Communications in Mathematics and Applications
Vol. 11, No. 2, pp. 271–279, 2020
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v11i2.1328

Research Article

Spectral Analysis of Klein-Gordon Difference Operator Given by a General Boundary Condition

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Abstract. In this study, we consider the spectral properties of the non-selfadjoint difference operator L generated in $l_2(\mathbb{N})$ by the difference expression

$$\triangle(a_{n-1}\triangle y_{n-1})+(v_n-\lambda)^2 y_n=0, \quad n\in\mathbb{N},$$

and a general boundary condition

$$\sum_{n=0}^{\infty} h_n y_n=0,$$

where $a_0=1$, $h_0\neq 0$ and $\{a_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ are complex sequences and $\{h_n\}_{n=1}^{\infty}\in l_1(\mathbb{N})\cap l_2(\mathbb{N})$. Along with the designation of the sets of eigenvalues and spectral singularities of the operator L, we investigate the quantitative properties of these sets under certain conditions using the uniqueness theorems of analytic functions.

Keywords. Eigenparameter; Spectral analysis; Eigenvalues; Spectral singularities; Discrete equation; Klein-Gordon equation

MSC. 39A10; 39A12; 47A10; 47A75

Received: November 28, 2019 **Accepted:** December 19, 2019

1. Introduction

A large number of subject in quantum physics results in determining the eigenvalues and eigenfuctions of differential operators. For instance, Hamiltonian of a quantum particle confined to a box involves a choice of boundary condition at the box ends. Since different choices of boundary condition imply different physical models, spectral theory of operators with boundary condition constitues a progressing field of investigation [19].

The study of the spectral analysis of the non-selfadjoint Sturm Liouville operator can be traced back to Naimark [17,18]. In his article [17] the *boundary value problem* (BVP)

$$\begin{cases} -y'' + q(x)y - \lambda^2 y = 0, & x \in \mathbb{R}_+, \\ y'(0) - hy(0) = 0 \end{cases}$$

where $h \in \mathbb{C}$ and q is a complex valued function has been taken into consideration. He showed that the spectrum of this BVP is composed of eigenvalues, spectral singularities and continuous spectrum. He also proved that these eigenvalues and spectral singularities are of finite number with finite multiplicity under certain conditions.

Krall [14, 15] studied the operator L_0 generated by the *boundary value problem* (BVP) including a differential Sturm-Liouville equation of the form

$$-y'' + q(x)y = \lambda y, \quad 0 \le x < \infty, \tag{1.1}$$

where the potential function q is an arbitrary measurable complex function satisfying

$$\int_0^\infty |q(x)|dx < \infty,$$

and the integral boundary condition

$$\int_0^\infty K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0, \tag{1.2}$$

where $K \in L^2(\mathbb{R}_+)$ is a complex valued function and λ is a spectral parameter and α, β are complex numbers with $|\alpha|^2 + |\beta|^2 \neq 0$. He extended the work of Naimark by applying a suitable boundary condition (1.2) and generated the ordinary and nonhomogeneous expansion for L_0. The adjoint L_0^* of the operator L_0 was obtained in [15]. Note that L_0^* deserves a special interest, since it is not purely a differential operator, i.e. L_0^* is the combination of a differential operator and one-dimensional vector in $L^2(\mathbb{R}_+)$.

Later on, the integral boundary condition has been applied to differential Klein-Gordon, quadratic pencil of Schrödinger type operators and quantitative spectral properties of the new boundary value problems has been studied in [7,9,16].

Spectral analysis of the operators including Sturm-Liouville, Klein-Gordon, quadratic pencil of Schrödinger and Dirac type equations within the context of determination of Jost solution and providing sufficient conditions guaranteeing the finiteness of the eigenvalues and the spectral singularities has been major topic of the papers [4,6–9,16,21–23].

As a result of wide application areas of difference equations from physics to engineering, investigation of discrete analogues of well known differential operators has become a popular research area in recent years. Some basic concepts of the discrete analogue of the differential Sturm-Liouville equation in connection with the classical moment problem and Toda lattices has been investigated in detail in [3,20].

Guseinov [12] took into consideration the inverse problem of scattering theory for the discrete analogue of the Sturm-Liouville equation

$$a_n y_{n+1} + b_n y_n + a_{n-1} y_{n-1} = \lambda y_n, \tag{1.3}$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are real sequences, $a_n > 0$ and

$$\sum_{n\in\mathbb{Z}} |n|(|1-a_n|+|b_n|) < \infty.$$

The Jost solution and some quantitative properties of the discrete analogues of the non-selfadjoint Sturm-Liouville, Klein-Gordon and quadratic pencil of Schrödinger type operators (which include the equation (1.3) as a special case) has been studied in [1, 2, 5, 10, 13, 24]. In particular, discrete analogue of the BVP (1.1)-(1.2) has been treated in [5].

