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Research Article

Generalized Apostol Type Polynomials Based on Twin-Basic Numbers

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Abstract. In this work, we consider a class of new generating function for (p,q)-analog of Apostol type polynomials of order α including Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α. By making use of their generating function, we derive some useful identities. We also introduce the generating functions of (p,q)-analogues of the Stirling numbers of second kind of order τ and the Bernstein polynomials by which we construct diverse correlations including aforementioned polynomials and the (p,q)-gamma function.

Keywords. (p,q)-calculus; Apostol-Bernoulli polynomials; Apostol-Euler polynomials; Apostol-Genocchi polynomials; Stirling numbers of second kind; Bernstein polynomials; Gamma function; Generating function; Cauchy product

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1. Introduction

Special functions possess a lot of importances in many fields of mathematics, physics, engineering and other related disciplines including the topics such as differential equations, mathematical analysis, functional analysis, mathematical physics, quantum mechanics and so on (cf. [1–33, 35–40], see also the references cited therein). Special polynomials in special functions have also intensive study fields (cf. [1,3,5,7–26,28–31,35–40], see also the references cited therein). One of the significant families of polynomials of special polynomials is the family of Apostol type polynomials that are primarily introduced by Apostol [3] in 1951 and also Srivastava [34] in 2000. The aforesaid type polynomials with their multifarious extensions have been studied and developed by several authors, some of whom are Açíkgöz, Apostol, Araci, Bayad, Dere, Duran, Natalini, Fugère, Gaboury, He, Jia, Kurt, Luo, Mahmudov, Özarslan, Simsek, Srivastava and Tremblay (see [3,5,7–9,11–13,15–19,21–24,29,30,34,38,40]). Diverse q-extensions of Apostol type Bernoulli, Euler and Genocchi polynomials were considered and investigated by many authors such as Choi, Anderson, Kurt, Mahmudov, Keleshteri, Luo, Simsek and Srivastava in [5, 8, 20, 25, 37]. Moreover, (p,q)-generalizations of Apostol type polynomials and numbers were worked and examined by Duran *et al.* in [9] and [11].

Unification of classical and Apostol type polynomials were given and discussed by El-Desouky, Gomaa, Kurt, Özarslan, Ozden, Simsek and Srivastava in [8,11,17,18,29–31]. Recently, unified q-polynomials and numbers were given and investigated by Kurt in [15]. Furthermore, (p,q)-analogues of the unified polynomials and numbers have been provided and analyzed by Duran *et al.* in [10] and [11].

Throughout of the paper, we make use of the following notations: $\mathbb{N} := \{1,2,3,\cdots\}$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials of degree α are defined as follows:

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x,\lambda)\frac{z^n}{n!} = \left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz}$$

$(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log\lambda| \text{ when } \lambda \in \mathbb{C}\setminus\{1\}; 1^{\alpha} := 1)$

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x,\lambda)\frac{z^n}{n!} = \left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{xz}$$

$(|z| < \pi \text{ when } \lambda = 1; |z| < |\log(-\lambda)| \text{ when } \lambda \in \mathbb{C}\setminus\{1\}; 1^{\alpha} := 1)$

and

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,\lambda)\frac{z^n}{n!} = \left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} e^{xz}.$$

$(|z| < \pi \text{ when } \lambda = 1; |z| < |\log(-\lambda)| \text{ when } \lambda \in \mathbb{C}\setminus\{1\}; 1^{\alpha} := 1)$

The (p,q)-Bernoulli polynomials $B_n^{(\alpha)}(x,y:p,q)$, the (p,q)-Euler polynomials $E_n^{(\alpha)}(x,y:p,q)$ and the (p,q)-Genocchi polynomials $G_n^{(\alpha)}(x,y:p,q)$ of degree α are defined by Duran *et al.* in [9], as follows:

$$\sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{z^m}{e_{p,q}(z) - \sum_{l=0}^{m-1}\frac{z^l}{[l]_{p,q}!}}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz), \tag{1.1}$$

$$\sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2}{e_{p,q}(z) + \sum_{l=0}^{m-1}\frac{z^l}{[l]_{p,q}!}1}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz) \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} G_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2z}{e_{p,q}(z)+\sum_{l=0}^{m-1}\frac{z^l}{[l]_{p,q}!}}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz), \tag{1.3}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0$ a nonnegative integer, and $p,q \in \mathbb{C}$ with the condition $0<|q|<|p|\leq 1$.

2. Background and Preliminaries

During the last century and also this century, applications of *quantum calculus* based on q-numbers have been studied and investigated successfully, densely and considerably (see [5, 7, 15, 20, 24–26, 35, 36, 38]). In conjunction with the motivation and inspiration of these applications, with the introduction of the (p,q)-number, many physicists and mathematicians have widely investigated the theory of *post quantum calculus* based on (p,q)-numbers along the traditional lines of classical and quantum cases. In recent years, Duran *et al.* [9] introduced (p,q)-extensions of Apostol-Bernoulli Apostol-Euler and Apostol-Genocchi polynomials and obtained some of their properties including addition theorems, difference equations, derivative properties, recurrence relationships, and so on. Also, (p,q)-analogues of some familiar formulae belonging to usual Apostol-Bernoulli, Euler and Genocchi polynomials including Cheon's main result [G. S. Cheon, *Appl. Math. Lett.* **16** (2003), 365 – 368] and the formula of Srivastava and Pintér [37] were provided. Duran *et al.* [10,11] studied on the unification of the (p,q)-polynomials and numbers covering (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials and also they gave many applicaitons for the mentioned unifications. Komatsu *et al.* [14] considered a (p,q)-analogue of the poly-Euler polynomials and numbers by using the (p,q)-polylogarithm function and then presented multifarious combinatorial identities and properties of the foregoing polynomials. Sadjang [33] satisfied some properties of the (p,q)-derivatives and the (p,q)-integrations. As an application, the author presented two (p,q)-Taylor formulas for polynomials and derived the fundamental theorem of (p,q)-calculus. Furthermore, Sadjang [32] provided (p,q)-extensions of the famous functions including gamma and beta functions and showed some applications.

Let us now brief some tools in (p,q)-calculus which will be useful in deriving the results of the paper. The (p,q)-number is defined by $[n]_{p,q}=\frac{p^n-q^n}{p-q}$ $(p\neq q)$ (see [4,6,9–11,14,32,33]). Obviously that when $p=1$, we have $[n]_q=\frac{1-q^n}{1-q}$ that stands for q-number. One can see that (p,q)-number is closely related to q-number with this relation $[n]_{p,q}=p^{n-1}[n]_{\frac{q}{p}}$. By appropriately using this obvious relation between the q-notation and its variant, the (p,q)-notation, most (if not all) of the (p,q)-results can be derived from the corresponding known q-results by merely changing the parameters and variables involved.

