1) (see Theorem 2).

The paper is organized as follows: In Section 2, we describe the Boltzmann equation with absorption term in Yang-Mills field. In Section 3, we introduce the function spaces and we give the energy estimates. In Section 4, we state and prove the existence and uniqueness theorem. In Section 5, we show the well-posedness of the solution.

In all what follows, Latin indices in lower case range from 1 to 3, Latin indices in upper case range either from 1 to N or from 1 to $N-1$.

2. Description of the Boltzmann Equation with Absorption Term in the Presence of a Yang-Mills Fields

2.1 The Boltzmann Equation with Absorption Term in Yang-Mills Fields

- In this section, unless otherwise specified, Greek indices range from 0 to 3. We use the Einstein summation convention i.e., $A_\alpha B^\alpha = \sum_\alpha A_\alpha B^\alpha$.
- We consider the collision evolution of a kind of fast moving massive and charged particles in the time-oriented Bianchi type 1 space-time with locally rotationally symmetric in the form:

$$g = -dt^2 + h^2(t)dx_1^2 + r^2(t)\left(dx_2^2 + dx_3^2\right), \tag{2}$$

where $h > 0$, $r > 0$ are two continuously differentiable functions of time t. We assume that $\frac{\dot{h}}{h}$ and $\frac{\dot{r}}{r}$ are bounded. Hence there exists a constant $C > 0$ such that: $|\frac{\dot{h}}{h}| \leq C$, $|\frac{\dot{r}}{r}| \leq C$. As a direct consequence, we have for $t \in \mathbf{R}^+$:

$$0 < h(t) \leq h_0 e^{ct}, \quad 0 < r(t) \leq r_0 e^{ct}, \quad \frac{1}{h(t)} \leq \frac{1}{h_0} e^{ct} \text{ and } \frac{1}{r(t)} \leq \frac{1}{r_0} e^{ct}, \tag{3}$$

where $h_0 = h(0)$, $r_0 = r(0)$.

• $(\mathcal{G},[\cdot])$ is a Lie algebra of a Lie group G, endowed with an Ad-invariant scalar product denoted by a dot "$\cdot$", which enjoys the following property:

$$u.[v,w]=[u,v].w, \quad \forall\, u,v,w\in\mathcal{G}, \tag{4}$$

where $[\,,\,]$ stands for the Lie brackets of the Lie algebra $\mathcal{G}$. We consider that $\mathcal{G}$ is a vector space on $\mathbf{R}$ with dimension $N\geq 2$ and (ε_I), $I=1,\dots,N$ an orthonormal basis of $\mathcal{G}$.

• The massive particles have the same rest mass $m>0$, normalized to the unity i.e., $m=1$. We denote by $T(\mathbf{R^4})$, the tangent bundle of $\mathbf{R^4}$ with coordinates (x^α,p^β), where $p=(p^\beta)=(p^0,\bar{p})$ stands for the momentum of each particule and $\bar{p}=(p^i)$, $i=1,2,3$. The charged particles move on the mass hyperboloid $\mathbf{P(R^4)}\subset T(\mathbf{R^4})$, whose equation is $\mathbf{P}_{t,x}(p)=g_{\alpha\beta}p^\alpha p^\beta=-1$ or equivalently, using the expression (2) of g, we have:

$$p^0=\left[1+h^2(t)(p^1)^2+r^2(t)\left((p^2)^2+(p^3)^2\right)\right]^{\frac{1}{2}}, \tag{5}$$

where the choice of $p^0>0$ symbolizes the fact that, naturally, the particles eject towards the future.

• Denote by A a Yang-Mills potential represented by a 1-form on $\mathbf{R^4}$ which takes its values in $\mathcal{G}$. Then the Yang-Mills potential is locally defined as follows:

$$A=A_\mu dx^\mu \ \text{ with } \ A_\mu=A^I_\mu\varepsilon_I, \quad I=1,2,\dots,N. \tag{6}$$

• Particles evolve in the space-time $(\mathbf{R^4},g)$, under the action of their own gravitational field represented by the metric tensor $g=(g_{\alpha\beta})$ given by (2) that informs about gravitational effects, and in addition, under the action of the non-Abelian forces generated by the Yang-Mills field $F=(F_{\alpha\beta})$, $(F_{\alpha\beta})$ a function from $\mathbf{R^4}$ to $\mathcal{G}$.

• The Yang-Mills field is the curvature of the Yang-Mills potential. It is represented by a $\mathcal{G}$-valued antisymmetric 2-form $F=(F^I_{\lambda\mu})$, linked to the potential $A=(A_\alpha)$ by:

$$F^I_{\lambda\mu}=\nabla_\lambda A^I_\mu-\nabla_\mu A^I_\lambda+C^I_{JH}A^J_\lambda A^H_\mu, \tag{7}$$

where C^I_{JH} are the structure constants of $\mathcal{G}$. We require that A satisfies the temporal gauge condition, which means that $A_0=0$.

• The non-Abelian charge of each particle is denoted q. We also suppose that q is an element of $\mathcal{G}$ whose given norm is $e>0$. To clarify this idea, q takes its values in an orbit of $\mathcal{G}$, which is a sphere ϑ whose equation is:

$$\vartheta: \quad q\cdot q=|q|^2=e^2, \tag{8}$$

where $|\cdot|$ stands for the norm deduced from the scalar product of $\mathcal{G}$. The relation (8) allows to express the component q^N of q as a function $\widetilde{q}=(q^I)$, $I=1,2,\dots,N-1$. We have:

$$q^N=\left[e^2-\sum_{I=1}^{N-1}(q^I)^2\right]^{\frac{1}{2}}. \tag{9}$$

• We denote by f the unknown distribution function which measures the probability density of the presence of particles in a given domain. f is a function defined on $T(\mathbf{R^4})\times\mathcal{G}$ and will be subject to the relativistic Boltzmann equation. Using relations (5), (9) and the fact that we are studying an homogeneous phenomenon, we obtain that the distribution function of Yang-Mills particules is definitely a function of independent variables $(t, p^i, q^I) = (t, \bar{p}, \widetilde{q})$. Then, $f = f(t, \bar{p}, \widetilde{q})$, $t \in \mathbf{R}$, $\bar{p} \in \mathbf{R^3}$, $\widetilde{q} \in \mathbf{R^{N-1}}$.

• The trajectories of particles with momentum $p = (p^\alpha) = (p^0, \bar{p})$ and charge $q = (\widetilde{q}, q^N)$ in a Yang-Mills field F, are no longer geodesics of space-time $(\mathbf{R^4}, g)$, but satisfy the following differential system:

$$\frac{dx^\alpha}{ds} = p^\alpha, \quad \frac{dp^\alpha}{ds} = P^\alpha = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + p^\beta q \cdot F^\alpha_\beta, \quad \frac{dq^I}{ds} = Q^I = -C^I_{JH} p^\alpha A^J_\alpha q^H, \tag{10}$$

where $\Gamma^\alpha_{\lambda\mu}$ are the Christoffel symbols of the Levi-Civita connection associated to g. The last equation in (10), called Wong's equation, expresses the fact that the covariant derivative of gauge of q along a trajectory is null. According to relations (5) and (9), the phase space of such Yang-Mills particles is in fact the subset $\mathbf{P}_{t,x} \times \vartheta$ of $T(\mathbf{R^4}) \times \mathcal{G}$.

• The Boltzmann equation with absorption term in f for the Yang-Mills charged particles in the Bianchi type 1 space-time can be written:

$$\frac{\partial f}{\partial t} + P^i \frac{\partial f}{\partial p^i} - Q^I \frac{\partial f}{\partial q^I} = \frac{1}{p^0} \mathcal{L}(f, f) - \frac{1}{p^0} \varrho(t, p, q) f, \tag{11}$$

where

$$P^i = \left(-2\Gamma^i_{i0} p^i - q.F^{i0} - \frac{p^j g^{ii} q.F_{ij}}{p^0} \right) \quad \text{and} \quad Q^I = \frac{p^i}{p^0} C^I_{JH} A^J_i q^H.$$

In (11), The left hand side is obtained from the Lie derivative of f with respect to the vectors field $Y = (p^\alpha, -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + p^\beta q \cdot F^\alpha_\beta, -C^I_{JH} p^\alpha A^J_\alpha q^H)$. $\mathcal{L}(f, f)$ represents the "collision operator" and $\varrho(t, p, q) \geq 0$ is the absorption rate of the medium at the instant t, for particles of momentum p and charges q. The function ϱ is a physical characteristic of the material and is given.