Note that the equation

$$y'' + [\lambda - p(x)]^2 y = 0, \quad x \in \mathbb{R}_+,$$

is called the Klein-Gordon s-wave equation in quantum physics for a particle of zero mass with static potential [6].

The present paper is motivated by the above mentioned studies.

In this study, we will consider the spectrum of the non-selfadjoint operator L generated by the discrete analogue of the Klein-Gordon equation

$$\triangle(a_{n-1}\triangle y_{n-1}) + (v_n - \lambda)^2 y_n = 0, \quad n \in \mathbb{N}, \tag{1.4}$$

and the general boundary condition

$$\sum_{n=0}^{\infty} h_n y_n = 0, \tag{1.5}$$

where $a_0 = 1$, $h_0 \neq 0$ and $\{a_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ are complex sequences and $\{h_n\}_{n=1}^{\infty} \in l_1(\mathbb{N}) \cap l_2(\mathbb{N})$.

Observe that the dependence on the spectral parameter λ is linear in the studies [2, 5, 13] while it is non-linear in (1.4). Thus, this study can be conceived as a generalization and extension of the papers [2, 5, 13] to the discrete Klein-Gordon operator case.

The remainder of the manuscript is organized as follows: In Section 2, we present the Jost solution and Green's function of the operator L. Section 3 deals with the eigenvalues and spectral singularities of L and investigate the quantitative properties of these eigenvalues and spectral singularities under certain conditions.

2. Solutions of *L*

Assume that

$$\sum_{n\in\mathbb{N}} n(|1-a_n|+|v_n|) < \infty, \tag{2.1}$$

holds. It is known from [1] that the equation (1.4) has the unique solution

$$f_n(z) = \alpha_n e^{inz}\left(1 + \sum_{m=1}^{\infty} K_{nm} e^{im\frac{z}{2}}\right), \quad n \in \mathbb{N}\cup\{0\}, \tag{2.2}$$

for $\lambda = 2\cos\left(\frac{z}{2}\right)$, $z \in \overline{\mathbb{C}}_+$. Note that the expressions of K_{nm} and α_n can be written uniquely in terms of (a_n) and (v_n). Moreover, the inequality

$$|K_{nm}| \le C \sum_{r=n+[|\frac{m}{2}|]}^{\infty} (|1-a_r|+|v_r|), \tag{2.3}$$

holds for the kernel K_{nm}, where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $C > 0$ is a constant. Therefore, $f_n(z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z \geq 0\}$. $f_n(z)$ is introduced as the Jost solution of the equation (1.4). Moreover, the following asymptotics is found

$$f_n(z) = \exp(inz)[1 + o(1)], \qquad n \to \infty,$$
$$f_n(z) = \alpha_n \exp(inz)[1 + o(1)], \quad \operatorname{Im} z \to \infty.$$

Let us define $\varphi_n(z)$ as the solution of (1.3) subject to the conditions

$$\varphi_0(z) = 0, \quad \varphi_1(z) = 1,$$

where

$$\varphi_n(z) = \widehat{\varphi}_n(\lambda) = \left\{\widehat{\varphi}_n\left(2\cos\frac{z}{2}\right)\right\}, \quad z \in \overline{\mathbb{C}}_+,\ n \in \mathbb{N} \cup \{0\}.$$

It is clear that, φ is entire function and

$$\varphi(z) = \varphi(z + 4\pi).$$

If we use the usual definition of Wronskian, we obtain

$$\begin{aligned} W[f,\varphi] = W[f_n(z),\varphi_n(z)] &= a_n[f_n(z)\varphi_{n+1}(z) - f_{n+1}(z)\varphi_n(z)] \\ &= f_0(z), \quad z \in \overline{\mathbb{C}}_+. \end{aligned}$$

Let us introduce the functions

$$N(z) := \sum_{n=0}^{\infty} h_n f_n(z),$$
$$\widetilde{N}(z) := \sum_{n=0}^{\infty} h_n \varphi_n(z),$$
$$S_k(z) := \frac{-1}{W[f,\varphi]}\left\{N(z)\varphi_{k+1}(z) - \widetilde{N}(z) f_{k+1}(z) - \sum_{n=k+1}^{\infty} h_n f_n(z)\varphi_{k+1}(z) + \sum_{n=k+1}^{\infty} h_n \varphi_n(z) f_{k+1}(z)\right\}.$$

We also define the semi-strips $P_0 := \{z : z \in \mathbb{C}, z = \xi + i\tau, -0 \leq \xi \leq 4\pi, \tau > 0\}$ and $P := P_0 \cup [0, 4\pi]$.