The (p,q)-derivative operator given by

$$D_{p,q;x}f(x) := D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x} \quad (x\neq 0) \text{ with } (D_{p,q}f)(0)=f'(0). \tag{2.1}$$

The (p,q)-power basis is also defined by

$$(x\oplus a)_{p,q}^n = (x+a)(px+aq)(p^2x+aq^2)\cdots(p^{n-1}x+aq^{n-1})$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k} \tag{2.2}$$

and

$$\begin{aligned}(x \ominus a)_{p,q}^{n} &= (x-a)(px-aq)(p^2x-aq^2)\cdots(p^{n-1}x-aq^{n-1}) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k (-1)^{n-k} a^{n-k},\end{aligned} \tag{2.3}$$

which are extended to

$$(x \oplus a)_{p,q}^{n} = \prod_{n=0}^{\infty} (p^n x + aq^n) \text{ and } (x \ominus a)_{p,q}^{n} = \prod_{n=0}^{\infty} (p^n x - aq^n), \tag{2.4}$$

where the convergence is required.

Here the notations $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q}$ ((p,q)-binomial coefficients) and $[n]_{p,q}!$ ((p,q)-factorial numbers) are $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} (n \geq k)$ and $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [2]_{p,q}[1]_{p,q}$ $(n \in \mathbb{N})$ with the initial condition $[0]_{p,q}! = 1$.

Let

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \text{ and } E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \tag{2.5}$$

denote two types of (p,q)-exponential functions satisfying the relations $e_{p,q}(x)E_{p,q}(-x) = 1$ and $e_{p^{-1},q^{-1}}(x) = E_{p,q}(x)$. Also, the following (p,q)-derivative representations

$$D_{p,q} e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q} E_{p,q}(x) = E_{p,q}(qx) \tag{2.6}$$

holds true.

The definite (p,q)-integral for a function f is defined by

$$\int_0^a f(x) d_{p,q} x = (p-q) a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} a\right), \quad \left(\left|\frac{q}{p}\right| < 1\right). \tag{2.7}$$

For further information (p,q)-calculus used in this paper, one can look at [4, 6, 9–11, 14, 32, 33] and cited references therein.

The (p,q)-gamma function is defined by (cf. [32])

$$\Gamma_{p,q}(x) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^x \ominus q^x)_{p,q}^{\infty}} (p-q)^{1-x}, \tag{2.8}$$

where $x \in \mathbb{C}$ and $0 < |q| < |p| \leq 1$. For $n \in \mathbb{N}$, $\Gamma_{p,q}(n+1) = [n]_{p,q}!$.

In the year 1903, the Swedish mathematician Gosta Mittag Leffler introduced the function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}; \mathrm{Re}(\alpha) > 0).$$

Motivated by this identity, the various generalizations have been studied. Now, we consider a new generalization of Mittag Leffler's function as follows:

Let $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ and $0 < |q| < |p| \leq 1$. We define the function $E_{\alpha,\beta}^{\gamma}(t; p,q)$ by the following relation:

$$E_{\alpha,\beta}^{\gamma}(t; p,q) = \sum_{n=0}^{\infty} \frac{(p^\gamma \ominus q^\gamma)_{p,q}^{n}}{(p \ominus q)_{p,q}^{n}} \frac{t^n}{\Gamma_{p,q}(\alpha n + \beta)}. \tag{2.9}$$

Substituting $p=1$, the function $E^{\gamma}_{\alpha,\beta}(t;p,q)$ reduces to the function $E^{\gamma}_{\alpha,\beta}(t;q)$ given in [5].

Upon setting $\alpha=\gamma=1$ and $\beta=m+1$, we obtain

$$\begin{aligned}E^{1}_{1,m+1}(t;p,q) &:= E_{m+1}(t;p,q)\\ &= \sum_{n=0}^{\infty}\frac{t^n}{\Gamma_{p,q}(n+m+1)}\\ &= \sum_{n=0}^{\infty}\frac{t^n}{[n+m]_{p,q}!}\\ &= \frac{1}{t^m}\sum_{k=m}^{\infty}\frac{t^k}{[k]_{p,q}!}\\ &= \frac{e_{p,q}(t)-\sum_{k=0}^{m-1}\frac{t^k}{[k]_{p,q}!}}{t^m}.\end{aligned}$$

In conjunction with these motivation, we consider the following (p,q)-Mittag-Leffler type function

$$E^{(\beta;\omega;\mu;\nu)}_{m+1}(t;p,q)=\frac{(2^{\mu}t^{\nu})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m}\frac{t^l}{[l]_{p,q}!}}, \tag{2.10}$$

where $m\in\mathbb{N}$, $\beta,\omega,\mu,\nu,p,q\in\mathbb{C}$ and $0<|q|<|p|\le 1$. When $p=1$ and $\omega=1$, then the function $E^{(\beta;\omega;\mu;\nu)}_{m+1}(t;p,q)$ reduces to the q-Mittag-Leffler type function function defined by Castilla *et al.* in [5].

Let $C[0,1]$ denote the set of all continuous functions on $[0,1]$. For any $f\in C[0,1]$, the notation $B_{n,p,q}(f;x)$ is called (p,q)-Bernstein operator of order n for f and is defined by Mursaleen *et al.* [27]

$$B_{n,p,q}(f;x)=\frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}f_k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}x^k(1\ominus x)^{n-k}_{p,q}=\sum_{k=0}^{n}f_k\mathfrak{B}^{p,q}_{n,k}(x), \tag{2.11}$$

where $f_k=f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right)$ and $\binom{k}{2}=\frac{k(k-1)}{2}$. For $n,r\in\mathbb{N}_0$, the (p,q)-Bernstein polynomials of degree n or (p,q)-Bernstein basis is defined by

$$\mathfrak{B}^{p,q}_{n,k}(x)=p^{\binom{k}{2}-\binom{n}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k\,(1\ominus x)^{n-k}_{p,q}\,. \quad \text{(cf. [2])} \tag{2.12}$$

The generating function of the (p,q)-Bernstein polynomials of degree n is introduced by Agyuz and Acikgoz [2] as follows:

$$p^{\binom{k}{2}-\binom{n}{2}}\frac{x^kt^k}{[k]_{p,q}!}\frac{e_{p,q}(t)}{e_{p,q}(xt)}=\sum_{n=k}^{\infty}\mathfrak{B}^{p,q}_{n,k}(x)\frac{t^n}{[n]_{p,q}!}. \tag{2.13}$$

When $p=1$, the mentioned polynomials reduces to the q-Bernstein polynomials given in [5].