• From the conservation law of momenta and charges, we have:

$$p + p_* = p' + p'_*, \tag{12a}$$

$$q + q_* = q' + q'_*. \tag{12b}$$

Given two functions f and g on $\mathbf{P}_{t,x} \times \vartheta$, the "collision operator" is often formally written as the difference between the gain term $\mathcal{L}^+$ and the loss term $\mathcal{L}^-$ (see [5]):

$$\mathcal{L}(f, g)(t, \bar{p}, \widetilde{q}) = \mathcal{L}^+(t, \bar{p}, \widetilde{q}) - \mathcal{L}^-(t, \bar{p}, \widetilde{q}), \tag{13}$$

where for the Yang-Mills charges particles we consider, we have:

$$\mathcal{L}^+(f, g) = \int_{\mathbf{R^3} \times \vartheta} \frac{hr^2}{p^0_*} d\bar{p}_* w_{\widetilde{q}_*} \int_{S^2 \times S^{N-2}} f(\bar{p}', \widetilde{q}') g(\bar{p}'_*, \widetilde{q}'_*) \sigma \, dw \, d\theta,$$

$$\mathcal{L}^-(f, g) = \int_{\mathbf{R^3} \times \vartheta} \frac{hr^2}{p^0_*} d\bar{p}_* w_{\widetilde{q}_*} \int_{S^2 \times S^{N-2}} f(\bar{p}, \widetilde{q}) g(\bar{p}_*, \widetilde{q}_*) \sigma \, dw \, d\theta$$

with:

• S^2 is the unit sphere of $\mathbf{R^3}$, whose element is denoted dw,

• S^{N-2} is the unit sphere of $\mathbf{R^{N-1}}$, whose element is denoted θ, and $d\theta$ his volume element,

• $\sigma = \sigma(t, \bar{p}, \widetilde{q}, \bar{p}_*, \widetilde{q}_*, \bar{p}', \widetilde{q}', \bar{p}'_*, \widetilde{q}'_*)$ is a positive regular function called the collision kernel or the differential cross-section of the collisions which measures interactions effects between particles and determines their nature. We require as assumption on σ, that:

$$(H_1): \begin{cases} \exists\, C>0,\, 0\le\sigma\le C \\ (1+|\bar{p}|)^l \|\partial_{(\bar{p},\tilde{q})}\sigma\|_{L^1(\Omega\times\mathcal{S})}\in L^\infty(\Omega),\ 0\le|\beta|\le m+3,\, 0\le l\le m+3 \\ (1+|\bar{p}|)^{|\beta|-1}\partial^\beta_{(\bar{p},\tilde{q})}\sigma\in L^\infty(\Omega\times\Omega\times\mathcal{S}),\, 1\le|\beta|\le m+3, \end{cases}$$

where $\beta\in\mathbf{N^{N+2}}$, $\Omega=\mathbf{R^3}\times\mathbf{R^{N-1}}$ and $\mathcal{S}=S^2\times S^{N-2}$, $\partial^\beta_{(\bar{p},\tilde{q})}$ the partial derivative of order β with respect to $(\bar{p},\tilde{q})$. These assumptions are closed to the $\mu-N$ regularity introduced by Choquet and Bancel in [5, 6] and used in [3, 4, 11, 15–17].

2.2 Parametrization of the Post-collisional Momenta and Post-collisional Charges

Suppose that the pre-collisional momenta p and p_* and the pre-collisional charges q and q_* are given. Then:

(a) The conservation law of momenta (12a) splits into:

$$p^0+p^0_*=p'^0+p'^0_*, \tag{14a}$$

$$\bar{p}+\bar{p}_*=\bar{p}'+\bar{p}'_*. \tag{14b}$$

Eq. (14a) express the conservation of the quantity $\tilde{e}=\sqrt{1+|\bar{p}|^2_g}+\sqrt{1+|\bar{p}_*|^2_g}$, called the elementary energy of the unit rest mass of particles. We parametrize (14b) by setting, following Noutchegueme et al. in [18], $\bar{p}'=\bar{p}+d(\bar{p},\bar{p}_*,\omega)w$ and $\bar{p}'_*=\bar{p}_*-d(\bar{p},\bar{p}_*,\omega)w$ with $w\in S^2$ in which, d is a regular function given by:

$$d(\bar{p},\bar{p}_*,\omega)=\frac{2\tilde{e}p^0p^0_*[\omega\cdot(\hat{\bar{p}}_*-\hat{\bar{p}})]}{\tilde{e}^2-[\omega\cdot(\bar{p}+\bar{p}_*)]^2}, \tag{15}$$

where $\hat{\bar{p}}=\frac{\bar{p}}{p^0}$, $\hat{\bar{p}}_*=\frac{\bar{p}_*}{p^0_*}$. The scalar product in (15) which is defined by $(\bar{p}\cdot\bar{p}_*)=h^2p^1p^1_*+r^2(p^2p^2_*+p^3p^3_*)$ gives for $(\bar{p}=\bar{p}_*):(\bar{p}\cdot\bar{p}_*)\equiv|\bar{p}|^2_g=h^2(p^1)^2+r^2[(p^2)^2+(p^3)^2]$.

The Jacobian of the transformation $(\bar{p},\bar{p}_*)\to(\bar{p}',\bar{p}'_*)$ is given by

$$\frac{\partial(\bar{p}',\bar{p}'_*)}{\partial(\bar{p},\bar{p}_*)}=-\frac{p'^0p'^0_*}{p^0p^0_*}. \tag{16}$$

(b) The conservation law of charges (12b) can also be split into

$$q^N+q^N_*=q'^N+q'^N_*, \tag{17}$$

$$\tilde{q}+\tilde{q}_*=\tilde{q}'+\tilde{q}'_*. \tag{18}$$

We also parametrize (18) by setting, following the sketch given in [18] for the parametrization of the post-collisional momenta: $\tilde{q}'=\tilde{q}+\eta(\tilde{q},\tilde{q}_*,\theta_1)\theta_1$, $\tilde{q}'_*=\tilde{q}_*-\eta(\tilde{q},\tilde{q}_*,\theta_1)\theta_1$ where $\theta_1\in S^{N-2}$, and we obtain η given by:

$$\eta(\tilde{q},\tilde{q}_*,\theta_1)=\frac{2(q^N+q^N_*)[(\tilde{q}_*\cdot\theta_1)q^N-(\tilde{q}\cdot\theta_1)q^N_*]}{[(q^N+q^N_*)^2+\{(\tilde{q}_*+\tilde{q})\cdot\theta_1\}^2]}. \tag{19}$$

We also obtain the Jacobian of the transformation $(\tilde{q},\tilde{q}_*)\mapsto(\tilde{q}',\tilde{q}'_*)$ which is given by:

$$J=\frac{\partial(\tilde{q}',\tilde{q}'_*)}{\partial(\tilde{q},\tilde{q}_*)}=-\frac{q'^Nq'^N_*}{q^Nq^N_*}. \tag{20}$$

3. Function Spaces and Energy Estimates

In this section, Greek indices are multi-indices bilonging to $\mathbf{N}^{\mathbf{N+2}}$, latin indices are either integers or real numbers. Especially, δ will denote a strictly positive real number. We set $\Omega = \mathbf{R}^3 \times \mathbf{R}^{\mathbf{N}-1}$ and $\mathcal{S} = S^2 \times S^{N-2}$.

Definition 1. Let $T > 0$, $l \in \mathbf{N}$ and $d \in \mathbf{R}^+$.

