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Research Article

Common Fixed Point Results for Three Multivalued ρ-Nonexpansive Mappings by Using Three Steps Iterative Scheme

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Abstract. The purpose of this research paper is to study the convergence and approximation of common fixed points for three multivalued ρ-nonexpansive mappings for three steps iterative scheme in modular function spaces. Further we construct a numerical example which illustrates our results.

Keywords. Common fixed point; Multivalued ρ-nonexpansive; Three steps iterative scheme; Modular function spaces

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1. Introduction

In 1950, Nakano [10] introduced the notion of modular spaces which was further generalized and redefined by Musielak and Orlicz [9] in 1959. Modular function spaces are the generalization of some class of Banach spaces due to which many analysts showed their interest to work in this field. Khamsi *et al*. [5] were the first who initiated the study of fixed point theory in these spaces in 1990. On the basis of their results, many work has been done in these spaces. Kozlowski [2,4,5,8] has contributed a lot in the study of fixed point theory in these spaces.

Until 2012, there was no result obtained for the approximation of fixed point in modular function spaces. In 2012, Dehaish and Kozlowski [2] tried to fill this gap by using Mann iteration for asymptotically pointwise nonexpansive mappings. In 2014, Abdou *et al.* [1] introduced the approximation of common fixed points of two ρ-nonexpansive mappings in these spaces by using Ishikawa iteration procedure. However, all the above work was done for single valued mappings.

In 2014, Khan and Abbas [6] were the first who gave the approximation theorems for fixed points of a multivalued ρ-nonexpansive mappings by using Mann iteration scheme in modular function spaces. In 2017, Khan *et al.* [7] gave some convergence theorems to approximate the fixed point of ρ-quasi-nonexpansive multivalued mappings in modular function spaces using a three step iterative process, where ρ satisfies Δ_2-condition. The results in [7] improved and generalized the results of Khan and Abbas [6]. In 2019, Panwar and Reena [11] proved some approximation results for fixed point of multivalued ρ-quasi-nonexpansive mappings for a newly defined hybrid iterative process in modular function spaces.

Motivated by the work done in this field, we prove some convergence results to approximate the common fixed point for three ρ-nonexpansive mappings by using three steps iterative scheme in these spaces. Our results extend, generalize and improve various results in existing literature.

In section 2, we provide some basic definitions, needed propositions and lemmas to prove our main results. In section 3, we study the convergence and approximation of common fixed points of three multivalued ρ-nonexpansive mappings for three steps iterative scheme in modular function spaces.

2. Preliminaries

Let Ω be a nonempty set and $\sum$ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a nontrivial δ-ring of subsets of Ω which means that $\mathcal{P}$ is closed under countable intersection, finite union and differences. Suppose that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \sum$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$. By ε we denote the linear space all simple functions with support from $\mathcal{P}$. Also, $\mathcal{M}_\infty$ denotes the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence

$$\{g_n\} \subset \varepsilon,\ |g_n| \le |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.$$

We define

$$\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(w)| < \infty \ \rho\text{-a.e.}\}.$$

Now, we recall definition of modular function.

Definition 2.1 ([8]). Let X (R or C) be a vector space. A functional ρ is called a modular if for arbitrary elements f and g of X, there hold the following:

(i) $\rho(f) = 0 \Longleftrightarrow f = 0$,

(ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$,

(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ whenever $\alpha, \beta \ge 0$, $\alpha + \beta = 1$.

If we replace (iii) by

(iv) $\rho(\alpha f+\beta g)\le\alpha\rho(f)+\beta\rho(g)$ whenever $\alpha,\beta\ge 0$, $\alpha+\beta=1$.

Then modular ρ is called convex.

Definition 2.2 ([8]). If ρ is convex modular in X, then the set defined by

$$L_\rho=\{f\in\mathcal{M}:\rho(\lambda f)\to 0 \text{ as } \lambda\to 0\}$$

is called modular function space. Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho=\inf\left\{\alpha>0:\rho\left(\frac{f}{\alpha}\right)\le\alpha\right\}.$$

In the case, ρ is convex modular

$$\|f\|_\rho=\inf\left\{\alpha>0:\rho\left(\frac{f}{\alpha}\right)\le 1\right\}$$

defines a norm on modular space L_ρ and it is called Luxemburge norm.

Definition 2.3 ([8]). Let $\rho:\mathcal{M}_\infty\to[0,\infty]$ be a nontrivial, convex and even function. Then ρ is a regular convex function pseudo modular if

1. $\rho(0)=0$;
2. ρ is monotone, i.e., $|f(w)|\le|g(w)|$ for any $w\in\Omega$ implies $\rho(f)\le\rho(g)$, where $f,g\in\mathcal{M}_\infty$;
3. ρ is orthogonally sub-additive, i.e., $\rho(f\chi_{A\cup B})\le\rho(f\chi_A)+\rho(f\chi_B)$ for any $A,B\in\sum$ such that $A\cap B\ne\phi$, $f\in\mathcal{M}_\infty$;
4. ρ has Fatou property, i.e., $|f_n(w)|\uparrow|f(w)|$ for $w\in\Omega$ implies $\rho(f_n)\uparrow\rho(f)$, where $f\in\mathcal{M}_\infty$;
5. ρ is order continuous in ε, i.e., $g_n\in\varepsilon$ and $|g_n(w)|\downarrow 0$ and $\rho(g_n)\downarrow 0$.