In the next section, we perform to define the family of unified (p,q)-analog of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α and to investigate some properties of them. Moreover, we consider (p,q) analog of a new generalization of Stirling numbers of the second kind of order μ by which we derive a relation including unified (p,q)-analog of Apostol type polynomials of order α.

3. Generalized (p,q)-extensions of Apostol Type Polynomials of Order α

Motivated and inspired by the following generating function introduced by Castilla *et al.* [5]

$$\left(\frac{(2^{\mu}t^{\nu})^{m}}{\lambda e_{q}(t)-\sum_{l=0}^{m-1}\frac{t^{l}}{[l]_{q}!}}\right)^{\alpha}e_{q}(xt)E_{q}(yt)=\sum_{n=0}^{\infty}T_{n,q}^{[m-1,\alpha]}(x,y;\lambda;\mu;\upsilon)\frac{t^{n}}{[n]_{q}!}$$

$$(|t|<|\log_{q}(-\lambda)| \text{ when } \lambda\in\mathbb{C}/\{-1,1\};1^{\alpha}=1;m\in\mathbb{N};\alpha,\lambda,\mu,\nu,q\in\mathbb{C} \text{ and } 0<|q|<1)$$

in this paper, we consider the following Definition 1 based on (p,q)-numbers.

Definition 1. Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. An extended unified (p,q)-analog of Apostol type polynomials of order α are defined via the following exponential generating function (in a suitable neigbourhood of $t=0$) including two different types of the (p,q)-exponential functions as given below:

$$\begin{aligned}\Theta_{m}^{(\alpha)}(x,y;t;\mu;\nu;\beta;\omega:p,q)&=\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\upsilon;\omega)\frac{t^{n}}{[n]_{p,q}!}\\&=\left(\frac{(2^{\mu}t^{\nu})^{m}}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^{l}}{[l]_{p,q}!}}\right)^{\alpha}e_{p,q}(xt)E_{p,q}(yt).\end{aligned}\tag{3.1}$$

When $x=y=0$, $T_{n,p,q}^{[m-1,\alpha]}(0,0;\beta;\mu;\upsilon;\omega):=T_{n,p,q}^{[m-1,\alpha]}(\beta;\mu;\upsilon;\omega)$ which are called an extended unified (p,q)-Apostol type numbers. We note that $T_{n,p,q}^{[m-1,1]}(x,y;\beta;\mu;\upsilon;\omega):=T_{n,p,q}^{[m-1]}(x,y;\beta;\mu;\upsilon;\omega)$ which are called an extended unified (p,q)-Apostol type polynomials.

Remark 1. When $p=1$ and $q\to 1$ in Definition 1, it reduces to Castilla's definition given in [5].

Remark 2. When $q\to p=1$ in Definition 1, we get an extended generalized Apostol type polynomials defined by

$$\sum_{n=0}^{\infty}T_{n}^{[m-1,\alpha]}(x,y;\beta;\mu;\upsilon;\omega)\frac{t^{n}}{n!}=\left(\frac{(2^{\mu}t^{\nu})^{m}}{\beta e^{t}-\omega\sum_{l=0}^{m-1}\frac{t^{l}}{l!}}\right)^{\alpha}e^{(x+y)t}.\tag{3.2}$$

We now give here some basic properties for $T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\upsilon;\omega)$ by the following four Lemmas 1-4 without proofs, since they can readily be proved by using Definition 1.

Lemma 1 (Addition property). *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. For $\alpha,\psi\in\mathbb{N}_0$, $T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\upsilon;\omega)$ satisfies the following relations:*

$$T_{n,p,q}^{[m-1,\alpha+\psi]}(x,y;\beta;\mu;\upsilon;\omega)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}T_{j,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;\upsilon;\omega)T_{n-j,p,q}^{[m-1,\mu]}(0,y;\beta;\mu;\upsilon;\omega),\tag{3.3}$$

$$T_{n,p,q}^{[m-1,\alpha+\psi]}(x,y;\beta;\mu;\upsilon;\omega)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}T_{j,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\upsilon;\omega)T_{n-j,p,q}^{[m-1,\mu]}(\beta;\mu;\upsilon;\omega),\tag{3.4}$$

$$T_{n,p,q}^{[m-1,\alpha+\psi]}(x,y;\beta;\mu;\upsilon;\omega)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}T_{j,p,q}^{[m-1,\alpha]}(\beta;\mu;\upsilon;\omega)T_{n-j,p,q}^{[m-1,\mu]}(x,y;\beta;\mu;\upsilon;\omega).\tag{3.5}$$

It immediately follows from Definition 1 that

$$T^{[m-1,0]}_{n,p,q}(x,y;\beta;\mu;v;\omega) = (x+y)^n_{p,q}, \tag{3.6}$$

$$T^{[m-1,0]}_{n,p,q}(x,0;\beta;\mu;v;\omega) = p^{\binom{n}{2}}x^n, \tag{3.7}$$

$$T^{[m-1,0]}_{n,p,q}(0,y;\beta;\mu;v;\omega) = q^{\binom{n}{2}}y^n. \tag{3.8}$$

So, we give the following results.

Lemma 2. *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. We have*

$$T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} x^{n-j}p^{\binom{n-j}{2}}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega), \tag{3.9}$$

$$T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} y^{n-j}q^{\binom{n-j}{2}}T^{[m-1,\alpha]}_{j,p,q}(x,0;\beta;\mu;v;\omega), \tag{3.10}$$

$$T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} (x\oplus y)^{n-j}_{p,q}T^{[m-1,\alpha]}_{j,p,q}(\beta;\mu;v;\omega). \tag{3.11}$$

Lemma 3 (Derivative properties). *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. We have*

$$D_{p,q;x}T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega) = [n]_{p,q}T^{[m-1,\alpha]}_{n-1,p,q}(px,y;\beta;\mu;v;\omega), \tag{3.12}$$

$$D_{p,q;y}T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega) = [n]_{p,q}T^{[m-1,\alpha]}_{n-1,p,q}(x,qy;\beta;\mu;v;\omega). \tag{3.13}$$

Lemma 4 (Integral representations). *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. Then,*

$$\int_a^b T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega)d_{p,q}x = \frac{T^{[m-1,\alpha]}_{n+1,p,q}\left(\frac{b}{p},y;\beta;\mu;v;\omega\right) - T^{[m-1,\alpha]}_{n+1,p,q}\left(\frac{a}{p},y;\beta;\mu;v;\omega\right)}{[n+1]_{p,q}}, \tag{3.14}$$

$$\int_c^d T^{[m-1,\alpha]}_{n,p,q}(x,y;\beta;\mu;v;\omega)d_{p,q}y = \frac{T^{[m-1,\alpha]}_{n+1,p,q}\left(x,\frac{d}{q};\beta;\mu;v;\omega\right) - T^{[m-1,\alpha]}_{n+1,p,q}\left(x,\frac{c}{q};\beta;\mu;v;\omega\right)}{[n+1]_{p,q}}. \tag{3.15}$$

Here is a difference relation as shown below.