1. $\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(\Omega) = \{u : \Omega \to \mathbf{R}, (1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}u \in L^2(\Omega),\ \beta \in \mathbf{N}^{\mathbf{N+2}},\ |\beta| \le l\}$, endowed with the norm
$$\|u\|_{\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(\Omega)} = \sum_{0\le|\beta|\le l} \left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}u\right\|_{L^2(\Omega)}.$$
2. $\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega) = \{u : [0,T]\times\Omega \to \mathbf{R}, u$ continuous, $u(t,\cdot) \in \mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(\Omega),\ \forall\ t \in [0,T]\}$, endowed with the norm
$$\| u \|_{\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)} = \sup_{0\le t\le T}\sum_{0\le|\beta|\le l} \left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}u(t,\cdot)\right\|_{L^2(\Omega)}.$$
3. $\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(\Omega) = \{u :\ \Omega \to \mathbf{R},\ (1+|\bar{p}|)^{d+|\beta|}D^{\beta}_{(\bar{p},\widetilde{q})}u \in L^2(\Omega),\ \beta \in \mathbf{N}^{\mathbf{N+2}},\ |\beta| \le l\}$, endowed with the norm
$$\|u\|_{\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(\Omega)} = \sum_{0\le|\beta|\le l} \left\|(1+|\bar{p}|)^{d+|\beta|}D^{\beta}_{(\bar{p},\widetilde{q})}u\right\|_{L^2(\Omega)}.$$
4. $\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega) = \{u : [0,T]\times\Omega \to \mathbf{R},\ u$ continuous, $u(t,\cdot) \in \mathbf{E}^{\mathbf{l}}{}_d(\Omega),\ \forall\ t \in [0,T]\}$, endowed with the norm
$$\| u \|_{\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)} = \sup_{0\le t\le T}\sum_{0\le|\beta|\le l} \left\|(1+|\bar{p}|)^{d+|\beta|}D^{\beta}_{(\bar{p},\widetilde{q})}u(t,\cdot)\right\|_{L^2(\Omega)}.$$
For $\delta > 0$, we also define
5. $\mathbf{K}^{\mathbf{l}}_{\mathbf{d},\delta}(0,T;\Omega) = \{u \in \mathbf{K}^{\mathbf{l}}{}_d(0,T;\Omega),\ \|u\|_{\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)} \le \delta\}$
6. $\mathbf{E}^{\mathbf{l}}_{\mathbf{d},\delta}(0,T;\Omega) = \{u \in \mathbf{E}^{\mathbf{l}}{}_d(0,T;\Omega),\ \|u\|_{\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)} \le \delta\}$.

$D^{\beta}_{(\bar{p},\widetilde{q})}$ denotes the derivative in the sens of distributions.

Lemma 1 ([10, 16])**.** *Let $T > 0$, $l \in \mathbf{N}$ and $d \in \mathbf{R}$;*

(i) $\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)$ *is a Banach space.*

(2i) $\mathbf{K}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)$ *is dense in* $\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T,\Omega)$ *for the norm* $\|.\|_{\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T,\Omega)}$.

(3i) $\mathbf{E}^{\mathbf{l}}{}_d(0,T;\Omega)$ *is a separable Hilbert space.*

(4i) *The space* $\mathbf{E}^{\mathbf{l}}_{\mathbf{d},\delta}(0,T;\Omega)$ *is a complete metric subspace of* $\mathbf{E}^{\mathbf{l}}_{\mathbf{d}}(0,T;\Omega)$.

Remark 1. (i) Since we are searching classical solution to the equation (1), the unknown function $f = f(t) = f(t,\bar{p},\widetilde{q})$ must be continuously differentiable and has to belong to the space $C^1_b(\mathbf{R}^{\mathbf{n}})$, with $n = N+4$. We have from Sobolev injection theorem
$$W^l_2 \hookrightarrow C^1_b(\mathbf{R}^{\mathbf{n}}) \text{ if } l > 1 + \frac{N+2}{2} = \frac{N+4}{2}.$$
Then, the smallest interger which satisfied this is $l = E(\frac{N+4}{2}) + 1$.

(2i) If $N=2m$ or $N=2m+1$ then, $l=m+3$. From now and on, we will use $l=m+3$.

(3i) For more details on the weighted Sobolec spaces, see [10].

Proposition 1. *Let* $d\in]\frac{N+4}{2},+\infty[$ *and* $f,\ g\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$. *Then, we have:*

$$\frac{1}{p^0}\mathcal{L}(f,g)\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega).$$

There exists $C=C(T)>0$ *such that:*

(i) $\left\|\frac{1}{p^0}\mathcal{L}(f,g)\right\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\le C\,\|f\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\,\|g\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}$

(2i) $\left\|\frac{1}{p^0}\mathcal{L}(f,f)-\frac{1}{p^0}\mathcal{L}(g,g)\right\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\le 2C\left(\|f\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}+\|g\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\right)\|f-g\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}.$

Proof. We recall that $\mathbf{K}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ is dense in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$. Let $f,\ g\in\mathbf{K}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ and $\beta\in\mathbf{N}^{\mathbf{m+3}}$, $0\le|\beta|\le m+3$, we have:

$$\frac{1}{p^0}\mathcal{L}(f,g)=\frac{1}{p^0}\mathcal{L}^+(f,g)-\frac{1}{p^0}\mathcal{L}^-(f,g).$$

We conclude using Theorem 16 in [11], that $\frac{1}{p^0}\mathcal{L}(f,g)\in\mathbf{K}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$, and there exists $C=C(T)>0$ such that $\left\|\frac{1}{p^0}\mathcal{L}(f,g)\right\|_{\mathbf{K}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\le C\,\|f\|_{\mathbf{K}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\,\|g\|_{\mathbf{K}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}$.

For (2i), we use the bilinearity of the collision operator and Lemma 17 in [11]. □

4. Existence and Uniqueness Theorem

Let $f_0\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$ be given. We will prove that, the Cauchy problem for the Boltzmann equation with absorption term that we write in the form :

$$\frac{\partial f}{\partial t}+P^i\frac{\partial f}{\partial p^i}+Q^I\frac{\partial f}{\partial q^I}+\frac{1}{p^0}\varrho f=\frac{1}{p^0}\mathcal{L}(f,f),\tag{21}$$

where $P^i=-\left(2\Gamma^i_{i0}p^i+q\cdot F^{i0}+\frac{p^j q\cdot F_{ij}}{p^0}\right)$ and $Q^I=-\frac{p^i}{p^0}C^I_{bc}A^b_i q^c$; has in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ a unique and bounded solution f such that $f(0,\bar{p},\tilde{q})=f_0$.

Since we are studying a homogeneous phenomenon, we will suppose that the Yang-Mills potential A and the Yang-Mills field F are two given continuous and bounded functions of time. So there exists two positive constants C_A and C_F such as:

$$|A|=\max_{1\le i\le 3,1\le a\le N}|A^a_i(t)|<C_A \quad\text{and}\quad |F|=\max_{1\le i,j\le 3,1\le a\le N}\left(|F^{i0}_a(t)|,|F^a_{ij}(t)|\right)<C_F.$$

We also assume that, the absorption rate ϱ is bounded and all its derivatives

$(1+|\bar{p}|)^{|\beta|-1}\partial^\beta_{(\bar{p},\tilde{q})}\varrho$ are bounded, for all $\beta\in\mathbf{N}^{\mathbf{N+2}}$ with $1\le|\beta|\le m+3$.

In all what follows, $C=C(h_0,r_0,T,C_F,C_A,e)$ is a position constant whose value may change from line to line.

Lemma 2. (i) *Let* $t\in[0,T[$. *The application* $t\to\bar{p}(t)$ *is uniformly bounded.*

(2i) P^i *and* Q^I *are bounded.*

(3i) $(1+|\bar{p}|)^{|\beta|-1}\partial^\beta_{(\bar{p},\tilde{q})}P^i$ *is bounded, for all* $\beta\in\mathbf{N}^{\mathbf{N+2}}$ *with* $1\le|\beta|\le m+3$, (4i) $(1+|\bar{p}|)^{|\beta|-1}\partial^{\beta}_{(\bar{p},\tilde{q})}Q^I$ *is bounded, for all* $\beta\in\mathbf{N}^{\mathbf{N+2}}$ *with* $1\le|\beta|\le m+3$.