A set $A\in\sum$ is said to be ρ-null if $\rho(g\chi_A)=0$ for every $A\in\varepsilon$. A property $p(w)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{w\in\Omega:p(w)$ does not holds$\}$ is ρ-null.

Definition 2.4 ([8]). A regular function pseudo modular ρ is a regular convex function modular if $\rho(f)=0$ implies $f=0$ a.e. The class of all nonzero regular convex function modular defined on Ω will be denoted by $\mathfrak{R}$.

Definition 2.5 ([2]). Let $\rho\in\mathfrak{R}$. We define the following uniform convexity type properties of the function modular ρ. Let $t\in(0,1)$, $r>0$, $\epsilon>0$. Define

$$D_1(r,\epsilon)=\{(f,g):f,g\in L_\rho,\rho(f)\le r,\rho(g)\le r,\rho(f-g)\ge\epsilon r\}.$$

Let

$$\delta_1^t(r,\epsilon)=\inf\left\{1-\frac{1}{r}\rho(tf+(1-t)g):(f,g)\in D_1(r,\epsilon)\right\}\ \text{ if }\ D_1(r,\epsilon)\ne\phi$$

and

$$\delta_1(r,\epsilon)=1\ \text{ if }\ D_1(r,\epsilon)=\phi.$$

We use the following notational convention: $\delta_1 = \delta_1^{\frac{1}{2}}$.

Definition 2.6 ([2]). A non-zero regular convex function modular ρ is said to satisfy (UC1) if every $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. Note that for every $r > 0$, $D_1(r,\epsilon) \neq \phi$ for $\epsilon > 0$ small enough.

Definition 2.7 ([2]). A non-zero regular convex function modular ρ is said to satisfy (UUC1) if for every $s \geq 0$, $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 0$ depending only upon s and ϵ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon)$ for any $r > s$.

Definition 2.8 ([8]). Let $\rho \in \mathfrak{R}$.

(1) A sequence $\{f_n\}$ is ρ-convergent to f, that is, $f_n \to f$ if and only if $\rho(f_n - f) \to 0$ as $n \to \infty$.

(2) A sequence $\{f_n\}$ is ρ-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $m,n \to \infty$.

(3) A set $B \subset L_\rho$ is called ρ-closed if for any sequence $\{f_n\} \subset B$, $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(4) A set $B \subset L_\rho$ is called ρ-bounded if ρ-diameter is finite; the ρ-diameter of B is defined as

$$\delta_\rho(B) = \sup\{\rho(f-g) : f,g \in B\}.$$

(5) A set $B \subset L_\rho$ is called ρ-compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0$ as $k \to \infty$.

(6) A set $B \subset L_\rho$ is called ρ-a.e. closed if for any sequence $\{f_n\} \subset B$ which ρ-a.e. converges $f_n \to f$ as $n \to \infty$ implies that f belongs to B.

(7) A set $B \subset L_\rho$ is called ρ-a.e. compact if for any sequence $\{f_n\} \subset B$, there exists a subsequence $\{f_{n_k}\}$ and $f \in B$ such that $\rho(f_{n_k} - f) \to 0$ a.e. as $k \to \infty$.

(8) Let $f \in L_\rho$ and $B \subset L_\rho$. The distance between f and B is defined as

$$d_\rho(f,B) = \inf\{\rho(f-g) : g \in B\}.$$

Proposition 2.9 ([8]). *Let $\rho \in \mathfrak{R}$.*

(i) *L_ρ is ρ-complete.*

(ii) *ρ-balls $B_\rho(f,r) = \{g \in L_\rho : \rho(f-g) \leq r\}$ are ρ-closed.*

(iii) *If $\rho(\alpha f_n) \to 0$ for $\alpha > 0$, then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0$ ρ-a.e. as $n \to \infty$.*

(iv) *$\rho(f) \leq \liminf_{n\to\infty} \rho(f_n)$ whenever $f_n \to f$ ρ-a.e. as $n \to \infty$ (Note: this property is equivalent to the Fatou property).*

(v) *Consider the set $L_\rho^0 = \{f \in L_\rho : \rho(f,\cdot) \text{ is order continuous}\}$ and*

$$E_\rho = \{f \in L_\rho : \lambda f \in L_\rho^0 \text{ for any } \lambda > 0\}.$$

Then we have $E_\rho \subset L_\rho^0 \subset L_\rho$.

Definition 2.10 ([8]). Let $\rho \in \mathfrak{R}$. Then ρ satisfies Δ_2-property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$ as $n \to \infty$.

Proposition 2.11 ([8]). *The following statements are equivalent:*

(i) *ρ satisfies Δ_2-condition.*

(ii) *$\rho(f_n - f) \to 0$ if and only if $\rho(\lambda(f_n - f)) \to 0$, for every $\lambda > 0$ if and only if*

$$\|f_n - f\|_\rho \to 0 \text{ as } n \to \infty.$$

Definition 2.12 ([6]). A set $C \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$, there exists an element $g \in C$ such that

$$\rho(f - g) = d_\rho(f, C) = \inf\{\rho(f - h) : h \in C\}.$$

$P_\rho(C)$ denotes the family of nonempty ρ-bounded ρ-proximinal subset of C and $C_\rho(C)$ denotes the family of ρ-bounded ρ-closed subsets of C. Let $H_\rho(\cdot,\cdot)$ be ρ-Hausdorff distance on $C_\rho(C)$, that is,

$$H_\rho(A,B) = \max\left\{\sup_{f\in A} \operatorname{dist}_\rho(f,B), \sup_{g\in B} \operatorname{dist}_\rho(g,A)\right\}, \quad A,B \in C_\rho(L_\rho).$$

Definition 2.13 ([6]). A multivalued mapping $T : C \to C_\rho(L_\rho)$ is said to be ρ-Lipschitzian if there exists a number $k \ge 0$ such that

$$H_\rho(T(f), T(g)) \le k\rho(f - g) \text{ for all } f, g \in C$$

(i) If $k = 1$, then T is called ρ-nonexpansive.

