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Research Article

General Iterative Scheme for Split Mixed Equilibrium Problems, Variational Inequality Problems and Fixed Point Problems in Hilbert Spaces

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Abstract. The purpose in this paper is to study the strong convergence of general iterative scheme to find a common element of the set of a finite family of nonexpansive mappings, the set of solutions of variational inequalities for a relaxed cocoercive mapping and the set of solutions of split mixed equilibrium problem. Our results extend recent results announced by many others.

Keywords. Split mixed equilibrium problem; Fixed point problem; Hilbert spaces; Relaxed cocoercive mapping; Finite family of nonexpansive mappings

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1. Introduction

Let H be a real Hilbert space which inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $A : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The *classical variational inequality* *problem*, denoted by $VI(C,A)$ is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \tag{1}$$

for all $v \in C$. For a given $z \in H, u \in C$ satisfies the inequality

$$\langle u-z, v-u\rangle \geq 0, \quad \forall\, v \in C, \tag{2}$$

if and only if $u = P_C z$. It is known that the projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{3}$$

for $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality problem (1) is equivalent to some fixed point problem.

The element $u \in C$ is a solution of the variational inequality (1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda A u)$, where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of variational inequalities and related optimization problem.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see, e.g., [19, 26, 34–36] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{4}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [35, 36], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A) S x_n + \alpha_n b, \quad n \geq 0, \tag{5}$$

converge strongly to the unique solution of the minimization problem (4) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [26] introduced a new iterative scheme by the viscosity approximation method which was first introduced by Moudafi [27]:

$$x_{n+1} = (I - \alpha_n A) S x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{6}$$

They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{8}$$

where C is the fixed point set of a nonexpansive mapping S, h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-cocoercive map, Takahashi and Toyoda [31] introduced the following iterative process:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n A x_n), \tag{9}$$

for every $n = 0,1,2,\dots$, where A is α-cocoercive, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They showed that, if $Fix(S)\cap VI(C,A)$ is nonempty, then the sequence $\{x_n\}$ generated by (9) converge weakly to some $z \in Fix(S)\cap VI(C,A)$. Recently, Iiduka and Takahashi [21] studied similar scheme as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)SP_C(x_n - \lambda_n A x_n), \tag{10}$$

for every $n = 0,1,2,\dots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They proved that the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)\cap VI(C,A)$. Very recently, Chen et al. [13] studied the following iterative process

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_C(x_n - \lambda_n A x_n), \quad n \ge 1, \tag{11}$$

and also obtained a strong convergence theorem by the so-called viscosity approxiamtion method [27].

Let $\varphi : C \to \mathbb{R}$ be a function, and $F : C \times C \to \mathbb{R}$ be a bifunction. The *mixed equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall\ y \in C. \tag{12}$$

The solution set of mixed equilibrium problem is denoted by $MEP(F,\varphi)$. In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem, which is to find $x \in C$ such that $F(x,y) \ge 0$, $\forall\ y \in C$. The solution set of equilibrium broblem is denoted by $EP(F)$.

The mixed equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, minimization problems, fixed point problems, Nash equilibrium problem in noncooperative games, and others ([4,7,15,20]).

In 1994, Censor and Elfving [8] firstly introduced the following split feasibility problem in finite-dimensional Hilbert spaces: Let H_1, H_2 be two Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* is formulated as finding a point x^* with the property

$$x^* \in C \ \text{ and } \ Ax^* \in Q. \tag{13}$$

The split feasibility problem can extensively be applied in fields such as intensity modulated radiation therapy, signal processing and image reconstruction, then the split feasibility has received so much attention by many scholars (see [9–12]).

In 2013, Kazmi and Rivi [23] introduced and studied the following split equilibrium problem: let $C \subseteq H_1$ and $Q \subseteq H_2$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and let $A : H_1 \to H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $x^* \in C$ such that

$$F_1(x^*,x) \ge 0, \forall\, x \in C \text{ and such that } y^* = Ax^* \in Q \text{ solves } F_2(y^*,y) \ge 0, \quad \forall\ y \in Q. \tag{14}$$

The solution set of the split equilibrium problem is denoted by

$$SEP(F_1,F_2) := \{x^* \in C : x^* \in EP(F_1) \text{ and } Ax^* \in EP(F_2)\}. \tag{15}$$

They gave an iterative algorithm to find the common element of sets of solution of the split equilibrium problem and hierarchical fixed point problem (refer to [5, 6] for more details).