Theorem 1. *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. We have*

$$\begin{aligned} T^{[m-1,\alpha-1]}_{n,p,q}(x,-1;\beta;\mu;v;\omega) = \frac{2^{-\mu m}[n]_{p,q}!}{[n+\nu m]_{p,q}!}\Big(&\beta T^{[m-1,\alpha]}_{n+\nu m,p,q}(x,0;\beta;\mu;v;\omega) \\ &-\omega\sum_{l=0}^{\min(n+\nu m,m-1)}\begin{bmatrix} n+\nu m \\ l \end{bmatrix}_{p,q} T^{[m-1,\alpha]}_{n+\nu m-l,p,q}(x,-1;\beta;\mu;v;\omega)\Big). \end{aligned} \tag{3.16}$$

Proof. By Definition 1, we see that

$$\sum_{n=0}^{\infty} T^{[m-1,\alpha-1]}_{n,p,q}(x,-1;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!} = \left(\sum_{n=0}^{\infty} T^{[m-1,\alpha]}_{n,p,q}(x,0;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right) \cdot\left(\frac{\beta e_{p,q}(t)E_{p,q}(-t)-\omega E_{p,q}(-t)\sum_{l=0}^{m}\frac{t^l}{[l]_{p,q}!}}{(2^{\mu}t^{\nu})^m}\right)$$

by the property $e_{p,q}(t)E_{p,q}(-t)=1$, we have

$$=2^{-\mu m}\left(\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right)\left(\beta-\omega E_{p,q}(-t)\sum_{l=0}^{m}\frac{t^l}{[l]_{p,q}!}\right)t^{-vm}$$

$$=2^{-\mu m}\sum_{n=0}^{\infty}\left(\beta T_{n,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;v;\omega)-\omega\sum_{l=0}^{\min(n,m-1)}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}T_{n-l,p,q}^{[m-1,\alpha]}(x,-1;\beta;\mu;v;\omega)\right)\frac{t^{n-vm}}{[n]_{p,q}!},$$

which gives the desired result. □

We give the following difference equation.

Theorem 2. *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,v,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. We have*

$$T_{n,p,q}^{[m-1,\alpha-1]}(0,y;\beta;\mu;v;\omega)=\frac{2^{-\mu m}[n]_{p,q}!}{[n+vm]_{p,q}!}\Bigg(\beta T_{n+vm,p,q}^{[m-1,\alpha]}(1,y;\beta;\mu;v;\omega)$$
$$-\omega\sum_{l=0}^{\min(n+vm,m-1)}\begin{bmatrix}n+vm\\l\end{bmatrix}_{p,q}T_{n+vm-l,p,q}^{[m-1,\alpha]}(0,y;\beta;\mu;v;\omega)\Bigg). \tag{3.17}$$

Proof. By Definition 1, we see that

$$\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,\alpha-1]}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,\alpha]}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right)\left(\frac{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}{(2^{\mu}t^{v})^m}\right)$$

by using Cauchy product on the above, we have

$$=2^{-\mu m}\left(\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,\alpha]}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right)\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)t^{-vm}$$

$$=2^{-\mu m}\sum_{n=0}^{\infty}\left(\beta T_{n,p,q}^{[m-1,\alpha]}(1,y;\beta;\mu;v;\omega)-\omega\sum_{l=0}^{\min(n,m-1)}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}T_{n-l,p,q}^{[m-1,\alpha]}(0,y;\beta;\mu;v;\omega)\right)\frac{t^{n-vm}}{[n]_{p,q}!}.$$ □

Upon setting $\alpha=1$ in (3.17), we give the following corollary.

Corollary 1. *We have*

$$y^n=\frac{2^{-\mu m}[n]_{p,q}!}{q^{\binom{n}{2}}[n+vm]_{p,q}!}\Bigg(\beta T_{n+vm,p,q}^{[m-1]}(1,y;\beta;\mu;v;\omega)$$
$$-\omega\sum_{l=0}^{\min(n+vm,m-1)}\begin{bmatrix}n+vm\\l\end{bmatrix}_{p,q}T_{n+vm-l,p,q}^{[m-1]}(0,y;\beta;\mu;v;\omega)\Bigg). \tag{3.18}$$

Here is a recurrence relation of extended unified (p,q)-analog of Apostol type polynomials by the following theorem.

Theorem 3. *Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q \in \mathbb{C}$ with $0<|q|<|p|\leq 1$. The following relationship holds true for the family of the polynomials $\{T_{n,p,q}^{[m-1]}(x,y;\beta;\mu;\nu;\omega)\}$:*

$$\omega \sum_{l=0}^{\min(n,m-1)} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} T_{n-l,p,q}^{[m-1]}(x,y;\beta;\mu;\nu;\omega) = \beta \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} T_{j,p,q}^{[m-1]}(x,y;\beta;\mu;\nu;\omega) - \frac{[n]_{p,q}!}{[n-\nu m]_{p,q}!} 2^{\mu m}(x\oplus y)_{p,q}^{n-\nu m}. \tag{3.19}$$