(ii) If $k < 1$, then T is called ρ-contractive.

Lemma 2.14 ([2]). *Let $\rho \in \Re$ and satisfy* (UUC1). *Let $\{t_n\} \subset (0,1)$ be bounded away from both* 0 *and* 1. *If there exists $R > 0$ such that*

$$\lim_{n\to\infty} \sup \rho(f_n) \le R, \quad \lim_{n\to\infty} \sup \rho(g_n) \le R \quad \text{and} \quad \lim_{n\to\infty} \rho(t_n f_n + (1 - t_n) g_n) = R,$$

then

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

The sequence $\{t_n\} \subset (0,1)$ is said to be bounded away from 0 *if there exists $a > 0$ such that $t_n \ge a$ for all $n \in \mathbb{N}$. Similarly, the sequence $\{t_n\} \subset (0,1)$ is said to be bounded away from* 1 *if there exists $b < 1$ such that $t_n \le b$ for all $n \in \mathbb{N}$.*

Lemma 2.15 ([6]). *Let $T : D \to P_\rho(D)$ be a multivalued mapping and*

$$P_\rho^T(f) = \{g \in T : \rho(f - g) = d_\rho(f, Tf)\}.$$

Then the following are equivalent:

(i) *$f \in F_\rho(T)$, that is, $f \in T(f)$,*

(ii) *$P_\rho^T = \{f\}$, that is, $f = g$ for each $g \in P_\rho^T(f)$,*

(iii) *$f \in F_\rho(P_\rho^T(f))$, that is, $f \in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T(f))$ where $F(P_\rho^T(f))$ denotes the set of fixed points of $P_\rho^T(f)$.*

Lemma 2.16. *Let $\rho \in \Re$ and satisfy $A, B \in P_\rho(L_\rho)$. For every $f \in A$, there exists $g \in B$ such that $\rho(f - g) \le H_\rho(A,B)$.*

Definition 2.17. A family of mappings $T_i : C \to P_\rho(C)$ is said to satisfy condition (II) if there exists a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0,\infty)$ such that

$$d_\rho(f, T_i(f)) \geq \varphi\Big(d_\rho\Big(f, \bigcap_{i=1}^{m} F_\rho(T_i)\Big)\Big).$$

3. Main Results

Let D_ρ be a non empty ρ-bounded, closed and convex subset of L_ρ and $T_1, T_2, T_3 : D_\rho \to P_\rho(D_\rho)$ be three multivalued mappings. Let $f_1 \in D_\rho$ and $\{f_n\} \subset D_\rho$ be defined by

$$\begin{cases} g_n = \gamma_n u_n + (1-\gamma_n) f_n \\ h_n = \beta_n v_n + (1-\beta_n) f_n \\ f_{n+1} = \alpha_n w_n + (1-\alpha_n) f_n, \quad n = 1,2,\dots \end{cases} \tag{1}$$

where $u_n \in P_\rho^{T_1}(f_n)$, $v_n \in P_\rho^{T_2}(g_n)$, $w_n \in P_\rho^{T_3}(h_n)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ which are bounded away from both 0 and 1.

Lemma 3.1. *Let $\rho \in \Re$ satisfies* (UUC1) *and D_ρ be nonempty ρ-bounded and convex subset of L_ρ. Suppose $T_1, T_2, T_3 : D_\rho \to P_\rho(D_\rho)$ are three multivalued mappings such that $P_\rho^{T_1}$, $P_\rho^{T_2}$ and $P_\rho^{T_3}$ are ρ-nonexpansive mappings and $F = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \neq \phi$. Then $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$.*

Proof. Let $p \in F$ be arbitrary. Then by Lemma 2.15, we have

$$P_\rho^{T_1}(p) = \{p\}, \quad P_\rho^{T_2}(p) = \{p\}, \quad P_\rho^{T_3}(p) = \{p\}.$$

From (1) and by convexity of ρ, we have

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho(\alpha_n w_n + (1-\alpha_n) f_n - p) \\ &\leq \alpha_n \rho(w_n - p) + (1-\alpha_n)\rho(f_n - p) \\ &\leq \alpha_n H_\rho(P_\rho^{T_3}(h_n), P_\rho^{T_3}(p)) + (1-\alpha_n)\rho(f_n - p) \\ &\leq \alpha_n \rho(h_n - p) + (1-\alpha_n)\rho(f_n - p). \end{aligned} \tag{2}$$

Again from (1) and by convexity of ρ, we get

$$\begin{aligned} \rho(h_n - p) &= \rho(\beta_n v_n + (1-\beta_n) f_n - p) \\ &\leq \beta_n \rho(v_n - p) + (1-\beta_n)\rho(f_n - p) \\ &\leq \beta_n H_\rho(P_\rho^{T_2}(g_n), P_\rho^{T_2}(p)) + (1-\beta_n)\rho(f_n - p) \\ &\leq \beta_n \rho(g_n - p) + (1-\beta_n)\rho(f_n - p). \end{aligned} \tag{3}$$