In 2016, Suantai et al. [30] proposed the iterative algorithm to solve the problems for finding a common elements the set of solution of the split equilibrium problem and the fixed point of a nonspreading multivalued mapping in Hilbert space, given sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in C \text{ arbitrarily}, \\ u_n = T^{F_1}_{r_n}(I-\gamma A^*(I-T^{F_2}_{r_n})A)x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n, \quad \forall\ n \in \mathbb{N}, \end{cases} \tag{16}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$ and $\gamma \in \left(0,\frac{1}{L}\right)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A, $C \subset H_1$, $Q \subset H_2$, $S : C \to K(C)$ is a $\frac{1}{2}$-nonspreading multivalued mapping, $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions. They showed that under certain conditions, the sequence $\{x_n\}$ converges weakly to an element of $Fix(S) \cap SEP(F_1,F_2)$.

Several iterative algorithms have been developed for solving split feasibility problems and related split equilibrium problems (see, e.g., [16, 17, 24]).

In this paper, we will consider a finite family of nonexpansive mapping. Let $K_i : C \to C$, where $i = 1,2,\dots,N$, be a finite family of nonexpansive mappings. Let $Fix(K_i)$ denote the fixed point set of K_i, that is, $Fix(K_i); = \{x \in C : K_i x = x\}$. Finding an optimal point in the intersection $\cap_{i=1}^N Fix(K_i)$ of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [3]). The problem of finding an optimal point that minimizes a given cost function over $\cap_{i=1}^N Fix(K_i)$ is of wide interdisciplinary interest and practical importance (see, e.g., [2, 14, 18]). A simple algorithmic solution to the problem of minimizing a quadratic function over $\cap_{i=1}^N Fix(K_i)$ is of extreme value in many applications including set theoretic signal estimation (see, e.g., [22, 37]).

We study the mapping W_n defined by

$$\begin{aligned} U_{n0} &= I, \\ U_{n1} &= \lambda_{n1}K_1U_{n0} + (1-\lambda_{n1})I, \\ U_{n2} &= \lambda_{n2}K_2U_{n1} + (1-\lambda_{n2})I, \\ &\ \vdots \\ U_{n,N-1} &= \lambda_{n,N-1}K_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n = U_{nN} &= \lambda_{nN}K_NU_{n,N-1} + (1-\lambda_{nN})I, \end{aligned} \tag{17}$$

where $\{\lambda_{n1}\},\{\lambda_{n2}\},\dots,\{\lambda_{nN}\} \in (0,1]$. Such a mapping W_n is called the W-mapping generated by $K_1,K_2,\dots,K_N$ and $\{\lambda_{n1}\},\{\lambda_{n2}\},\dots,\{\lambda_{nN}\}$. Nonexpansivity of each K_i ensures the nonexpansivity of W_n. Moreover, in [1, Lemma 3.1], it is shown that $Fix(W_n) = \cap_{i=1}^N Fix(K_i)$.

Motivated and inspired by the above results and related literature, we propose an iterative algorithm for finding a common element of the set of solutions of split mixed equilibriumproblems and the set of fixed points of finite family of nonexpansive mappings in real Hilbert spaces. Then we prove some strong convergence theorem which extend and improve the corresponding results of Kazmi and Rizvi [23] and Suantai et al. [30] and many others.

2. Preliminaries

In this section, we collect some notations and lemmas. Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is also well known [28] that Hilbert space H satisfies *Opail's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{18}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 1. *In a real Hilbert space H, the following inequalities hold:*

(1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \ \forall\ x, y \in H;$

(2) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \ \forall\ x, y \in H;$

(3) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \ \forall\ \lambda \in [0,1], \ \forall\ x, y \in H.$

An element $x \in C$ is called a *fixed point* of S if $x \in Sx$. The set of all fixed point of S is denoted by $Fix(S)$, that is $Fix(S) = \{x \in C : x \in Sx\}$.

Recall that the following definitions:

(1) S is called v-strongly monotone, if each $x, y \in C$, we have

$$\langle Sx - Sy, x - y\rangle \ge v\|x-y\|^2,$$

for constant $v > 0$. This implies that

$$\langle Sx - Sy\rangle \ge v\|x-y\|,$$

that is, S is v-expansive and when $v = 1$, it is expansive.