Proof. Since

$$\frac{\omega \sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)e_{p,q}(t)} = \frac{\beta}{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)} - \frac{1}{e_{p,q}(t)},$$

we have

$$\frac{(2^\mu t^\nu)^m e_{p,q}(xt)E_{p,q}(yt)\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)e_{p,q}(t)} = \beta\frac{(2^\mu t^\nu)^m e_{p,q}(xt)E_{p,q}(yt)}{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)} - \frac{(2^\mu t^\nu)^m e_{p,q}(xt)E_{p,q}(yt)}{e_{p,q}(t)},$$

$$\frac{(2^\mu t^\nu)^m e_{p,q}(xt)E_{p,q}(yt)}{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)}\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!} = \beta\frac{(2^\mu t^\nu)^m e_{p,q}(xt)E_{p,q}(yt)}{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)}e_{p,q}(t) - \left(2^\mu t^\nu\right)^m e_{p,q}(xt)E_{p,q}(yt),$$

which means that

$$\omega\sum_{n=0}^{\infty}\sum_{l=0}^{\min(n,m-1)} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} T_{n-l,p,q}^{[m-1]}\left(x,y;\beta;\mu;\nu;\omega\right)\frac{t^n}{[n]_{p,q}!} = \beta\sum_{n=0}^{\infty} T_{n,p,q}^{[m-1]}\left(x,y;\beta;\mu;\nu;\omega\right)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} p^{\binom{n}{2}}\frac{t^n}{[n]_{p,q}!} - 2^{\mu m}\sum_{n=0}^{\infty}(x\oplus y)_{p,q}^n\frac{t^{n+\nu m}}{[n]_{p,q}!}.$$

Using Cauchy product and then equating the coefficients of $\frac{t^n}{[n]_{p,q}}$ completes the proof. □

We provide now the following explicit formula for unified (p,q)-analog of Apostol type polynomials of order α.

Theorem 4. *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\leq 1$. The unified polynomial $T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\nu;\omega)$ holds the following relation:*

$$T_{n,p,q}^{[m-1,\alpha]}\left(x,y;\beta;\mu;\nu;\omega\right) = \frac{2^{\mu m}[n]_{p,q}!}{[n+\nu m]_{p,q}!}\sum_{j=0}^{n+\nu m}\begin{bmatrix} n+\nu m \\ j \end{bmatrix}_{p,q} T_{n,p,q}^{[m-1]}\left(\beta;\mu;\nu;\omega\right)s^{u-n} \cdot\left(\beta\sum_{k=0}^{u}\begin{bmatrix} u \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} s^{-k} T_{u-k,p,q}^{[m-1,\alpha]}\left(x,y;\beta;\mu;\nu;\omega\right) - \omega\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix} u \\ k \end{bmatrix}_{p,q} s^{-k} T_{u-k,p,q}^{[m-1,\alpha]}\left(x,y;\beta;\mu;\nu;\omega\right)\right). \tag{3.20}$$

Proof. The proof of this theorem is based on the following equality:

$$\sum_{n=0}^{\infty} T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\nu;\omega)\frac{t^n}{[n]_{p,q}!} = \left(\frac{(2^{\mu}t^{\nu})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}\right)^{\alpha} e_{p,q}(tx)E_{p,q}(yt) \cdot \frac{\beta e_{p,q}(t/s)-\omega\sum_{l=0}^{m-1}\frac{(t/s)^l}{[l]_{p,q}!}}{(2^{\mu}(t/s)^{\nu})^m}\frac{(2^{\mu}(t/s)^{\nu})^m}{\beta e_{p,q}(t/s)-\omega\sum_{l=0}^{m-1}\frac{(t/s)^l}{[l]_{p,q}!}}. \qquad \square$$

The following theorem involves the recurrence relationships for extended unified (p,q)-analog of Apostol type polynomials of order α.

Theorem 5. *Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\beta,\mu,\nu,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. For $x,y\in\mathbb{R}$, the following formulas are valid:*

$$T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\nu;\omega) = \frac{2^{\mu m}[n]_{p,q}!}{[n+\nu m]_{p,q}!}\sum_{j=0}^{n+\nu m}\begin{bmatrix} n+\nu m \\ j\end{bmatrix}_{p,q} T_{n,p,q}^{[m-1]}(xs,0;\beta;\mu;\nu;\omega)s^{u-n} \cdot\left(\beta\sum_{k=0}^{u}\begin{bmatrix} u \\ k\end{bmatrix}_{p,q} p^{\binom{k}{2}}s^{-k}T_{u-k,p,q}^{[m-1,\alpha]}(0,y;\beta;\mu;\nu;\omega) -\omega\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix} u \\ k\end{bmatrix}_{p,q} s^{-k}T_{u-k,p,q}^{[m-1,\alpha]}(0,y;\beta;\mu;\nu;\omega)\right) \tag{3.21}$$

and

$$T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\nu;\omega) = \frac{2^{\mu m}[n]_{p,q}!}{[n+\nu m]_{p,q}!}\sum_{j=0}^{n+\nu m}\begin{bmatrix} n+\nu m \\ j\end{bmatrix}_{p,q} T_{n,p,q}^{[m-1]}(0,ys;\beta;\mu;\nu;\omega)s^{u-n} \cdot\left(\beta\sum_{k=0}^{u}\begin{bmatrix} u \\ k\end{bmatrix}_{p,q} p^{\binom{k}{2}}s^{-k}T_{u-k,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;\nu;\omega) -\omega\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix} u \\ k\end{bmatrix}_{p,q} s^{-k}T_{u-k,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;\nu;\omega)\right). \tag{3.22}$$

Proof. The proof of this theorem is based on the following equalities:

$$\sum_{n=0}^{\infty} T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\nu;\omega)\frac{t^n}{[n]_{p,q}!} = \left(\frac{(2^{\mu}t^{\nu})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}\right)^{\alpha} \cdot E_{p,q}(yt)\frac{\beta e_{p,q}(t/s)-\omega\sum_{l=0}^{m}\frac{(t/s)^l}{[l]_{p,q}!}}{(2^{\mu}(t/s)^{\nu})^m} \cdot\frac{(2^{\mu}(t/s)^{\nu})^m}{\beta e_{p,q}(t/s)-\omega\sum_{l=0}^{m}\frac{(t/s)^l}{[l]_{p,q}!}}e_{p,q}\left(\frac{t}{s}xs\right)$$

and

$$\sum_{n=0}^{\infty} T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;\nu;\omega)\frac{t^n}{[n]_{p,q}!} = \left(\frac{(2^{\mu}t^{\nu})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}\right)^{\alpha} \cdot e_{p,q}(xt)\frac{\beta e_{p,q}(t/s)-\omega\sum_{l=0}^{m}\frac{(t/s)^l}{[l]_{p,q}!}}{(2^{\mu}(t/s)^{\nu})^m}$$

$$\cdot \frac{(2^{\mu}(t/s)^{\nu})^{m}}{\beta e_{p,q}(t/s)-\omega\sum_{l=0}^{m}\frac{(t/s)^{l}}{[l]_{p,q}!}}E_{p,q}\left(\frac{t}{s}ys\right). \qquad \square$$

4. Multifarious Correlation Formulas for the Polynomials $T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega)$

This part provides several correlation formulas and identities including (p,q)-analogues of the some special polynomials for the polynomials $T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega)$.