Using (1) and convexity of ρ, we have

$$\begin{aligned} \rho(g_n - p) &= \rho((1-\gamma_n) f_n + \gamma_n u_n - p) \\ &\leq (1-\gamma_n)\rho(f_n - p) + \gamma_n \rho(u_n - p) \\ &\leq (1-\gamma_n)\rho(f_n - p) + \gamma_n H_\rho(P_\rho^{T_1}(f_n), P_\rho^{T_1}(p)) \end{aligned}$$

$$\leq (1-\gamma_n)\rho(f_n - p) + \gamma_n \rho(f_n - p)$$
$$\leq \rho(f_n - p). \tag{4}$$

Using (2), (3) and (4), we obtain

$$\rho(f_{n+1} - p) \leq \rho(f_n - p).$$

This shows that the sequence $\{\rho(f_n - p)\}$ is decreasing. Hence

$$\lim_{n\to\infty} \rho(f_n - p) \text{ exists for all } p \in F. \qquad \square$$

Theorem 3.2. *Let $\rho \in \Re$ satisfy* (UUC1) *and D_ρ be nonempty ρ-bounded and convex subset of L_ρ. Suppose $T_1, T_2, T_3 : D_\rho \to P_\rho(D_\rho)$ are three multivalued mappings such that $P_\rho^{T_1}$, $P_\rho^{T_2}$ and $P_\rho^{T_3}$ are ρ-nonexpansive mappings and $F = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \neq \phi$. Let $f_1 \in C$ and $\{f_n\}$ be given by* (1)*. Then $\{f_n\}$ is a common ρ-approximate sequence of T_1, T_2 and T_3.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$. Let

$$\lim_{n\to\infty} \rho(f_n - p) = R. \tag{5}$$

From (4) and (5), we get

$$\lim_{n\to\infty} \sup \rho(g_n - p) \leq R. \tag{6}$$

From (3) and (4), we have

$$\rho(h_n - p) \leq \rho(f_n - p).$$

This implies that

$$\lim_{n\to\infty} \sup \rho(h_n - p) \leq \lim_{n\to\infty} \sup \rho(f_n - p)$$
$$\lim_{n\to\infty} \sup \rho(h_n - p) \leq R \tag{7}$$

Also,

$$\rho(v_n - p) \leq H_\rho(P_\rho^{T_2}(g_n), P_\rho^{T_2}(p))$$
$$\leq \rho(g_n - p) \leq \rho(f_n - p)$$

which implies that

$$\lim_{n\to\infty} \sup \rho(v_n - p) \leq \lim_{n\to\infty} \sup \rho(f_n - p)$$
$$\lim_{n\to\infty} \sup \rho(v_n - p) \leq R. \tag{8}$$

Similarly, we can show that

$$\lim_{n\to\infty} \sup \rho(w_n - p) \leq R. \tag{9}$$

Since the sequence $\{\alpha_n\} \subset (0,1)$ is bounded away from 0 and 1, so there exists $\alpha \in (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

Now,

$$\rho(f_{n+1} - p) = \rho(\alpha_n w_n + (1-\alpha_n) f_n - p)$$

$$
\begin{aligned}
&= \rho(\alpha_n(w_n - p) + (1-\alpha_n)(f_n - p)) \\
&\le \alpha_n \rho(w_n - p) + (1-\alpha_n)\rho(f_n - p)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \inf \rho(f_{n+1} - p) &\le \lim_{n\to\infty} \inf(\alpha_n \rho(w_n - p) + (1-\alpha_n)\rho(f_n - p)) \\
&\le \lim_{n\to\infty} \inf \alpha_n \rho(w_n - p) + \lim_{n\to\infty} \inf(1-\alpha_n)\rho(f_n - p) \\
R &\le \lim_{n\to\infty} \inf \alpha \rho(w_n - p) + (1-\alpha)R \\
R &\le \lim_{n\to\infty} \inf \rho(w_n - p) \qquad (10)
\end{aligned}
$$

From (9) and (10), we get

$$\lim_{n\to\infty} \rho(w_n - p) = R.$$

Since $w_n \in P_\rho^{T_3}(h_n)$, then

$$\rho(w_n - p) \le H_\rho(P_\rho^{T_3}(h_n), P_\rho^{T_3}(p)) \le \rho(h_n - p),$$

which implies that

$$\lim_{n\to\infty} \inf \rho(h_n - p) \ge R. \qquad (11)$$

Using (7) and (11), we get

$$\lim_{n\to\infty} \rho(h_n - p) = R. \qquad (12)$$

Since the sequence $\{\beta_n\} \subset (0,1)$ is bounded away from 0 and 1, so there exists $\beta \in (0,1)$ such that

$$\lim_{n\to\infty} \beta_n = \beta.$$

Then,

$$
\begin{aligned}
\rho(h_n - p) &= \rho(\beta_n v_n + (1-\beta_n) f_n - p) \\
&= \rho(\beta_n(v_n - p) + (1-\beta_n)(f_n - p)) \\
&\le \beta_n \rho(v_n - p) + (1-\beta_n)\rho(f_n - p)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \inf \rho(h_n - p) &\le \lim_{n\to\infty} \inf(\beta_n \rho(v_n - p) + (1-\beta_n)\rho(f_n - p)) \\
&\le \lim_{n\to\infty} \inf \beta_n \rho(v_n - p) + \lim_{n\to\infty} \inf(1-\beta_n)\rho(f_n - p) \\
R &\le \lim_{n\to\infty} \inf \beta \rho(v_n - p) + (1-\beta)R \\
R &\le \lim_{n\to\infty} \inf \rho(v_n - p) \qquad (13)
\end{aligned}
$$