Proof. (1) For (i), see [2].

(2) (ii) is immediate using (i) and the fact that the Yang-Mills potential A and the Yang-Mills field F are bounded by hypothesis.

(3) For all $\beta\in\mathbf{N}^{\mathbf{N+2}}$ such that $1\le|\beta|\le m+3$, we have:

$$\partial^{\beta}_{(\bar{p},\tilde{q})}P^i=-2\Gamma^i_{i0}\partial^{\beta}_{(\bar{p},\tilde{q})}(p^i)-\partial^{\beta}_{(\bar{p},\tilde{q})}(q\cdot F^{i0})-\partial^{\beta}_{(\bar{p},\tilde{q})}\left(\frac{p^j q\cdot F_{ij}}{p^0}\right).$$

Knowing that the application $t\to\bar{p}(t)$ is uniformly bounded on $[0,T[$, immediately

$$(1+|\bar{p}|)^{|\beta|-1}|2\Gamma^i_{i0}\partial^{\beta}_{(\bar{p},\tilde{q})}(p^i)+\partial^{\beta}_{(\bar{p},\tilde{q})}(q\cdot F^{i0})|\le C. \tag{1B}$$

Furthermore, the Leibniz formula applied twice successively gives

$$\begin{aligned}\partial^{\beta}_{(\bar{p},\tilde{q})}\left(\frac{p^j g^{ii} q.F_{ij}}{p^0}\right)&=\sum_{k\le\beta}C^k_{\beta}\partial^k_{(\bar{p},\tilde{q})}\left(\frac{1}{p^0}\right)\sum_{\lambda\le\beta-k}C^{\lambda}_{\beta-k}\partial^{\lambda}_{(\bar{p},\tilde{q})}p^j\partial^{\beta-k-\lambda}_{(\bar{p},\tilde{q})}\left(q.F_{ij}\right)\\&=\sum_{k\le\beta}C^k_{\beta}(1+|\bar{p}|)^{|k|}\partial^k_{(\bar{p},\tilde{q})}\left(\frac{1}{p^0}\right)\sum_{\lambda\le\beta-k}C^{\lambda}_{\beta-k}\partial^{\lambda}_{(\bar{p},\tilde{q})}p^j\partial^{\beta-k-\lambda}_{(\bar{p},\tilde{q})}\left(q.F_{ij}\right)\frac{1}{(1+|\bar{p}|)^{|k|}}.\end{aligned}$$

Invoking Lemma 13 in [11] and the fact that $\frac{1}{(1+|\bar{p}|)^{|k|}}<1$, we get:

$$\left|(1+|\bar{p}|)^{|\beta|-1}\partial^{\beta}_{(\bar{p},\tilde{q})}\left(\frac{p^j g^{ii} q.F_{ij}}{p^0}\right)\right|\le C(1+|\bar{p}|)^{|\beta|-1}\sum_{k\le\beta}C^k_{\beta}\frac{1}{p^0}\sum_{\lambda\le\beta-k}C^{\lambda}_{\beta-k}|\partial^{\lambda}_{(\bar{p},\tilde{q})}p^j||\partial^{\beta-k-\lambda}_{(\bar{p},\tilde{q})}(q\cdot F_{ij})|,$$

we conclude that:

$$\left|(1+|\bar{p}|)^{|\beta|-1}\partial^{\beta}_{(\bar{p},\tilde{q})}\left(\frac{p^j g^{ii} q.F_{ij}}{p^0}\right)\right|\le C. \tag{2B}$$

(1B) and (2B) complete the proof of (3i). The proof for (4i) is similar to (3i). □

We define recursively the following sequence $(f_n)_{n\in\mathbf{N}}$ by:

$$\frac{\partial f_{n+1}}{\partial t}+P^i\frac{\partial f_{n+1}}{\partial p^i}+Q^I\frac{\partial f_{n+1}}{\partial q^I}+\frac{1}{p^0}\varrho f_{n+1}=\frac{1}{p^0}\mathcal{L}(f_n,f_n), \tag{22}$$

where $f_{n+1}(0,\bar{p},\tilde{q})=f_0$.

We note that, for a given f_n, (22) is a linear partial differential equation with f_{n+1} as unknown and the initial data f_0. Let $f_n\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$ be given, we will prove that the linearized Boltzmann equation (22) has in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ a unique bounded solution. To proceed, we use the Faedo-Galerkin scheme, which consists in finding the approximate solutions of the problem, estimating them uniformly and passing to the limit in a suitable weak sense, to get the expected solution (see [14]).

Construction of the sequence. Let $(v_k)_{k\in\mathbf{N}^*}$ be an Hilbertian basis of $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ (since it is a separable Hilbert space). We follow the method used in [3] for the construction of the sequence $(f^M_{n+1})_M$ which will converge to the solution f_{n+1} of (22). We then write

$$f^M_{n+1}=\sum_{k=1}^{M}\lambda_k v_k,\quad M\in\mathbf{N}^*, \tag{23}$$

where the coefficients λ_k are derivables functions of t and are given as solutions of the M ordinary differential equations of the system:

$$\left(\partial_t f_{n+1}^M/v_j\right)+\left(P^i\partial_{p^i}f_{n+1}^M/v_j\right)+\left(Q^I\partial_{q^I}f_{n+1}^M/v_j\right)+\left(\frac{1}{p^0}\varrho f_{n+1}^M/v_j\right)=\left(\frac{1}{p^0}\mathcal{L}(f_n,f_n)/v_j\right) \tag{24}$$

and where $(./\cdot)$ stands for the scalar product in $\mathbf{E_d^{m+3}}(\Omega)$. The initial data are:

$$\lambda_j(0)=(f_0/v_j), \tag{25}$$

Thus we obtain that:

$$\partial_t f_{n+1}^M+P^i\partial_{p^i}f_{n+1}^M+Q^I\partial_{q^I}f_{n+1}^M+\frac{1}{p^0}\varrho f_{n+1}^M=\frac{1}{p^0}\mathcal{L}(f_n,f_n). \tag{26}$$

So f_{n+1}^M is solution of the linearized Boltzmann equation (22), with the initial data:

$$f_{n+1}^M(0)=\sum_{k=1}^{M}\lambda_k(0)v_k=\sum_{k=1}^{M}(f_0/v_k)v_k.$$

We present in the following propositions, a list of energy estimates useful for the boundedness of the sequence $(f_{n+1}^M)_M$.

Proposition 2 (see [11]). *Let $d\in]\frac{N+4}{2},\infty[$, $f_{n+1}^M\in\mathbf{E_d^{m+3}}(\Omega)$, $\alpha,\beta\in\mathbf{N^{N+2}}$ such that $|\alpha|\le|\beta|\le m+3$. Then*

$$\left|\left((1+|\bar p|)^{d+|\beta|}\partial^{\beta}_{(\bar p,\tilde q)}\left[P^i\partial_{p^i}f_{n+1}^M\right]/(1+|\bar p|)^{d+|\beta|}\partial_{(\bar p,\tilde q)}f_{n+1}^M\right)_{L^2(\Omega)}\right|$$
$$\le C\left(\sum_{|\alpha|\le|\beta|}\left\|(1+|\bar p|)^{d+|\alpha|}\partial^{\alpha}_{(\bar p,\tilde q)}f_{n+1}^M\right\|_{L^2}\right)\left\|(1+|\bar p|)^{d+|\beta|}\partial^{\beta}_{(\bar p,\tilde q)}f_{n+1}^M\right\|_{L^2(\Omega)},$$

where (/) stands for the scalar product in $L^2(\Omega)$.

The proof of this proposition is given by Lemmas 3, 4 and 5.