For all $z \in P$ and $f_0(z) \neq 0$, the Green's function of the operator L is obtained by the standard techniques as

$$G_{nk}(z) = G_{nk}^{(1)}(z) + G_{nk}^{(2)}(z),$$

where

$$G_{nk}^{(1)}(z) := \frac{f_n(z) S_k(z)}{N(z)}, \tag{2.4}$$

and

$$G_{nk}^{(2)}(z) := \begin{cases} 0, & k < n, \\ \frac{\varphi_{k+1}(z) f_n(z) - \varphi_n(z) f_{k+1}(z)}{W[f,\varphi]}, & k \geq n. \end{cases} \tag{2.5}$$

Hence, for $\phi = \{\phi_k\} \in l_2(\mathbb{N})$, $k \in \mathbb{N} \cup \{0\}$, we obtain that

$$\left(R_\lambda(L)\phi\right)_n := \sum_{k=0}^{\infty} G_{nk}(z)\phi_{k+1}, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.6}$$

is the resolvent of the operator L.

3. Eigenvalues and Spectral Singularities of *L*

Let us denote the set of eigenvalues and spectral singularities of L by σ_d and σ_{ss}, respectively. From (2.4)-(2.6) and definition of the eigenvalues and the spectral singularities, we get

$$\sigma_d = \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in P_0, N(z) = 0\right\}, \tag{3.1}$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in P, N(z) = 0\right\}. \tag{3.2}$$

We define the sets

$$A_1 := \{z : z \in P_0, N(z) = 0\},$$

$$A_2 := \{z : z \in P, N(z) = 0\},$$

and A_3 and A_4 as the sets of accumulation points of the sets A_1 and A_2, respectively, and A_5 as the set of zeros in P_0 of $N(z)$ with infinite multiplicity. It can be seen that

$$A_1 \cap A_5 = \emptyset, \quad A_3 \subset A_2, \quad A_4 \subset A_2, \quad A_5 \subset A_2,$$

and the linear Lebesgue measures of A_2, A_3, A_4 and A_5 are zero. From the continuity of the all derivatives of $N(z)$ on the real axis, we find

$$A_3 \subset A_5 \text{ and } A_4 \subset A_5. \tag{3.3}$$

Also, we can rewrite the sets of eigenvalues and spectral singularities of L as

$$\sigma_d = \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in A_1\right\},$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in A_2\right\}.$$

Theorem 3.1. *If the conditions* (2.1) *and* $\{h_n\}_{n=1}^{\infty} \in l_1(\mathbb{N}) \cap l_2(\mathbb{N})$ *hold, then*

(i) *The set of eigenvalues of L is bounded, countable and its limit points can lie only in* $[-2,2]$.

(ii) $\sigma_{ss} \subset [-2,2]$, $\sigma_{ss} = \overline{\sigma_{ss}}$, *and* $\mu(\sigma_{ss}) = 0$ *where* μ *denotes the linear Lebesgue measure.*

Proof. From (2.3), we get the analycity of $N(z)$ in the upper half-plane and continuity of $N(z)$ in the real axis. In addition to this, the asymptotic

$$N(z) = \alpha_0 h_0 [1 + o(1)], \quad \operatorname{Im} z > 0,\ \operatorname{Im} z \to \infty, \tag{3.4}$$

satisfies. Making use of (3.1), (3.2) and (3.4) and uniqueness theorems of analytic functions [11], we obtain (i) and (ii). □

Definition 3.1. *The multiplicity of a zero of* $N(z)$ *in* P *is called the multiplicity of the corresponding eigenvalue or spectral singularity of the operator* L.

Now, we will consider the condition

$$\sum_{n=1}^{\infty} e^{\varepsilon n} (|1 - a_n| + |v_n| + |h_n|) < \infty, \quad \varepsilon > 0. \tag{3.5}$$

Theorem 3.2. *If* (3.5) *holds, then L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Proof. From (2.3) and (3.5), we get

$$|K_{nm}| \le C \exp\left(\frac{-\varepsilon}{2}(n+m)\right), \tag{3.6}$$

for all $C > 0$ is a constant, $n = 0,1,2,\dots$ and $m = 1,2,\dots$. Using (3.6) and (2.2), we have

$$|N(z)| \le \sum_{m=1}^{\infty} e^{-m\left(\frac{\varepsilon}{4}+\operatorname{Im}\frac{z}{2}\right)}. \tag{3.7}$$