From (8) and (13), we get

$$\lim_{n\to\infty} \rho(v_n - p) = R.$$

Since $v_n \in P_\rho^{T_2}(g_n)$, then

$$\rho(v_n - p) \le H_\rho(P_\rho^{T_2}(g_n), P_\rho^{T_2}(p)) \le \rho(g_n - p),$$

which implies that

$$\lim_{n\to\infty} \inf \rho(g_n - p) \ge R. \qquad (14)$$

By (6) and (14), we get

$$\lim_{n\to\infty} \rho(g_n - p) = R\,.$$

From (5), we have

$$\lim_{n\to\infty} \rho(\gamma_n u_n + (1-\gamma_n) f_n - p) = R$$

or

$$\lim_{n\to\infty} \rho(\gamma_n (u_n - p) + (1-\gamma_n)(f_n - p)) = R\,. \tag{15}$$

Since $u_n \in P_\rho^{T_1}(f_n)$, then

$$\rho(u_n - p) \le H_\rho(P_\rho^{T_1}(f_n), P_\rho^{T_1}(p)) \le \rho(f_n - p),$$

which implies that

$$\lim_{n\to\infty} \sup \rho(u_n - p) \le R\,. \tag{16}$$

Then from (5), (15), (16) and Lemma 2.14, we have

$$\lim_{n\to\infty} \rho(f_n - u_n) = 0\,. \tag{17}$$

Since $u_n \in P_\rho^{T_1}(f_n)$, so $\lim\limits_{n\to\infty} d_\rho(f_n, T_1(f_n))$. Therefore, it follows from (17) that $\{f_n\}$ is ρ-approximate sequence of T_1.

From (12),

$$\lim_{n\to\infty} \rho(\beta_n v_n + (1-\beta_n) f_n - p) = R$$

or

$$\lim_{n\to\infty} \rho(\beta_n (u_n - p) + (1-\beta_n)(f_n - p)) = R\,. \tag{18}$$

From (5), (7), (18) and Lemma 2.14, we have

$$\lim_{n\to\infty} \rho(f_n - v_n) = 0\,. \tag{19}$$

Also,

$$R = \lim_{n\to\infty} \rho(f_{n+1} - p) = \lim_{n\to\infty} \rho(\alpha_n w_n + (1-\alpha_n) f_n - p)\,. \tag{20}$$

Using (5), (9), (20) and Lemma 2.14, we have

$$\lim_{n\to\infty} \rho(f_n - w_n) = 0 \tag{21}$$

From (1)

$$\rho(h_n - f_n) \le \rho(v_n - f_n)\,.$$

By (19),

$$\lim_{n\to\infty} \rho(h_n - f_n) = 0\,.$$

Again from (1)

$$\rho(g_n - f_n) \le \rho(u_n - f_n)\,.$$

By (17),

$$\lim_{n\to\infty} \rho(g_n - f_n) = 0\,.$$

Since $P_\rho^{T_2}$ is nonexpansive, then

$$H_\rho(P_\rho^{T_2}(g_n), P_\rho^{T_2}(f_n)) \le \rho(g_n - f_n),$$

which implies that

$$H_\rho(P_\rho^{T_2}(g_n), P_\rho^{T_2}(f_n)) = 0. \tag{22}$$

Since $v_n \in P_\rho^{T_2}$,

$$d_\rho(f_n, P_\rho^{T_2}(g_n)) \le \rho(f_n - v_n).$$

Then by (19), we get

$$d_\rho(f_n, P_\rho^{T_2}(g_n)) = 0. \tag{23}$$

Now,

$$d_\rho(f_n, P_\rho^{T_2}(f_n)) \le d_\rho(f_n, P_\rho^{T_2}(g_n)) + d_\rho(P_\rho^{T_2}(g_n), P_\rho^{T_2}(f_n)).$$

Using (22) and (23), we obtained that

$$d_\rho(f_n, P_\rho^{T_2}(f_n)) = 0. \tag{24}$$

But $d_\rho(f_n, T_2(f_n)) \le d_\rho(f_n, P_\rho^{T_2}(f_n))$, therefore from (24), it follows that $\{f_n\}$ is ρ-approximate sequence of T_2. Hence, $\{f_n\}$ is ρ-approximate sequence of T_1 and T_2.
Since $\lim_{n\to\infty} \rho(h_n - f_n) = 0$ and $P_\rho^{T_3}$ is a nonexpansive mapping,

$$H_\rho(P_\rho^{T_3}(h_n), P_\rho^{T_3}(f_n)) \le \rho(h_n - f_n)$$

which implies that

$$\lim_{n\to\infty} H_\rho(P_\rho^{T_2}(h_n), P_\rho^{T_2}(f_n)) = 0$$

$$d_\rho(P_\rho^{T_3}(h_n), P_\rho^{T_3}(f_n)) = 0. \tag{25}$$

Since $w_n \in P_\rho^{T_3}(h_n)$

$$d_\rho(f_n, P_\rho^{T_3}(h_n)) \le \rho(f_n - h_n).$$

Then by (21)

$$d_\rho(f_n, P_\rho^{T_3}(h_n)) = 0. \tag{26}$$

Then

$$d_\rho(f_n, P_\rho^{T_3}(f_n)) \le d_\rho(f_n, P_\rho^{T_3}(h_n)) + d_\rho(P_\rho^{T_3}(h_n), P_\rho^{T_3}(f_n)).$$