(2) S is said to be v-cocoercive [32, 33], if for each $x, y \in C$, we have

$$\langle Sx - Sy\rangle \ge v\|Sx - Sy\|^2,$$

for constant $v > 0$. Clearly, every v-cocoercive map S is $\frac{1}{v}$-Lipschitz continuous.

(3) S is said to be relaxed u-cocoercive, if there exists a constant $u > 0$ such that

$$\langle Sx - Sy, x - y\rangle \ge (-u)\|Sx - Sy\|^2, \quad \forall\ x, y \in C.$$

(4) S is called relaxed (u, v)-cocoercive, if there exists two constants $u, v > 0$ such that

$$\langle Sx - Sy, x - y\rangle \ge (-u)\|Sx - Sy\|^2 + v\|x-y\|^2, \quad \forall\ x, y \in C,$$

for $u = 0, S$ is v-strongly monotone. This class of maps is more general than the class of strongly monotone maps. It is easy to see that we have the following implication:

v-strongly monotonicity $\Rightarrow$ relaxed (u,v)-cocoercivity.

(5) A mapping $S : C \to C$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \ \forall \ x, y \in C$.

(6) A mapping $f : H \to H$ is said to be a contraction if there exists a coefficient $\alpha (0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \ \forall \ x, y \in H.$$

(7) An operator B is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2, \ \forall \ x \in H.$$

(8) A set valued mapping $S : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Sx$ and $g \in Sy$ imply

$$\langle x - y, f - g\rangle \geq 0.$$

A monotone mapping $S : H \to 2^H$ is maximal if the graph $G(S)$ of S is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping S is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f -g\rangle \geq 0$ for every $(y, g) \in G(S)$ implies $f \in Sx$. Let B be a monotone map of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall \ u \in C\}$ and define

$$Sv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then S is maximal monotone and $0 \in Sv$ if and only if $v \in VI(C,B)$ (see [29]).

For solving the mixed equilibrium problem, we assume that the bifunction $F_1 : C \times C \to \mathbb{R}$ satisfies the following assumption:

Assumption 1. *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ be the bifunction, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is convex and lower continuous satisfies the following conditions:*

(A1) *$F_1(x, x) = 0, \ \forall \ x \in C$;*

(A2) *F_1 is monotone, i.e., $F_1(x, y) + F_1(y, x) \leq 0, \ \forall \ x, y \in C$;*

(A3) *for each $x, y, z \in C$, $\lim_{t \downarrow 0} F_1(tz + (1 - t)x, y) \leq F_1(x, y)$;*

(A4) *for each $x \in C, y \mapsto F_1(x, y)$ is convex and lower semicontinuous;*

(B1) *for each $x \in H_1$ and fixed $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,*

$$F_1(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) *C is bounded set.*

Lemma 2 ([25]). *Let C be a nonempty closed and convex subset of a Hilbert space H_1. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfies Assumption 1 and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $C \cap dom\,\varphi \neq \emptyset$. For $r > 0$ and $x \in H_1$. Define a mapping $T_r^{F_1} : H_1 \to C$ as follows:*

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z,y) + \varphi(y) - \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$

for all $x \in H_1$. Assume that either (B1) *or* (B2) *holds. Then the following conclusions hold:*

(1) *for each $x \in H_1, T_r^{F_1} \neq \emptyset$;*

(2) *$T_r^{F_1}$ is single-valued;*

(3) *$T_r^{F_1}$ is firmly nonexpansive, i.e., for any $x, y \in H_1$,*

$$\|T_r^{F_1}x - T_r^{F_1}y\|^2 \leq \langle T_r^{F_1}x - T_r^{F_1}y, x - y\rangle;$$

(4) *$Fix(T_r^{F_1}) = MEP(F_1, \varphi)$;*

(5) *$MEP(F_1, \varphi)$ is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 1 and $\phi : Q \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function such that $Q \cap dom\,\varphi \neq \emptyset$, where Q is a nonempty closed and convex subset of a Hilbert space H_2. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(v) = \left\{ w \in Q : F_2(w,d) + \phi(d) - \phi(w) + \frac{1}{r}\langle d - w, w - v\rangle \geq 0, \forall\, d \in Q \right\}.$$

Then we have the following:

(6) *for each $v \in H_2, T_s^{F_2} \neq \emptyset$;*

(7) *$T_s^{F_2}$ is single-valued;*

(8) *$T_s^{F_2}$ is firmly nonexpansive;*

(9) *$Fix(T_s^{F_2}) = MEP(F_2, \phi)$;*

(10) *$MEP(F_2, \phi)$ is closed and convex.*

Lemma 3 ([34, 35]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequences in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 4 ([26]). *Assume B is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

3. Main Result

Theorem 1. *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, let $K_1, K_2, K_3, \dots, K_N$ be a finite family of nonexpansive mapping of C into H_1 and let D be a μ-Lipschitzian, relaxed (μ, v)-cocoercive map of C into H_1. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 1, let $\varphi : C \times \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \times \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex functions such that $C \cap dom\,\varphi \neq \emptyset$ and $Q \cap dom\,\varphi \neq \emptyset$, respectively, and F_2 is upper semicontinuous in the first argument. Let f be a contraction of H_1 into itself with coefficient α $(0 < \alpha < 1)$ and let B be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $\|B\| \le 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\Theta = \cap_{i=1}^{N} Fix(K_i) \cap SMEP(F_1, \varphi, F_2, \phi) \cap VI(C, D) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in H_1$ and*

$$\begin{cases} y_n = T_{r_n}^{F_1}(I - \xi A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - s_n D)y_n, \quad n \ge 1, \end{cases} \tag{19}$$

where $\{\alpha_n\} \subset [0,1], \{r_n\} \subset (0,\infty), \{s_n\} \subset [0,\infty)$ and $\xi \in \left(0, \frac{1}{L}\right)$ such that L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Assume that the following conditions are satisfied:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum\limits_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\{s_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le \frac{2(v - u\mu^2)}{\mu^2}$, $v \ge u\mu^2$;

(C4) $\sum\limits_{n=0}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$, *for all* $i = 1, 2, \dots, N$;

(C5) $\liminf\limits_{n\to\infty} r_n > 0$.

Then, the both sequence $\{x_n\}$ and $\{y_n\}$ generated by (19) converges strongly to $q \in \Theta$ where $q = P_\Theta(\gamma f + (I - B))(q)$ which solves the following variational inequality

$$\langle \gamma f(q) - Bq, p - q \rangle \le 0, \quad \forall\, p \in \Theta. \tag{20}$$

Proof. Since $\alpha_n \to 0$as $n \to \infty$ by the condition (C1), we may assume, without loss of generality, that $\alpha_n < \|B\|^{-1}$ for all n. From Lemma 4, we know that $0 < \rho \le \|B\|^{-1}$ that $\|I - \rho D\| \le 1 - \rho\bar{\gamma}$. First, we show that $I - s_n D$ is nonexpansive. Indeed, from the relaxed (u, v)-cocoercive and μ-Lipschitzian definition on D and condition (C3), we have

$$\begin{aligned} \|(I - s_n D)x - (I - s_n D)y\|^2 &= \|(x - y) - s_n(Dx - Dy)\|^2 \\ &= \|x - y\|^2 - 2s_n\langle x - y, Dx - Dy \rangle + s_n^2\|Dx - Dy\|^2 \\ &\le \|x - y\|^2 - 2s_n[-u\|Dx - Dy\|^2 + v\|x - y\|^2] + s_n^2\|Dx - Dy\|^2 \\ &\le \|x - y\|^2 + 2s_n\mu^2 u\|x - y\|^2 - 2s_n v\|x - y\|^2 + \mu^2 s_n^2\|x - y\|^2 \\ &= (1 + 2s_n\mu^2 u - 2s_n v + \mu^2 s_n^2)\|x - y\|^2 \\ &\le \|x - y\|^2, \end{aligned} \tag{21}$$

which implies that the mapping $I - s_n D$ is nonexpansive.