Lemma 3. *Let $d\in]\frac{N+4}{2},\infty[$ and $f_{n+1}^M\in\mathbf{E}_d^{m+3}(\Omega)$, then :*

$$\left|\left((1+|\bar p|)^d\left[P^i\frac{\partial f_{n+1}^M}{\partial p^i}\right]/(1+|\bar p|)^d f_{n+1}^M\right)_{L^2(\Omega)}\right|\le C\left\|(1+|\bar p|)^d f_{n+1}^M\right\|^2_{L^2(\Omega)}.$$

Proof. We have $\partial_{p^i}[(1+|\bar p|)^d f_{n+1}^M]=\partial_{p^i}[(1+|\bar p|)^d]f_{n+1}^M+(1+|\bar p|)^d\partial_{p^i}f_{n+1}^M$, from where

$$\left|\left((1+|\bar p|)^d P^i\left[\frac{\partial f_{n+1}^M}{\partial p^i}\right]/(1+|\bar p|)^d f_{n+1}^M\right)_{L^2}\right|\le|B_1|+|B_2|, \tag{27}$$

with
$$\begin{cases}B_1=\left(P^i\partial_{p^i}[(1+|\bar p|)^d f_{n+1}^M]/(1+|\bar p|)^d f_{n+1}^M\right)_{L^2(\Omega)}\\ B_2=\left(P^i\partial_{p^i}[(1+|\bar p|)^d]f_{n+1}^M/(1+|\bar p|)^d f_{n+1}^M\right)_{L^2(\Omega)}\end{cases}$$

For the term B_1, we have:

$$B_1=\left(\partial_{p^i}[(1+|\bar p|)^d f_{n+1}^M]/P^i(1+|\bar p|)^d f_{n+1}^M\right)_{L^2(\Omega)}$$
$$=-\left((1+|\bar p|)^d f_{n+1}^M/\partial_{p^i}(P^i)(1+|\bar p|)^d f_{n+1}^M+P^i\partial_{p^i}[(1+|\bar p|)^d f_{n+1}^M]\right)_{L^2(\Omega)}.$$

Using the symmetry and bilinearity of the scalar product to the second member of the previous equality, one has :

$$B_1 = -\frac{1}{2}\left(\partial_{p^i} P^i [(1+|\bar{p}|)^d f^M_{n+1}] / (1+|\bar{p}|)^d f^M_{n+1}\right)_{L^2(\Omega)}.$$

According to Lemma 2, $\partial_{p^i}(P^i)$ is bounded. Hence,

$$|B_1| \leq \frac{1}{2}\left|\partial_{p^i} P^i\right| \parallel (1+|\bar{p}|)^d f^M_{n+1} \parallel^2_{L^2(\Omega)} \leq C \left\|(1+|\bar{p}|)^d f^M_{n+1}\right\|^2_{L^2(\Omega)}. \tag{28}$$

For the last term B_2 in (27), we have $\partial_{p^i}[(1+|\bar{p}|)^d] = \frac{p^i d(1+|\bar{p}|)^d}{|\bar{p}|(1+|\bar{p}|)}$, which implies that:

$$|B_2| \leq \left|\frac{P^i p^i d(1+|\bar{p}|)^d}{|\bar{p}|(1+|\bar{p}|)}\right| \left\|(1+|\bar{p}|)^d f^M_{n+1}\right\|^2_{L^2(\Omega)} \leq C \left\|(1+|\bar{p}|)^d f^M_{n+1}\right\|^2_{L^2(\Omega)} \tag{29}$$

Eqs. (27), (28) and (29) complete the proof of Lemma 3. □

Lemma 4. *Let* $d \in]\frac{N+4}{2}, \infty[$ *and* $f^M_{n+1} \in \mathbf{E}^{m+3}_d(\Omega)$. *Then :*

$$\left|\left((1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})}\left[P^i \frac{\partial f^M_{n+1}}{\partial p^i}\right] / (1+|\bar{p}|)^d \partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right)_{L^2(\Omega)}\right|$$
$$\leq C\left\{\sum_{i=1}^{3}\left\|(1+|\bar{p}|)^{d+1}\partial_{p^i} f^M_{n+1}\right\|_{L^2(\Omega)}\right\} \times \left\|(1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right\|_{L^2(\Omega)}.$$

Proof. We have $\partial_{(\bar{q},\tilde{q})}[P^i \partial_{p^i} f^M_{n+1}] = \partial_{(\bar{q},\tilde{q})} P^i \partial_{p^i} f^M_{n+1} + P^i \partial_{(\bar{q},\tilde{q})}\left(\partial_{p^i} f^M_{n+1}\right)$, from where

$$\left|\left((1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})}\left[P^i \frac{\partial f^M_{n+1}}{\partial p^i}\right] / (1+|\bar{p}|)^d \partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right)_{L^2(\Omega)}\right| \leq |B_3| + |B_4|, \tag{30}$$

with $\begin{cases} B_3 = \left((1+|\bar{p}|)^{d+1} P^i \partial_{(\bar{q},\tilde{q})}\left(\partial_{p^i} f^M_{n+1}\right) / \partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right)_{L^2(\Omega)} \\ B_4 = \left((1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})}[P^i]\partial_{p^i} f^M_{n+1} / (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right)_{L^2(\Omega)} \end{cases}$

For the term B_4, since $\partial_{(\bar{q},\tilde{q})} P^i$ is bounded (Lemma 2), we get:

$$|B_4| \leq C \sum_{i=1}^{3} \parallel (1+|\bar{p}|)^{d+1}\partial_{p^i} f^M_{n+1} \parallel_{L^2} \times \parallel (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1} \parallel^2_{L^2(\Omega)}. \tag{31}$$

Now, for the term B_3, we have

$$\partial_{p^i}[(1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1}] = (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})}\left(\partial_{p^i} f^M_{n+1}\right) + \partial_{p^i}[(1+|\bar{p}|)^{d+1}]\partial_{(\bar{q},\tilde{q})} f^M_{n+1}.$$

Thus, $|B_3| \leq |B'_3| + |B''_3|$, with $B'_3 = \left(P^i \partial_{p^i}[(1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1}] / (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right)_{L^2(\Omega)}$ and $B''_3 = \left(P^i \partial_{p^i}[(1+|\bar{p}|)^{d+1}]\partial_{(\bar{q},\tilde{q})} f^M_{n+1} / (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1}\right)_{L^2(\Omega)}$.

Where, we have on the one hand

$$|B'_3| \leq C \parallel (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1} \parallel^2_{L^2(\Omega)}, \tag{32}$$

and on the other hand $\partial_{p^i}[(1+|\bar{p}|)^{d+1}] = \frac{(d+1)p^i(1+|\bar{p}|)^{d+1}}{|\bar{p}|(1+|\bar{p}|)}$, which implies that:

$$|B''_3| \leq \left|\frac{(d+1)P^i p^i (1+|\bar{p}|)^{d+1}}{|\bar{p}|(1+|\bar{p}|)}\right| \parallel (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1} \parallel^2_{L^2(\Omega)}$$
$$\leq C \parallel (1+|\bar{p}|)^{d+1}\partial_{(\bar{q},\tilde{q})} f^M_{n+1} \parallel^2_{L^2(\Omega)}. \tag{33}$$

The inequality (31), (32) and (33) end the proof of lemma 4. □

Lemma 5. *Let* $d \in]\frac{N+4}{2}, \infty[$, $f^M_{n+1} \in \mathbf{E}^{m+3}_d(\Omega)$ *and* $\beta \in \mathbf{N}^{N+2}$, $|\beta| \le m+2$ *if*

$$\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}\left[P^i\frac{\partial f^M_{n+1}}{\partial p^i}\right] / (1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)} \le C\left(\sum_{|\alpha|\le|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^{\alpha}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2}\right)\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}$$

then, $\forall\ \beta' \in \mathbf{N}^{N+2}$, $|\beta'| = 1$, *we have:*

$$\left((1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}\left[P^i\frac{\partial f^M_{n+1}}{\partial p^i}\right] / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)} \le C\left(\sum_{|\alpha|\le|\beta+\beta'|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^{\alpha}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2}\right)\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}$$