By (25) and (26), we obtained that

$$d_\rho(f_n, P_\rho^{T_3}(f_n)) = 0. \tag{27}$$

But $d_\rho(f_n, T_3(f_n)) \le d_\rho(f_n, P_\rho^{T_3}(f_n))$, therefore from (27), it follows that $\{f_n\}$ is ρ-approximate sequence of T_3. Hence, $\{f_n\}$ is ρ-approximate sequence of T_1, T_2 and T_3. □

Theorem 3.3. *Let $\rho \in \Re$ satisfy* (UUC1) *and D_ρ be nonempty ρ-bounded and convex subset of L_ρ. Suppose $T_1, T_2, T_3 : D_\rho \to P_\rho(D_\rho)$ are three multivalued mappings such that $P_\rho^{T_1}, P_\rho^{T_2}$ and $P_\rho^{T_3}$ are ρ-nonexpansive mappings and $F = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \ne \phi$. Let $f_1 \in C$ and $\{f_n\}$ be given by* (1)*. Then $\{f_n\}$ converges to a common fixed point of T_1, T_2 and T_3.*

Proof. Using the compactness of D_ρ, there must be a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in D_\rho$ such that $\lim_{n\to\infty} \rho(f_{n_k} - f) = 0$ as $k \to \infty$. We show that f is a common fixed point of T_1, T_2 and T_3, i.e., $f \in F_\rho(T_1)$, $f \in F_\rho(T_2)$ and $f \in F_\rho(T_3)$. Let $g \in P_\rho^{T_1}(f)$, $h \in P_\rho^{T_2}(f)$ and $w \in P_\rho^{T_3}(f)$ be arbitrary. Then by Lemma 2.16 $g_k \in P_\rho^{T_1}(f_{n_k})$, $h_k \in P_\rho^{T_2}(f_{n_k})$ and $w_k \in P_\rho^{T_3}(f_{n_k})$ such that

$$\rho(g_k - g) \le H_\rho(P_\rho^{T_1}(f_{n_k}), P_\rho^{T_1}(f)),$$
$$\rho(h_k - g) \le H_\rho(P_\rho^{T_2}(f_{n_k}), P_\rho^{T_2}(f))$$

and

$$\rho(w_k - g) \le H_\rho(P_\rho^{T_3}(f_{n_k}), P_\rho^{T_3}(f)).$$

We have

$$\begin{aligned}
\rho\left(\frac{f-g}{3}\right) &= \rho\left(\frac{f - f_{n_k}}{3} + \frac{f_{n_k} - g_k}{3} + \frac{g_k - g}{3}\right) \\
&\le \frac{1}{3}\rho(f - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - g_k) + \frac{1}{3}\rho(g_k - g) \\
&\le \rho(f - f_{n_k}) + d_\rho(f_{n_k}, P_\rho^{T_1}(f_{n_k})) + \rho(g_k - g) \\
&\le \rho(f - f_{n_k}) + d_\rho(f_{n_k}, P_\rho^{T}(f_{n_k})) + H_\rho(P_\rho^{T_1}(f_{n_k}), P_\rho^{T_1}(f)) \\
&\le \rho(f - f_{n_k}) + d_\rho(f_{n_k}, P_\rho^{T_1}(f)) + \rho(f - f_{n_k}) \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Hence $f = g$ a.e. Since $g \in P_\rho^{T_1}(f)$ was arbitrary, we have $P_\rho^{T_1} = \{f\}$. Thus by using Lemma 2.15, $f \in F_\rho(T_1)$. Similarly, we can show that $f \in F_\rho(T_2)$ and $f \in F_\rho(T_3)$. □

Theorem 3.4. *Let $\rho \in \Re$ satisfy* (UUC1) *and D_ρ be nonempty ρ-bounded and convex subset of L_ρ. Suppose $T_1, T_2, T_3 : D_\rho \to P_\rho(D_\rho)$ are three multivalued mappings such that $P_\rho^{T_1}$, $P_\rho^{T_2}$ and $P_\rho^{T_3}$ are ρ-nonexpansive mappings and $F = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \ne \phi$. Let $f_1 \in C$ and $\{f_n\}$ be given by* (1). *Suppose that T_1, T_2 and T_3 satisfy condition* (II). *Then the sequence $\{f_n\}$ converges to a fixed point of F.*

Proof. By using Lemma 3.1, we obtained that $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$.
If $\lim_{n\to\infty} \rho(f_n - p) = 0$, then nothing to do. Assume that $\lim_{n\to\infty} \rho(f_n - p) = R > 0$. By same lemma, we have

$$\rho(f_{n+1} - p) \le \rho(f_n - p) \text{ for all } p \in F.$$

This implies that

$$d_\rho(f_{n+1}, F) \le d_\rho(f_n, F)$$

so that $\lim_{n\to\infty} d_\rho(f_n, F)$ exists. By Theorem 3.2 and condition (II)

$$0 = \lim_{n\to\infty} d_\rho(f_n, T_1(f_n)) \ge \lim_{n\to\infty} \varphi(d_\rho(f_n, F)).$$