Next, we show that $A^*(I-T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $I-T_{r_n}^{F_2}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned}\|A^*(I-T_{r_n}^{F_2})Ax-A^*(I-T_{r_n}^{F_2})Ay\|^2 &= \langle A^*(I-T_{r_n}^{F_2})(Ax-Ay),A^*(I-T_{r_n}^{F_2})(Ax-Ay)\rangle\\ &= \langle (I-T_{r_n}^{F_2})(Ax-Ay),AA^*(I-T_{r_n}^{F_2})(Ax-Ay)\rangle\\ &\le L\langle (I-T_{r_n}^{F_2})(Ax-Ay),(I-T_{r_n}^{F_2})(Ax-Ay)\rangle\\ &= L\|(I-T_{r_n}^{F_2})(Ax-Ay)\|^2\\ &\le L\langle Ax-Ay,(I-T_{r_n}^{F_2})(Ax-Ay)\rangle\\ &= L\langle x-y,A^*(I-T_{r_n}^{F_2})Ax-A^*(I-T_{r_n}^{F_2})Ay\rangle \end{aligned} \tag{22}$$

for all $x,y\in H_1$. This implies that $A^*(I-T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $\xi\in\left(0,\frac{1}{L}\right)$, it follows that $I-\xi A^*(I-T_{r_n}^{F_2})A$ is a nonexpansive mapping. Next, we divide the proof into several steps.

Step 1. We will prove that $\{x_n\}$ is bounded.
Indeed, take $p\in\Theta$ arbitrarily. Then we have $p=T_{r_n}^{F_1}p$ and $p=(I-\xi A^*(I-T_{r_n}^{F_2}A)p$. By nonexpansiveness of $I-\xi A^*(I-T_{r_n}^{F_2})A$, it implies that

$$\begin{aligned}\|y_n-p\| &= \|T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n-T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)p\|\\ &\le \|(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n-(I-\xi A^*(I-T_{r_n}^{F_2})A)p\|\\ &\le \|x_n-p\|. \end{aligned} \tag{23}$$

Putting $\rho_n=P_C(I-s_nD)y_n$, we have

$$\|\rho_n-p\| = \|(I-s_nD)y_n-p\| \le \|y_n-p\| \le \|x_n-p\|. \tag{24}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(\gamma f(W_nx_n)-Bq)+(I-\alpha_nB)(W_n\rho_n-p)\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|+\|I-\alpha_nB\|\|W_n\rho_n-p\|\\ &\le \alpha_n[\gamma\|f(W_nx_n)-f(p)\|+\|\gamma f(p)-Bp\|]+(1-\alpha_n\bar{\gamma})\|\rho_n-p\|\\ &\le [1-(\bar{\gamma}-\gamma\alpha)\alpha_n]\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\|\\ &= [1-(\bar{\gamma}-\gamma\alpha)\alpha_n]\|x_n-p\|+\frac{\alpha_n(\bar{\gamma}-\gamma\alpha)}{\bar{\gamma}-\gamma\alpha}\|\gamma f(p)-Bp\|\\ &\le \max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-Bp\|}{\bar{\gamma}-\gamma\alpha}\right\} \end{aligned} \tag{25}$$

which give that

$$\|x_n-p\| \le \max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-Bp\|}{\bar{\gamma}-\gamma\alpha}\right\}, \quad n\ge 0. \tag{26}$$

Therefore, we obtain that $\{x_n\}$ is bounded, so is $\{y_n\}$.

Step 2. We will prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
Note that,

$$\begin{aligned}\|\rho_{n+1}-\rho_n\| &= \|P_C(I-s_{n+1}D)y_{n+1}-P_C(I-S_nD)y_n\|\\ &\le \|(I-s_{n+1}D)y_{n+1}-(I-S_nD)y_n\| \end{aligned}$$

$$
\begin{aligned}
&= \|(I-s_{n+1}D)y_{n+1}-(I-s_{n+1}D)y_n+(s_n-s_{n+1})Dy_n\| \\
&\le \|y_{n+1}-y_n\|+|s_n-s_{n+1}|\|Dy_n\|. \qquad (27)
\end{aligned}
$$

Observe that,

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)x_{n+1}-T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n\| \\
&\le \|x_{n+1}-x_n\|. \qquad (28)
\end{aligned}
$$

Substituting (28) into (27), we have

$$
\|\rho_{n+1}-\rho_n\| \le \|x_{n+1}-x_n\|+|s_n-s_{n+1}|\|Dy_n\|. \qquad (29)
$$