Proof. We set $\lambda = \beta+\beta'$. According to the Leibniz formula, we have:

$$\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}\left[P^i\frac{\partial f^M_{n+1}}{\partial p^i}\right] = \sum_{k\le\lambda} C^k_\lambda \partial^k_{(\bar{p},\tilde{q})}P^i\partial^{\lambda-k}_{(\bar{p},\tilde{q})}\left(\frac{\partial f^M_{n+1}}{\partial p^i}\right),$$

which implies

$$\left|\left((1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}\left[P^i\frac{\partial f^M_{n+1}}{\partial p^i}\right] / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)}\right| \le |K_1|+|K_2|, \tag{34}$$

where
$$\begin{cases} K_1 = \left((1+|\bar{p}|)^{d+|\beta|+1}P^i\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}\left(\frac{\partial f^M_{n+1}}{\partial p^i}\right) / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)} \\ K_2 = \left((1+|\bar{p}|)^{d+|\beta|+1}\sum_{k\le\lambda,\ |k|\ge1}C^k_\lambda\partial^k_{(\bar{p},\tilde{q})}P^i\partial^{\lambda-k}_{(\bar{p},\tilde{q})}\left(\frac{\partial f^M_{n+1}}{\partial p^i}\right) / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)} \end{cases}.$$

Estimation of K_2. According to Lemma 2, $(1+|\bar{p}|)^{|k|-1}\partial^k_{(\bar{p},\tilde{q})}\widetilde{P^i}$ is bounded. Thus,

$$\begin{aligned} |K_2| &\le C\sum_{k\le\lambda,\ |k|\ge1}C^k_\lambda\left|\left((1+|\bar{p}|)^{d+|\beta|-|k|+2}\partial^{\lambda-k}_{(\bar{p},\tilde{q})}\left(\frac{\partial f^M_{n+1}}{\partial p^i}\right) / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)}\right| \\ &= C\sum_{k\le\lambda,\ |k|\ge1}C^k_\lambda\left|\left((1+|\bar{p}|)^{d+|\beta|-|k|+2}\partial^{\lambda+\gamma-k}_{(\bar{p},\tilde{q})}\left(f^M_{n+1}\right) / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)}\right|, \end{aligned}$$

with $\gamma \in \mathbf{N}^3$ such that $|\gamma| = 1$. Hence,

$$|K_2| \le C\sum_{k\le\lambda,|k|\ge1}\left\|(1+|\bar{p}|)^{d+|\beta|-|k|+2}\partial^{\lambda+\gamma-k}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)} \tag{35}$$

Estimation of K_1. Deriving the function $(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M$, with respect to p^i, we find:

$$|K_1| \le |K'_1|+|K''_1|, \tag{36}$$

where
$$\begin{cases} K'_1 = \left(P^i\partial_{p^i}\left[(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right] / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)} \\ K''_1 = \left(P^i\partial_{p^i}\left[(1+|\bar{p}|)^{d+|\beta|+1}\right]\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1} / (1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)} \end{cases}.$$

Since P^i is bounded, one has:

$$K_1' \leq C \left\| (1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|^2_{L^2(\Omega)}. \tag{37}$$

furthermore, $\partial_{p^i}\left[(1+|\bar{p}|)^{d+|\beta|+1}\right] = \frac{(d+|\beta|+1)p^i(1+|\bar{p}|)^{d+|\beta|+1}}{|\bar{p}|(1+|\bar{p}|)}$, implies that:

$$K_1'' \leq C \left\| (1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|^2_{L^2(\Omega)}. \tag{38}$$

With (36), (37) and (38), we get:

$$K_1 \leq C \left\| (1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|^2_{L^2(\Omega)}. \tag{39}$$

The inequalities (34), (35) and (39) completes the proof of Lemma 5. □

Proposition 3. *Let $d \in]\frac{N+4}{2}, +\infty[$, $f^M_{n+1} \in \mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$, $\alpha,\beta \in \mathbf{N}^{\mathbf{N+2}}$ such that $|\alpha| \leq |\beta| \leq m+3$. Then*

$$\left| \left((1+|\bar{p}|)^{d+|\beta|} \partial^{\beta}_{(\bar{p},\tilde{q})} \left[Q^I \partial_{q^I} f^M_{n+1} \right] / (1+|\bar{p}|)^{d+|\beta|} \partial^{\beta}_{(\bar{p},\tilde{q})} f^M_{n+1} \right)_{L^2(\Omega)} \right|$$
$$\leq C \left(\sum_{|\alpha|\leq|\beta|} \left\| (1+|\bar{p}|)^{d+|\alpha|} \partial^{\alpha}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|_{L^2(\Omega)} \right) \left\| (1+|\bar{p}|)^{d+|\beta|} \partial^{\beta}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|_{L^2(\Omega)},$$

where (/) stands for the scalar product on $L^2(\Omega)$.

Proof. The proof of this proposition is similar to the one of Proposition 2. □

Proposition 4. *Let $d \in]\frac{N+4}{2}, +\infty[$, $f^M_{n+1} \in \mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$, $\alpha,\beta \in \mathbf{N}^{\mathbf{N+2}}$ such that $|\alpha| \leq |\beta| \leq m+3$. Then*

$$\left| \left((1+|\bar{p}|)^{d+|\beta|} \partial^{\beta}_{(\bar{p},\tilde{q})} \left[\frac{1}{p^0} \varrho f^M_{n+1} \right] / (1+|\bar{p}|)^{d+|\beta|} \partial^{\beta}_{(\bar{p},\tilde{q})} f^M_{n+1} \right)_{L^2(\Omega)} \right|$$
$$\leq C \left(\sum_{|\alpha|\leq|\beta|} \left\| (1+|\bar{p}|)^{d+|\alpha|} \partial^{\alpha}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|_{L^2(\Omega)} \right) \left\| (1+|\bar{p}|)^{d+|\beta|} \partial^{\beta}_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|_{L^2(\Omega)},$$

where (/) stands for the scalar product on $L^2(\Omega)$.

Proof. For $|\beta| = 0$, the result is obvious, using the fact that $\frac{1}{p^0}$ and ϱ are bounded.

For $|\beta| = 1$, let $i_1 = 1,2,3$ and $a_1 = 1,\dots,N-1$.

We have $\partial_{p^{i_1}}\left(\frac{\varrho}{p^0} f^M_{n+1}\right) = \partial_{p^{i_1}}\left(\frac{\varrho}{p^0}\right) f^M_{n+1} + \frac{\varrho}{p^0}\partial_{p^{i_1}}\left(f^M_{n+1}\right)$ and $\partial_{q^{a_1}}\left(\frac{\varrho}{p^0} f^M_{n+1}\right) = \frac{1}{p^0}\partial_{q^{a_1}}(\varrho) f^M_{n+1} + \frac{\varrho}{p^0}\partial_{q^{a_1}}\left(f^M_{n+1}\right)$. Since $\partial_{p^{i_1}}\left(\frac{\varrho}{p^0}\right)$, $\frac{\varrho}{p^0}$ and $\frac{1}{p^0}\partial_{q^{a_1}}(\varrho)$ are bounded, we obtain:

$$\left| \left((1+|\bar{p}|)^{d+1} \partial_{(\bar{p},\tilde{q})} \left[\frac{\varrho}{p^0} f^M_{n+1} \right] / (1+|\bar{p}|)^{d} \partial_{(\bar{p},\tilde{q})} f^M_{n+1} \right)_{L^2(\Omega)} \right| \tag{40}$$
$$\leq C \left\{ \sum_{|\alpha|\leq 1} \left\| (1+|\bar{p}|)^{d+1} \partial_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|_{L^2(\Omega)} \right\} \times \left\| (1+|\bar{p}|)^{d+1} \partial_{(\bar{p},\tilde{q})} f^M_{n+1} \right\|_{L^2(\Omega)}$$

Now, we suppose that $\forall\ \alpha,\beta \in \mathbf{N}^{\mathbf{N+2}}$, $|\alpha| \leq |\beta| \leq m+2$, the Proposition 4 is true. Let $\beta' \in \mathbf{N}^{\mathbf{N+2}}$, $|\beta'| = 1$. We set $\lambda = \beta + \beta'$, proceeding like in the proof of Lemma 23 in [11], we write:

$$\left| \left((1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'}_{(\bar{p},\tilde{q})} \left[\frac{\varrho}{p^0} f^M_{n+1} \right] / (1+|\bar{p}|)^{d+|\beta|+1} \partial^{\beta+\beta'}_{(\bar{p},\tilde{q})} f^M_{n+1} \right)_{L^2(\Omega)} \right| \leq |K_3| + |K_4|, \tag{41}$$

where $\begin{cases} K_3 = \left((1+|\bar{p}|)^{d+|\beta|+1}\frac{\varrho}{p^0}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)/(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right)_{L^2(\Omega)} \\ K_4 = \left((1+|\bar{p}|)^{d+|\beta|+1}\sum_{k\le\lambda,\ |k|\ge1}C^k_\lambda\partial^k_{(\bar{p},\widetilde{q})}\frac{\varrho}{p^0}\partial^{\lambda-k}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)/(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right)_{L^2(\Omega)}. \end{cases}$