Since φ is increasing and $\varphi(0) = 0$ so that $\lim_{n\to\infty} \varphi(d_\rho(f_n, F)) = 0$.
Let $\epsilon > 0$ be arbitrary. Then there exists an integer $m_0 \in \mathbb{N}$ such that $d_\rho(f_n, F) < \frac{\epsilon}{2}$, for all $n \ge m_0$. Particularly, $\inf\{\rho(f_{m_0}) : p \in F\} < \frac{\epsilon}{2}$. Thus, there exists a $p_0 \in F$ such that

$$\rho(f_{m_0} - p_0) < \epsilon$$

$$\rho\left(\frac{f_n - f_m}{2}\right) \le \frac{1}{2}\rho(f_n - p_0) + \frac{1}{2}\rho(f_m - p_0)$$
$$\le \frac{1}{2}\rho(f_{m_0} - p_0) + \frac{1}{2}\rho(f_{m_0} - p_0) \le \epsilon.$$

Since ρ satisfies Δ_2-condition, by Proposition 2.11, we get $\{f_n\}$ is a ρ-Cauchy sequence in D_ρ. As L_ρ is complete and D_ρ is ρ-closed, then there must exists an $f \in C$ such that $\rho(f_n - f) \to 0$ as $n \to \infty$. By Theorem 3.3 the required result is proved. □

Example 3.5. Let the real number system $\mathbb{R}$ be space modulared as $\rho(f) = |f|$.
Define $D_\rho = \{f \in L_\rho : 0 \le f \le 3\}$ and $T_1, T_2, T_3 : D_\rho \to P_\rho(D_\rho)$ as:

$$T_1 = \left[0, \frac{f+1}{2}\right], \quad T_2 = \left[0, \frac{f+3}{4}\right] \quad \text{and} \quad T_3 = \left[0, \frac{f+2}{3}\right].$$

Clearly, D_ρ is a nonempty ρ-compact, ρ-bounded and convex subset of $L_\rho = \mathbb{R}$. Define a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{4}$. Note that $d_\rho(f, T_i(f)) \ge \phi\left(d_\rho\left(f, \bigcap_{i=1}^{i=3} F_\rho(T_i)\right)\right)$ for all $f \in D_\rho$ as follows. If $f \in F = [0,1]$, then obviously

$$d_\rho(f, T_i(f)) = 0 = \phi\left(d_\rho\left(f, \bigcap_{i=1}^{i=3} F_\rho(T_i)\right)\right).$$

If $f \in (3,\infty)$, then

$$d_\rho(f, T_1(f)) = d_\rho\left(f, \left[0, \frac{f+1}{2}\right]\right) = \left|f - \frac{f+1}{2}\right| = \frac{f-1}{2},$$
$$d_\rho(f, T_2(f)) = d_\rho\left(f, \left[0, \frac{f+3}{4}\right]\right) = \left|f - \frac{f+3}{4}\right| = \frac{3(f-1)}{4},$$
$$d_\rho(f, T_3(f)) = d_\rho\left(f, \left[0, \frac{f+2}{3}\right]\right) = \left|f - \frac{f+2}{3}\right| = \frac{2(f-1)}{3},$$
$$\phi(d_\rho(f, F)) = \phi(d_\rho(f, [0,1])) = \phi(|f-1|) = \frac{f-1}{4}$$

so $d_\rho(f, T_i(f)) \ge \phi\left(d_\rho\left(f, \bigcap_{i=1}^{i=3} F_\rho(T_i)\right)\right)$ for all $f \in D_\rho$ and hence the condition (II) is satisfied.
Also, $P_\rho^{T_1}(f) = \{f\}$ when $f \in F_\rho(T_1)$. If $f \not\in F_\rho(T_1) = [0,1]$, then

$$\begin{aligned}
P_\rho^{T_1}(f) &= \{g \in T_1(f) : \rho(f-g) = d_\rho(f, T_1)\} \\
&= \left\{g \in T_1(f) : |f-g| = d_\rho\left(f, \left[0, \frac{f+1}{2}\right]\right)\right\} \\
&= \left\{g \in T_1(f) : |f-g| = \left|f - \frac{f+1}{2}\right| = \left|\frac{f-1}{2}\right|\right\} \\
&= \left\{g \in T_1(f) : f-g = \frac{f-1}{2}\right\}
\end{aligned}$$

because $f > g$ for all $g \in T_1(f)$ where $f \in (1,3]$.

$$P_\rho^{T_1}(f) = \left\{g = \frac{f+1}{2}\right\}.$$

Similarly, $P_\rho^{T_2}(f)=\{f\}$ when $f\in F_\rho(T_2)$. If $f\notin F_\rho(T_2)=[0,1]$, then

$$\begin{aligned}
P_\rho^{T_2}(f)&=\{g\in T_2(f):\rho(f-g)=d_\rho(f,T_2)\}\\
&=\left\{g\in T_2(f):|f-g|=d_\rho\left(f,\left[0,\frac{f+3}{4}\right]\right)\right\}\\
&=\left\{g\in T_2(f):|f-g|=\left|f-\frac{f+3}{4}\right|=\left|\frac{3(f-1)}{4}\right|\right\}\\
&=\left\{g\in T_2(f):f-g=\frac{3(f-1)}{4}\right\}
\end{aligned}$$

because $f>g$ for all $g\in T_2(f)$ where $f\in(1,3]$.