It is seen from (3.7) that $N(z)$ has analytic continuation to the half-plane $\operatorname{Im} z > \frac{-\varepsilon}{2}$. Because of $N(z)$ is a 4π periodic, the limit points of its zeros in P cannot lie in $[0,4\pi]$. For this reason and using Theorem 3.1, we find the finiteness of eigenvalues and spectral singularities of L. □

The condition (3.5) ensures the analytic continuation of $N(z)$ from the real axis to the lower half-plane. Now, we will consider the condition

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\beta}} (|1-a_n| + |v_n| + |h_n|) < \infty, \quad \varepsilon > 0, \ \frac{1}{2} \le \beta < 1, \tag{3.8}$$

which is weaker then (3.5). Obviously, $N(z)$ is analytic in the upper half-plane and infinitely differentiable on the real axis. However $N(z)$ does not have an analytic continuation from the real axis to the lower half-plane. Thus, a different method to investigate the finiteness of the eigenvalues and spectral singularities of L has to be thought. We will use the following lemma.

Lemma 3.1 ([2])**.** *Assume that the 4π periodic function ξ is analytic in the open half-plane, all of its derivatives are continuous in the closed upper half-plane and*

$$\sup_{z\in P} |\xi^{(k)}(z)| \le \eta_k, \quad k \in \mathbb{N} \cup \{0\}. \tag{3.9}$$

If the set G with linear Lebesgue measure zero is the set of all zeros of the function ξ with infinite multiplicity in P, if

$$\int_0^{\omega} \ln t(s)\, d\mu(G_s) > -\infty,$$

where $t(s) = \inf\limits_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, $\mu(G_s)$ is the Lebesgue measure of the s-neighborhood of G, and $\omega \in (0,4\pi)$ is an arbitrary constant, then $\xi \equiv 0$.

Theorem 3.3. *Under the condition* (3.8)*, $A_5 = \emptyset$.*

Proof. We will apply the previous lemma to our case. Using (3.8), (2.2) and (2.3), the following inequality can be written

$$|N^{(k)}(z)| \le \eta_k, \quad k \in \mathbb{N} \cup \{0\},$$

for the k.th derivative of $N(z)$ where

$$\eta_k = 2^k C \sum_{m=1}^{\infty} m^k \exp(-\varepsilon m^{\beta}),$$

and $C > 0$ is a constant. Also, we obtain the following estimation

$$\eta_k \le 2^k C \int_0^{\infty} x^k e^{-\varepsilon x^{\beta}} dx \le D d^k k! k^{k\frac{1-\beta}{\beta}}, \tag{3.10}$$

where D and d are constants depending C, ε and β.

Hence, we can write that

$$\int_0^{\omega} \ln t(s) d\mu(A_{5,s}) > -\infty, \tag{3.11}$$

where $t(s) = \inf_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, $\mu(A_{5,s})$ is the Lebesgue measure of the s-neighborhood of A_5 and η_k is defined by (3.10).

Now, we have

$$t(s) \le D \exp\left\{-\frac{1-\beta}{\beta} e^{-1} d^{-\frac{\beta}{1-\beta}} s^{-\frac{\beta}{1-\beta}}\right\}, \tag{3.12}$$

by (3.10). From (3.11) and (3.12), we get

$$\int_0^{\omega} s^{-\frac{\beta}{1-\beta}} d\mu(A_{5,s}) < \infty. \tag{3.13}$$

Since $\frac{\beta}{1-\beta} \ge 1$ (3.13) holds for arbitrary s if and only if $\mu(A_{5,s}) = 0$ or $A_5 = \emptyset$. □

Now, we can present the major theorem of our study using the previous result.

Theorem 3.4. *Assume that* (3.8) *holds, then L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. We are supposed to show that the function $N(z)$ has a finite number of zeros with finite multiplicities in P. From (3.3) and the previous theorem, it is seen that $A_3 = A_4 = \emptyset$. Hence, the bounded sets A_1 and A_2 do not have limit points, i.e., $N(z)$ has only finite number of zeros in P. Since, $A_5 = \emptyset$, these zeros are of finite multiplicity. □

4. Conclusion

In this paper, we mainly take into consideration the spectrum of the boundary value problem including Klein-Gordon difference operator and a general boundary condition. We present the Jost solution of the problem, and state the Green's function. After determining the sets of eigenvalues and spectral singularities of the problem, we investigate the quantitative properties of these sets under the Naimark's and Pavlov's conditions using the uniqueness theorems of analytic functions.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers for their valuable comments and constructive suggestions to improve the quality of this paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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