Since $\frac{\varrho}{p^0}$ and $(1+|\bar{p}|)^{|k|}\partial^k_{(\bar{p},\widetilde{q})}\left(\frac{\varrho}{p^0}\right)$ are bounded, we obtain:

$$|K_3| \le C\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right\|^2_{L^2(\Omega)} \tag{42}$$

$$|K_4| \le C\sum_{k\le\lambda,|k|\ge1}\left\|(1+|\bar{p}|)^{d+|\beta|-|k|+1}\partial^{\lambda-k}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right\|_{L^2(\Omega)}\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right\|_{L^2(\Omega)} \tag{43}$$

The inequalities (42), (43) and (41) imply that

$$\left((1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left[\varrho\frac{\partial f^M_{n+1}}{\partial p^i}\right]/(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right)_{L^2(\Omega)}$$
$$\le C\left(\sum_{|\alpha|\le|\beta+\beta'|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^{\alpha}_{(\bar{p},\widetilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}\right)\left\|(1+|\bar{p}|)^{d+|\beta|+1}\partial^{\beta+\beta'}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right\|_{L^2(\Omega)}.$$

This complete the proof of Proposition 4. □

In the next proposition, we prove that the sequence $(f^M_{n+1})_{M\in\mathbf{N}^*}$ is bounded in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$.

Proposition 5. *Let* $d\in]\frac{N+4}{2},+\infty[$, $f_n\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$, $\beta\in\mathbf{N}^{\mathbf{N+2}}$ *such that* $|\beta|\le m+3$ *and* $T>0$. *We have:*

$$\left\|f^M_{n+1}\right\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)} \le C, \quad \forall M\in\mathbf{N}^*.$$

Proof. Since $f_n\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$, it results from Proposition 1 that $\frac{1}{p^0}\mathcal{L}(\widetilde{f_n},\widetilde{f_n})\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ and so

$$(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}\left(\frac{1}{p^0}\mathcal{L}(\widetilde{f_n},\widetilde{f_n})\right)\in L^2(\Omega),\quad \forall\beta\in\mathbf{N}^{\mathbf{m+3}} \text{ such that } |\beta|\le m+3.$$

Since $f^M_{n+1}\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$, we also have

$$(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}f^M_{n+1}\in L^2(\Omega),\quad \forall\beta\in\mathbf{N}^{\mathbf{m+3}} \text{ such that } |\beta|\le m+3.$$

Consequently, we can consider the scalar product in $L^2(\Omega)$ and using the bilinearity of the scalar product, which yields :

$$\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}\left[\partial_t f^M_{n+1}\right]/H\right)_{L^2(\Omega)}+\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}\left[P^i\partial_{p^i}f^M_{n+1}\right]/H\right)_{L^2(\Omega)}$$
$$+\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}\left[Q^I\partial_{q^I}f^M_{n+1}\right]/H\right)_{L^2(\Omega)}+\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}\left[\frac{1}{p^0}\varrho f^M_{n+1}\right]/H\right)_{L^2(\Omega)}$$
$$=\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}[\frac{1}{p^0}\mathcal{L}(f_n,f_n)]/H\right)_{L^2(\Omega)}, \tag{44}$$

where $H=(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}f^M_{n+1}$. Noting that $(1+|\bar{p}|)^{d+|\beta|}$ does not depend on t, we have:

$$(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}[\partial_t f^M_{n+1}]=\partial_t\left[(1+|\bar{p}|)^{d+\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}\left(f^M_{n+1}\right)\right].$$

The relation (44), using the Cauchy-Schwartz inequality, becomes

$$\frac{1}{2}\frac{d}{dt}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})}[f^M_{n+1}]\right\|^2_{L^2(\Omega)}$$

$$
\begin{aligned}
&\leq -\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}[P^i\partial_{p^i}f^M_{n+1}]/(1+|\bar{p}|)^{d+\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)}\\
&\quad-\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}[Q^I\partial_{q^I}f^M_{n+1}]/(1+|\bar{p}|)^{d+\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)}\\
&\quad-\left((1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}[\frac{1}{p^0}\varrho f^M_{n+1}]/(1+|\bar{p}|)^{d+\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right)_{L^2(\Omega)}\\
&\quad+\left\|\frac{1}{p^0}\mathcal{L}(f_n,f_n)\right\|_{E^{m+3}_d(\Omega)}\times\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}.
\end{aligned}\tag{45}
$$

Introducing Propositions 2, 3 and 4 in (45), we get after simplification:

$$
\begin{aligned}
&\frac{d}{dt}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}\\
&\qquad\leq C\left(\sum_{|\alpha|\leq|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial^{\alpha}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}\right)+\left\|\frac{1}{p^0}\mathcal{L}(f_n,f_n)\right\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}.
\end{aligned}\tag{46}
$$

Summing (46) over $|\beta|=0,1,2,\dots,m+3$, we get:

$$
\begin{aligned}
&\frac{d}{dt}\left(\sum_{|\beta|\leq m+3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}\right)\\
&\qquad\leq C\left(\sum_{|\beta|\leq m+3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\alpha}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}+\left\|\frac{1}{p^0}\mathcal{L}(f_n,f_n)\right\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\right).
\end{aligned}\tag{47}
$$

By the Gronwall lemma, using Proposition 1, the above inequality yields:

$$
\sum_{|\beta|\leq m+3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\tilde{q})}f^M_{n+1}\right\|_{L^2(\Omega)}\leq C\left[\|f_0\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}+T\|f_n\|^2_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\right].\tag{48}
$$

Since $f_n,f_0\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$, we conclude that $\|f^M_{n+1}\|_{\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)}\leq C$. □

Theorem 1. *Let $f_0\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$, $f_n\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$ be given. Then the linearized Boltzmann equation with absorption term* (22) *has in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ a unique and bounded solution.*

Proof. The proof of this theorem will be done into two steps.

Existence: According to Proposition 5, the sequence $(f^M_{n+1})_M$ is bounded in the reflexive Hilbert space $\mathbf{E}_{d^{m+3}}(\Omega)$. Accordingly, $(f^M_{n+1})_M$ admits a subsequence $\left(f^{M_k}_{n+1}\right)_{M_k}$ which converges weakly to f_{n+1} in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$. Hence f_{n+1} is a solution of the linearized Boltzmann equation (22) such that $f_{n+1}(0,\bar{p},\tilde{q})=f_0$.

Uniqueness: We assume that, there is another solution g_{n+1} of (22) with the same initial data f_0. Setting $h_{n+1}=f_{n+1}-g_{n+1}$, h_{n+1} satisfies

$$
\begin{cases}\frac{\partial h_{n+1}}{\partial t}+P^i\frac{\partial h_{n+1}}{\partial p^i}+Q^I\frac{\partial h_{n+1}}{\partial q^I}+\frac{1}{p^0}\varrho h_{n+1}=0\\ h_{n+1}(0,\bar{p},\tilde{q})=0.\end{cases}\tag{49}
$$

In the sequel, proceeding as in the proof of Proposition 5, we show that $h_{n+1}=0$. Then, $f_{n+1}=g_{n+1}$ and the solution is unique. This ends the proof of Theorem 1. □

Theorem 2. *Let $f_0\in\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d},\delta}(\Omega)$ be given. Then the Boltzmann equation with absorption term* (21) *has in $\mathbf{E}^{\mathbf{m+3}}_{\mathbf{d}}(\Omega)$ a local unique solution f such that $f(0,\bar{p},\tilde{q})=f_0(\bar{p},\tilde{q})$.*

Proof. From (22), we define the following Cauchy problems

$$((P_n))_{n\in\mathbf{N}^*}: \quad \begin{cases} \frac{\partial f_{n+1}}{\partial t} + P^i \frac{\partial f_{n+1}}{\partial p^i} + Q^I \frac{\partial f_{n+1}}{\partial q^I} + \frac{1}{p^0}\varrho f_{n+1} = \frac{1}{p^0}\mathcal{L}(f_n, f_n) \\ f_{n+1}(0,\bar{p},\widetilde{q}) = f_0 . \end{cases}$$

We use Theorem 1, for constructing a sequence $(f_n)_n$ of solutions for the Cauchy problems $((P_n))_{n\in\mathbf{N}^*}$.