Let us define a novel (p,q)-analog of Stirling numbers of the second kind as follows.

Definition 2. Let $m\in\mathbb{N}$, $\tau\in\mathbb{N}_0$, $\beta,\omega,p,q\in\mathbb{C}$ with $0<|q|<|p|\le 1$. A new (p,q)-extension of the Stirling numbers $S_{p,q}(n,\tau;a,b,m,\beta,\omega)$ of the second kind of order τ is defined by means of the following generating function:

$$\sum_{n=0}^{\infty}S_{p,q}(n,\tau;a,b,m,\beta,\omega)\frac{t^n}{[n]_{p,q}!}=\frac{\left(\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)^{\tau}}{[\tau]_{p,q}!}. \tag{4.1}$$

A formula including the family of extended unified polynomials $\{T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega)\}$ and the generalized (p,q)-Stirling numbers $S_{p,q}(n,v;a,b,\beta)$ of the second kind of order v is presented in following Theorem 6.

Theorem 6. *Let $m\in\mathbb{N}$, $\tau,n,\phi,v\in\mathbb{N}_0$, $\beta,\omega,\mu,p,q\in\mathbb{C}$ with $n>vm\phi$ and $0<|q|<|p|\le 1$. The following correlation*

$$\begin{aligned}&T_{n-vm\phi,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega)\\&=2^{-\mu m\phi}[\phi]_{p,q}!\sum_{j=0}^{n}\frac{[n-vm\phi]_{p,q}!}{[n]_{p,q}!}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}T_{n-j,p,q}^{[m-1,\alpha+\phi]}(x,y;\beta;\mu;v;\omega)S_{p,q}(j,\tau;a,b,m,\beta,\omega)\end{aligned} \tag{4.2}$$

is true.

Proof. It follows directly from Definitions 1 and 2. $\square$

In the case when $\alpha=0$ in Theorem 6, we have the following corollary.

Corollary 2. *The following correlation holds true:*

$$2^{\mu m\phi}\frac{[vm\phi]_{p,q}!}{[\phi]_{p,q}!}(x\oplus y)_{p,q}^{n-vm\phi}=\sum_{j=0}^{n}\frac{\begin{bmatrix}n\\j\end{bmatrix}_{p,q}}{\begin{bmatrix}n\\vm\phi\end{bmatrix}_{p,q}}T_{n-j,p,q}^{[m-1,\phi]}(x,y;\beta;\mu;v;\omega)S_{p,q}(j,\tau;a,b,m,\beta,\omega). \tag{4.3}$$

We provide the following results taken from the reference [9].

Lemma 5. *The following relations are valid:*

$$y^{n-m}=\frac{[n-m]_{p,q}!}{q^{\binom{n-m}{2}}[n]_{p,q}!}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}B_k^{[m-1,1]}(0,y:p,q)$$

$$-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{n-k}^{[m-1,1]}(0,y:p,q) \quad (n\geq m),$$

$$y^n = \frac{2^{-m}}{q^{\binom{n}{2}}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} E_k^{[m-1,1]}(0,y:p,q)$$

$$-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} E_{n-k}^{[m-1,1]}(0,y:p,q) \quad (n\geq 0),$$

$$y^{n-m} = \frac{2^{-m}[n-m]_{p,q}!}{q^{\binom{n-m}{2}}[n]_{p,q}!}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} G_k^{[m-1,1]}(0,y:p,q)$$

$$-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} G_{n-k}^{[m-1,1]}(0,y:p,q) \quad (n\geq m).$$

We give the following theorem including many different polynomials and (p,q)-gamma function.

Theorem 7. *The following expressions hold true:*

$$\begin{aligned} T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega) = {} & \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} T_{j,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;v;\omega) \\ & \cdot\left(\frac{\Gamma_{p,q}(n-j-m+1)}{q^{\binom{n-j-m}{2}}\Gamma_{p,q}(n-j+1)}\sum_{k=0}^{n-j}\begin{bmatrix} n-j \\ k \end{bmatrix}_{p,q} p^{\binom{n-j-k}{2}} B_k^{[m-1,1]}(0,y:p,q)\right. \\ & \left. -\sum_{k=0}^{\min(n-j,m-1)}\begin{bmatrix} n-j \\ k \end{bmatrix}_{p,q} B_{n-j-k}^{[m-1,1]}(0,y:p,q)\right) \end{aligned}$$

$$\begin{aligned} T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega) = {} & \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} T_{j,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;v;\omega) \\ & \cdot\left(\frac{2^{-m}}{q^{\binom{n}{2}}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} E_k^{[m-1,1]}(0,y:p,q)\right. \\ & \left. -\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} E_{n-k}^{[m-1,1]}(0,y:p,q)\right) \end{aligned}$$

$$\begin{aligned} T_{n,p,q}^{[m-1,\alpha]}(x,y;\beta;\mu;v;\omega) = {} & \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} T_{j,p,q}^{[m-1,\alpha]}(x,0;\beta;\mu;v;\omega) \\ & \cdot\left(\frac{2^{-m}\Gamma_{p,q}(n-j-m+1)}{q^{\binom{n-j-m}{2}}\Gamma_{p,q}(n-j+1)}\sum_{k=0}^{n-j}\begin{bmatrix} n-j \\ k \end{bmatrix}_{p,q} p^{\binom{n-j-k}{2}} G_k^{[m-1,1]}(0,y:p,q)\right. \\ & \left. -\sum_{k=0}^{\min(n-j,m-1)}\begin{bmatrix} n-j \\ k \end{bmatrix}_{p,q} G_{n-j-k}^{[m-1,1]}(0,y:p,q)\right) \end{aligned}$$

Proof. The proofs of the relationships given in theorem above are based on the Lemma 2, Lemma 5 and eq. (2.8). So, we omit them. □

We now state the following results given as Theorems 8, 9 and 10.

Theorem 8. *The following correlation*

$$T^{[m-1,\alpha]}_{n,p,q}(1,y;\beta;\mu;v;\omega)-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}T^{[m-1,\alpha]}_{n-k,p,q}(0,y;\beta;\mu;v;\omega)$$
$$=\sum_{j=0}^{n-m}\begin{bmatrix} n-m\\ j\end{bmatrix}_{p,q}\frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n-m+1)}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega)B^{[m-1,-1]}_{n-m-j}(p,q)$$

holds true for $n\geq m$.