$$P_\rho^{T_2}(f)=\left\{g=\frac{f+3}{4}\right\}$$

and, $P_\rho^{T_3}(f)=\{f\}$ when $f\in F_\rho(T_3)$. If $f\notin F_\rho(T_3)=[0,1]$, then

$$\begin{aligned}
P_\rho^{T_3}(f)&=\{g\in T_3(f):\rho(f-g)=d_\rho(f,T_3)\}\\
&=\left\{g\in T_3(f):|f-g|=d_\rho\left(f,\left[0,\frac{f+2}{3}\right]\right)\right\}\\
&=\left\{g\in T_3(f):|f-g|=\left|f-\frac{f+2}{3}\right|=\left|\frac{2(f-1)}{3}\right|\right\}\\
&=\left\{g\in T_3(f):f-g=\frac{2(f-1)}{3}\right\}
\end{aligned}$$

because $f>g$ for all $g\in T_3(f)$ where $f\in(1,3]$.

$$P_\rho^{T_3}(f)=\left\{g=\frac{f+2}{3}\right\}.$$

Now, we prove that $P_\rho^{T_1},P_\rho^{T_2}$ and $P_\rho^{T_3}$ are nonexpansive for all $f\in D_\rho$.
First of all, we take $P_\rho^{T_1}$. If $f\in[0,1]$, then the proof is trivial. So, we take $f\in(1,3]$.

$$\begin{aligned}
H_\rho(P_\rho^{T_1}(f),P_\rho^{T_1}(p))&=H_\rho\left(\frac{f+1}{2},p\right)\\
&=\left|\frac{f+1}{2}-p\right|\\
&\le|f-p|\quad\text{for all } f\in(1,3].
\end{aligned}$$

This shows that T_1 is nonexpansive for all $f\in D_\rho$.
Now, we take $P_\rho^{T_2}$. If $f\in[0,1]$, then the proof is trivial. So, we take $f\in(1,3]$.

$$\begin{aligned}
H_\rho(P_\rho^{T_2}(f),P_\rho^{T_2}(p))&=H_\rho\left(\frac{f+3}{4},p\right)\\
&=\left|\frac{f+3}{4}-p\right|\\
&\le|f-p|\quad\text{for all } f\in(1,3].
\end{aligned}$$

Hence, T_2 is nonexpansive for all $f\in D_\rho$.
Lastly, we take $P_\rho^{T_3}$. If $f\in[0,2]$, then the proof is trivial. So, we take $f\in(1,3]$.

$$H_\rho(P_\rho^{T_3}(f),P_\rho^{T_3}(p))=H_\rho\left(\frac{f+2}{3},p\right)$$

$$= \left|\frac{f+2}{3} - p\right|$$
$$\leq |f-p| \quad \text{for all } f \in (1,3].$$

This proves that T_3 is nonexpansive for all $f \in D_\rho$.

Now, we generate the sequence (1) and show that it converges strongly to a common fixed point of T_1, T_2 and T_3. Choose $f_1 = 2 \in D_\rho = [0,3]$ and take $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$ for all $n \in \mathbb{N}$.

Table 1 shows that the sequence $\{f_n\}$ generated from (1) converges to a common fixed point of T_1, T_2 and T_3.

Table 1. Computing common fixed point of mappings T_1, T_2 and T_3

n	f_n	u_n	g_n	v_n	h_n	w_n
1	2	1.5	1.75	1.1875	1.59375	1.197916667
2	1.598958333	1.299479167	1.44921875	1.112304688	1.35563151	1.118543837
3	1.358751085	1.179375543	1.269063314	1.067265828	1.213008457	1.071002819
4	1.214876952	1.107438476	1.161157714	1.040289428	1.12758319	1.04252773
5	1.128702341	1.064351171	1.096526756	1.024131689	1.076417015	1.025472338
6	1.07708734	1.03854367	1.057815505	1.014453876	1.045770608	1.015256869
7	1.046172104	1.023086052	1.034629078	1.00865727	1.027414687	1.009138229
8	1.027655167	1.013827583	1.020741375	1.005185344	1.016420255	1.005473418
9	1.016564293	1.008282146	1.012423219	1.003105805	1.009835049	1.00327835
10	1.009921321	1.004960661	1.007440991	1.001860248	1.005890784	1.001963595
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	1.000058957	1.000029479	1.000044218	1.000011054	1.000035006	1.000011669
⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	1.000004545	1.000002272	1.000003409	1.000000852	1.000002698	1.000000899
⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	1.00000035	1.000000175	1.000000263	1.000000066	1.000000208	1.000000069
⋮	⋮	⋮	⋮	⋮	⋮	⋮
35	1.000000027	1.000000014	1.00000002	1.000000005	1.000000016	1.000000005
⋮	⋮	⋮	⋮	⋮	⋮	⋮
40	1.000000002	1.000000001	1.000000002	1	1.000000001	1
41	1.000000001	1.000000001	1.000000001	1	1.000000001	1
42	1.000000001	1	1.000000001	1	1	1
43	1	1	1	1	1	1
44	1	1	1	1	1	1
45	1	1	1	1	1	1

4. Conclusion

We have proved convergence and approximation of common fixed points for three multivalued ρ-nonexpansive mappings for three steps iterative scheme in modular function spaces and numerical assertion empathized the validity of our results. We may suggest to the reader that using the above ideas, one can prove the convergence and approximation of common fixed points for a finite family of multivalued ρ-quasi nonexpansive mappings. We would like to suggest the readers to combine the ideas studied, for example, in [7, 11].

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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