For f_0, there exists a unique and bounded solution f_1 for the Cauchy problem (P_0) in $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$, for f_1 there exists a unique and bounded solution f_2 for the Cauchy problem (P_1) in $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$. So, recursively, we construct the sequence $(f_n)_{n\in\mathbf{N}} \subset \mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$ of solutions for the Cauchy problems $((P_n))_{n\in\mathbf{N}^*}$.

Existence: We have to prove that the sequence (f_n) is bounded in $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$.

Suppose $\|f_n\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}} \le \delta$. Combining (48) and (47), one has:

$$\frac{d}{dt}\left(\sum_{|\beta|\le m+3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^{\beta}_{(\bar{p},\widetilde{q})} f^M_{n+1}\right\|_{L^2(\Omega)}\right)$$
$$\le C\left(\|f_0\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} + T\|f_n\|^2_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} + \left\|\frac{1}{p^0}\mathcal{L}(f_n,f_n)\right\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)}\right).$$

Integrating this inequality on $[0,t]$, $0 \le t < T$, we obtain using $\left\|\frac{1}{p^0}\mathcal{L}(f_n,f_n)\right\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} \le \delta^2$, that:

$$\left\|f^M_{n+1}\right\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} \le \|f_0\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} + C\left(\|f_0\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} T + T^2\delta^2 + \delta^2 T\right) \tag{50}$$

Since $\delta > 0$ is given, if we take $f_0 \in \mathbf{E}^{\mathbf{m}+3}_{\mathbf{d},\delta}(\Omega)$ and $T > 0$ such that :

$$\| f_0 \|_{\mathbf{E}_{d^{m+3}(\Omega)}} \le \frac{\delta}{2} \quad \text{and} \quad C\left(\|f_0\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} T + T^2\delta^2 + \delta^2 T\right) \le \frac{\delta}{2}, \tag{51}$$

Eq. (50) implies that $\|f^M_{n+1}\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} \le \delta$. Since $f^M_{n+1} \rightharpoonup f_{n+1}$ in $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$ (Theorem 1), it results that $\|f_{n+1}\|_{\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)} \le \delta$. This implies that the sequence $(f_n)_n$ is bounded.

Consequently, we can choose a weak convergent subsequence (f_{n_k}) of the bounded sequence (f_n) in the reflexive Hilbert space $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$ which converges weakly to the solution f of Boltzmann equation (21) in $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d}}(\Omega)$ such that $f(0,\bar{p},\widetilde{q}) = f_0$.

Uniqueness: Let f and g belonging to $\mathbf{E}^{\mathbf{m}+3}_{\mathbf{d},\delta}(\Omega)$ two solutions of the Boltzmann equation (21) with the same initial data f_0. Setting $h = f - g$, then we get

$$\begin{cases} \frac{\partial G}{\partial t} + P^i\partial_{p^i} G + Q^I\partial_{q^I} G + \frac{1}{p^0}\varrho G = \frac{1}{p^0}\mathcal{L}(f,G) + \frac{1}{p^0}\mathcal{L}(G,g) \\ G(0) = 0 . \end{cases} \tag{52}$$

The inequality (47) gives

$$\frac{d}{dt}\left(\sum_{|\beta|\le m+3}\left\|(1+|\bar{p}|)^{d+|\beta|}D^{\beta}_{(\bar{p},\widetilde{q})} G\right\|_{L^2(\Omega)}\right)$$
$$\le C\left(\sum_{|\beta|\le m+3}\left\|(1+|\bar{p}|)^{d+|\beta|}D^{\beta}_{(\bar{p},\widetilde{q})} G\right\|_{L^2(\Omega)}\right) + \left\|(1+|\bar{p}|)^{d+|\beta|}D^{\beta}_{(\bar{p},\widetilde{q})}\left(\frac{1}{p^0}\mathcal{L}(f,G) + \frac{1}{p^0}\mathcal{L}(G,g)\right)\right\|_{L^2(\Omega)}.$$

Integrating on $[0,t]$ for $t \in [0,T]$ and applying Gronwall lemma, knowing that $G(0,\bar{p},\widetilde{q}) = 0$, we obtain:

$$\sum_{|\beta| \le m+3} \left\| (1+|\bar{p}|)^{d+|\beta|} D^{\beta}_{(\bar{p},\widetilde{q})} G \right\|_{L^2(\Omega)}$$
$$\le C \int_0^t \left\| (1+|\bar{p}|)^{d+\beta|} D^{\beta}_{(\bar{p},\widetilde{q})} \left(\frac{1}{p^0} \mathcal{L}(f,G) + \frac{1}{p^0} \mathcal{L}(G,g) \right)(\tau) \right\|_{L^2(\Omega)} d\tau. \tag{53}$$

Then, taking the supremum in (53), for t and using Proposition 1, we get:

$$\| G \|_{\mathbf{E_d^{m+3}}(\Omega)} \le CT \| G \|_{\mathbf{E_d^{m+3}}(\Omega)} . \tag{54}$$

If $T > 0$ is chosen such that $T \times C < 1$, then we have $\| G \|_{\mathbf{E_d^{m+3}}(\Omega)} = 0$. Thus $f = g$. □

5. Well-posedness of the Solution

Theorem 3. *Let $f_0 \in \mathbf{E^{m+3}_{d,\delta}}(\Omega)$ be given. The solution f of the Boltzmann equation with absorption term* (21) *given by Theorem* 2 *satisfies*

$$\|f\|_{\mathbf{E_d^{m+3}}(\Omega)} \le C \|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} . \tag{55}$$

Proof. We have by the inequality (50) of Theorem 2, when M goes to infinty

$$\|f_{n+1}\|_{\mathbf{E_d^{m+3}}(\Omega)} \le \|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} + C \left(\|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} T + T^2\delta^2 + \delta^2 T \right).$$

Since the sequence (f_n) converges to f (Theorem 2), the above inequality yields

$$\|f\|_{\mathbf{E_d^{m+3}}(\Omega)} \le \|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} + C \left(\|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} T + T^2\delta^2 + \delta^2 T \right). \tag{56}$$

The quantity $C = C(h_0, r_0, T, C_F, C_A, e)$ continuously depends on its variables, which are continuous functions of t on the compact $[0,T]$. Then there exists an absolute constant C_0 which is the maximum value of $C = C(h_0, r_0, T, C_F, C_A, e)$ on $[0,T]$. Thus, (56) can be written as

$$\|f\|_{\mathbf{E_d^{m+3}}(\Omega)} \le \|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} + C_0 T \left(\|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)} + T\delta^2 + \delta^2 \right). \tag{57}$$

If we choose T such that $T\delta^2 + \delta^2 \le \|f_0\|_{\mathbf{E_d^{m+3}}(\Omega)}$, we then obtain $\|f\|_{\mathbf{E_d^{m+3}}} \le C \|f_0\|_{\mathbf{E_d^{m+3}}}$. Which yields to (55). □

6. Conclusion

In the present paper, we have obtained a local in time classical solution for the Boltzmann equation with absorption term in the presence of a given Yang-Mills field, on a Bianchi type 1 space-time. In our future investigations, we will coupled this equation with the Yang-Mills system. In this case, the Yang-Mills field F and the Yang-Mills potential A also become unknown functions; the current which is the source of the Yang-Mills field is generated by a distribution function, subject to the Boltzmann equation. The coupled system obtained is of great interest in order to understand certain physical phenomena linked to our universe.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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