Proof. Using Definition 1 and Eq. (1.1), we attain

$$\sum_{n=0}^{\infty}T^{[m-1,\alpha]}_{n,p,q}(1,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}-\sum_{n=0}^{\infty}\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}T^{[m-1,\alpha]}_{n-k,p,q}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{(2^{\mu}t^{v})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m}\frac{t^l}{[l]_{p,q}!}}\right)^{\alpha}E_{p,q}(yt)\frac{\left(e_{p,q}(t)-\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)}{t^m}t^m$$
$$=\left(\sum_{n=0}^{\infty}T^{[m-1,\alpha]}_{n,p,q}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}B^{[m-1,-1]}_{n}(p,q)\frac{t^n}{[n]_{p,q}!}\right)t^m$$
$$=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n-m}\begin{bmatrix} n-m\\ j\end{bmatrix}_{p,q}\frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n-m+1)}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega)B^{[m-1,-1]}_{n-m-j}(p,q)\right)\frac{t^n}{[n]_{p,q}!},$$

which completes the proof of this theorem. □

Theorem 9. *The following correlation*

$$T^{[m-1,\alpha]}_{n,p,q}(1,y;\beta;\mu;v;\omega)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}T^{[m-1,\alpha]}_{n-k,p,q}(0,y;\beta;\mu;v;\omega)$$
$$=2^m\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_{p,q}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega)E^{[m-1,-1]}_{n-j}(p,q)$$

holds true for $n\geq m$.

Proof. Using Definition 1 and eq. (1.2), we acquire

$$\sum_{n=0}^{\infty}T^{[m-1,\alpha]}_{n,p,q}(1,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}T^{[m-1,\alpha]}_{n-k,p,q}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{(2^{\mu}t^{v})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m}\frac{t^l}{[l]_{p,q}!}}\right)^{\alpha}E_{p,q}(yt)\frac{\left(e_{p,q}(t)+\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)}{2^m}2^m$$
$$=\left(\sum_{n=0}^{\infty}T^{[m-1,\alpha]}_{n,p,q}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}E^{[m-1,-1]}_{n}(p,q)\frac{t^n}{[n]_{p,q}!}\right)$$
$$=2^m\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_{p,q}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega)E^{[m-1,-1]}_{n-j}(p,q)\right)\frac{t^n}{[n]_{p,q}!},$$

which completes the proof of this theorem. □

Theorem 10. *The following correlation*

$$T^{[m-1,\alpha]}_{n,p,q}(1,y;\beta;\mu;v;\omega)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} T^{[m-1,\alpha]}_{n-k,p,q}(0,y;\beta;\mu;v;\omega)$$
$$=2^m\sum_{j=0}^{n-m}\begin{bmatrix} n-m \\ j \end{bmatrix}_{p,q}\frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n-m+1)}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega)G^{[m-1,-1]}_{n-m-j}(p,q)$$

holds true for $n\geq m$.

Proof. Using Definition 1 and eq. (1.3), we obtain

$$\sum_{n=0}^{\infty}T^{[m-1,\alpha]}_{n,p,q}(1,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} T^{[m-1,\alpha]}_{n-k,p,q}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{(2^{\mu}t^{v})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m}\frac{t^l}{[l]_{p,q}!}}\right)^{\alpha}E_{p,q}(yt)\frac{\left(e_{p,q}(t)+\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}\right)}{t^m2^m}2^mt^m$$
$$=\left(\sum_{n=0}^{\infty}T^{[m-1,\alpha]}_{n,p,q}(0,y;\beta;\mu;v;\omega)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}G^{[m-1,-1]}_{n}(p,q)\frac{t^n}{[n]_{p,q}!}\right)2^mt^m$$
$$=\sum_{n=0}^{\infty}\left(2^m\sum_{j=0}^{n-m}\begin{bmatrix} n-m \\ j \end{bmatrix}_{p,q}\frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n-m+1)}T^{[m-1,\alpha]}_{j,p,q}(0,y;\beta;\mu;v;\omega)G^{[m-1,-1]}_{n-m-j}(p,q)\right)\frac{t^n}{[n]_{p,q}!},$$

which completes the proof of this theorem. □

Recall from (2.13) that

$$p^{\binom{k}{2}-\binom{n}{2}}\frac{x^kt^k}{[k]_{p,q}!}\frac{e_{p,q}(t)}{e_{p,q}(xt)}=p^{\binom{k}{2}-\binom{n}{2}}\frac{x^kt^k}{[k]_{p,q}!}e_{p,q}(t)E_{p,q}(-xt)\tag{4.4}$$
$$=\sum_{n=k}^{\infty}\mathfrak{B}^{p,q}_{n,k}(x)\frac{t^n}{[n]_{p,q}!}.$$

We immediately see that $\mathfrak{B}^{p,q}_{n,k}(x)=0$ for $n<k$.

Remark 3. Setting $p=1$ in (4.4), the polynomials $\mathfrak{B}^{p,q}_{n,k}(x)$ reduces to the q-Bernstein polynomials $\mathfrak{B}^{q}_{n,k}(x)$ (cf. [5]).

Remark 4. When $q\to p=1$ in (4.4), the polynomials $\mathfrak{B}^{p,q}_{n,k}(x)$ reduces to the classical Bernstein polynomials $\mathfrak{B}_{n,k}(x)$ (cf. [1]).

We now give the following theorem.

Theorem 11. *The following correlation*

$$\mathfrak{B}^{p,q}_{n,k}(x)=p^{\binom{k}{2}-\binom{n}{2}}x^k\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j \end{bmatrix}_{p,q}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}T^{[m-1,-d]}_{j,p,q}(\beta;\mu;v;\omega)T^{[m-1,d]}_{n-j,p,q}(1,-x;\beta;\mu;v;\omega)\tag{4.5}$$

holds true for $n\geq k\geq 0$.

Proof. Using Definition 1 and eq. 1.3, we obtain

$$\sum_{n=k}^{\infty}\mathfrak{B}^{p,q}_{n,k}(x)\frac{t^n}{[n]_{p,q}!}$$

$$= p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k t^k}{[k]_{p,q}!}\left(\frac{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}{(2^{\mu}t^{\nu})^m}\right)^d\left(\frac{(2^{\mu}t^{\nu})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}\right)^d e_{p,q}(t)E_{p,q}(-xt)$$
$$= p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k t^k}{[k]_{p,q}!}\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,-d]}(\beta;\mu;\upsilon;\omega)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,d]}(1,-x;\beta;\mu;\upsilon;\omega)\frac{t^n}{[n]_{p,q}!}$$
$$= p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k}{[k]_{p,q}!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j\end{bmatrix}_{p,q}T_{j,p,q}^{[m-1,-d]}(\beta;\mu;\upsilon;\omega)\,T_{n-j,p,q}^{[m-1,d]}(1,-x;\beta;\mu;\upsilon;\omega)\frac{t^{n+k}}{[n]_{p,q}!},$$

which implies the asserted result (4.5). □

The immediate result of the Theorem 11 is given below.

Corollary 3. *We have*

$$\mathfrak{B}_{n,k}^{p,q}(x) = p^{\binom{k}{2}-\binom{n}{2}}x^k\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j\end{bmatrix}_{p,q}\begin{bmatrix} n \\ k\end{bmatrix}_{p,q}T_{j,p,q}^{[m-1,-1]}(\beta;\mu;\upsilon;\omega)\,T_{n-j,p,q}^{[m-1,1]}(1,-x;\beta;\mu;\upsilon;\omega)$$

We give the following correlation.

Theorem 12. *For $n \geq k \geq 0$, we have*

$$\mathfrak{B}_{n,k}^{p,q}(x) = p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k}{2^{\mu m\tau}}\sum_{j=0}^{n-k+\nu m\tau}\begin{bmatrix} n-k+\nu m\tau \\ j\end{bmatrix}_{p,q}$$
$$\cdot S_{p,q}(j,\tau;a,b,m,\beta,\omega)\,T_{n-k+\nu m\tau-j,p,q}^{[m-1,d]}(1,-x;\beta;\mu;\upsilon;\omega)$$
$$\cdot\frac{\Gamma_{p,q}(n+1)\Gamma_{p,q}(\tau+1)}{\Gamma_{p,q}(n-k+\nu m\tau+1)\Gamma_{p,q}(k+1)}. \tag{4.6}$$

Proof. Using Definition 1 and Eq. 1.3, we obtain

$$\sum_{n=k}^{\infty}\mathfrak{B}_{n,k}^{p,q}(x)\frac{t^n}{[n]_{p,q}!} = p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k t^k}{[k]_{p,q}!}\left(\frac{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}{(2^{\mu}t^{\nu})^m}\right)^{\tau}$$
$$\cdot\left(\frac{(2^{\mu}t^{\nu})^m}{\beta e_{p,q}(t)-\omega\sum_{l=0}^{m-1}\frac{t^l}{[l]_{p,q}!}}\right)^{\tau} e_{p,q}(t)E_{p,q}(-xt)$$
$$= p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k t^{k-\nu m\tau}[\tau]_{p,q}!}{[k]_{p,q}!2^{\mu m\tau}}\sum_{n=0}^{\infty}S_{p,q}(n,\tau;a,b,m,\beta,\omega)$$
$$\cdot\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}T_{n,p,q}^{[m-1,\tau]}(1,-x;\beta;\mu;\upsilon;\omega)\frac{t^n}{[n]_{p,q}!}$$
$$= p^{\binom{k}{2}-\binom{n}{2}}\frac{x^k[\tau]_{p,q}!}{[k]_{p,q}!2^{\mu m\tau}}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j\end{bmatrix}_{p,q}S_{p,q}(j,\tau;a,b,m,\beta,\omega)$$
$$\cdot T_{n-j,p,q}^{[m-1,d]}(1,-x;\beta;\mu;\upsilon;\omega)\frac{t^{n+k-\nu m\upsilon}}{[n]_{p,q}!},$$

which gives the desired result (4.6). □

Remark 5. When $p=1$ in (4.6), one gets that

$$\mathfrak{B}_{n,k}^{q}(x)=\frac{x^{k}}{2^{\mu m\tau}}\sum_{j=0}^{n-k+vm\tau}\begin{bmatrix} n-k+vm\tau \\ j \end{bmatrix}_{q} S_{q}\left(j,\tau;a,b,m,\beta,\omega\right)T_{n-k+vm\tau-j,q}^{[m-1,d]}\left(1,-x;\beta;\mu;v;\omega\right) \cdot\frac{\Gamma_{q}(n+1)\Gamma_{q}(\tau+1)}{\Gamma_{q}(n-k+vm\tau+1)\Gamma_{q}(k+1)},$$

where $\mathfrak{B}_{n,k}^{q}(x)$ denotes the q-Bernstein polynomials (cf. [5]), $\left[\begin{smallmatrix} - \\ - \end{smallmatrix}\right]_q$ denotes the q-binomial coefficients (cf. [4, 7, 15, 18, 19, 24, 26, 32, 34, 36]), $S_q(j,\tau;a,b,m,\beta,\omega)$ denotes the generalized q-Stirling numbers of the second kind, $T_{n-k+vm\tau-j,q}^{[m-1,d]}$ denotes the extended generalized q-extensions for Apostol type polynomials (cf. [5]) and $\Gamma_q(n+1)$ denotes the q-gamma function (cf. [7, 33]).

Remark 6. Also taking $q\to p=1$ in (4.6), we get

$$B_{n,k}(x)=\frac{x^{k}}{2^{\mu m\tau}}\sum_{j=0}^{n-k+vm\tau}\binom{n-k+vm\tau}{j}S\left(j,\tau;a,b,m,\beta,\omega\right)T_{n-k+vm\tau-j}^{[m-1,d]}\left(1,-x;\beta;\mu;v;\omega\right) \cdot\frac{\Gamma(n+1)\Gamma(\tau+1)}{\Gamma(n-k+vm\tau+1)\Gamma(k+1)},$$

where $B_{n,k}(x)$ denotes the classical Bernstein polynomials (cf. [1]), $\binom{\cdots}{\cdots}$ denotes the usual binomial coefficients, $S\left(j,\tau;a,b,m,\beta,\omega\right)$ denotes the generalized Stirling numbers of the second kind, $T_{n-k+vm\tau-j}^{[m-1,d]}$ denotes the extended generalized Apostol type polynomials in (3.2) and $\Gamma(n+1)$ denotes the usual gamma function [32].

5. Conclusion

In the present paper, by defining a novel generating function for (p,q)-analog of Apostol type polynomials of order α, several useful formulas heve been obtained related to the extensions of Apostol type polynomials. Also, by means of the (p,q)-generalizations of the Stirling numbers of second kind of order τ and the Bernstein polynomials in conjunction with some properties of them, diverse functional equations have been derived. Then, multifarious interesting and new correlations including not only aforesaid polynomials but also the (p,q)-gamma function have been investigated. Via the results derived in this paper, the (p,q)-polynomials studied in the present paper can be examined in the approximation theory.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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