Observe that

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|(I-\alpha_{n+1}B)(W_{n+1}\rho_{n+1}-W_n\rho_n)-(\alpha_{n+1}-\alpha_n)BW_n\rho_n \\
&\quad +\gamma[\alpha_{n+1}(f(W_{n+1}x_{n+1})-f(W_nx_n))+f(W_nx_n)(\alpha_{n+1}-\alpha_n)]\| \\
&\le (1-\alpha_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|)+|\alpha_{n+1}-\alpha_n|\|BW_n\rho_n\| \\
&\quad +\gamma[\alpha_{n+1}\alpha(\|x_{n+1}-x_n\|+\|W_{n+1}x_n-W_nx_n\|)+|\alpha_{n+1}-\alpha_n|\|f(W_nx_n)\|]. \qquad (30)
\end{aligned}
$$

Next, we estimate $\|W_{n+1}x_n-W_nx_n\|$ and $\|W_{n+1}\rho_n-W_n\rho_n\|$. It follows from the definition of W_n that

$$
\begin{aligned}
\|W_{n+1}\rho_n-W_n\rho\| &= \|\lambda_{n+1,N}K_NU_{n+1,N-1}\rho_n+(1-\lambda_{n+1,N})\rho_n-\lambda_{n,N}K_NU_{n,N-1}\rho_n-(1-\lambda_{n,N})\rho_n\| \\
&\le |\lambda_{n+1,N}-\lambda_{n,N}|\|\rho_n\|+\|\lambda_{n+1,N}K_NU_{n+1,N-1}\rho_n-\lambda_{n,N}K_NU_{n,N-1}\rho_n\| \\
&\le |\lambda_{n+1,N}-\lambda_{n,N}|\|\rho_n\|+\|\lambda_{n+1,N}(K_NU_{n+1,N-1}\rho_n-K_NU_{n,N-1}\rho_n)\| \\
&\quad +|\lambda_{n+1,N}-\lambda_{n,N}|\|K_NU_{n,N-1}\rho_n\| \\
&\le M_1|\lambda_{n+1,N}-\lambda_{n,N}|+\lambda_{n+1,N}\|U_{n+1,N-1}\rho_n-U_{n,N-1}\rho_n\|, \qquad (31)
\end{aligned}
$$

where M_1 is an appropriate constant such that

$$
M_1 \ge \max\Big\{\sup_{n\ge1}\{\|\rho_n\|\},\sup_{n\ge1}\{\|K_NU_{n,N-1}\rho_n\|\}\Big\}.
$$

Next, we consider

$$
\begin{aligned}
&\|U_{n+1,N-1}\rho_n-U_{n,N-1}\rho_n\| \\
&= \|\lambda_{n+1,N-1}K_{N-1}U_{n+1,N-2}\rho_n+(1-\lambda_{n+1,N-1})\rho_n-\lambda_{n,N-1}K_{N-1}U_{n,N-2}\rho_n-(1-\lambda_{n,N-1})\rho_n\| \\
&\le |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|\rho_n\|+\|\lambda_{n+1,N-1}K_{N-1}U_{n+1,N-2}\rho_n-\lambda_{n,N-1}K_{N-1}U_{n,N-2}\rho_n\| \\
&\le |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|\rho_n\|+\lambda_{n+1,N-1}\|K_{N-1}U_{n+1,N-2}\rho_n-K_{N-1}U_{n,N-2}\rho_n\| \\
&\quad +|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|K_{N-1}U_{n,N-2}\rho_n\| \\
&\le M_2|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+\|U_{n+1,N-2}\rho_n-U_{n,N-2}\rho_n\|, \qquad (32)
\end{aligned}
$$

where M_2 is an appropriate constant such that

$$
M_2 \ge \max\Big\{\sup_{n\ge1}\{\|\rho_n\|\},\sup_{n\ge1}\{\|K_{N-1}U_{n,N-2}\rho_n\|\}\Big\}.
$$

In a similar way, we obtain

$$
\|U_{n+1,N-1}\rho_n-U_{n,N-1}\rho_n\| \le M_3\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|, \qquad (33)
$$

where M_3 is an appropriate constant such that

$$M_3 \geq \max\left\{\sup_{n\geq 1}\{\|\rho_n\|\}, \sup_{n\geq 1}\{\|K_i U_{n,i-1}\rho_n\|\}|,\, i=1,2,\dots,N\right\}.$$

Substituting (33) into (31)

$$\begin{aligned}\|W_{n+1}\rho_n - W_n\rho_n\| &\leq M_1|\lambda_{n+1,N}-\lambda_{n,N}| + \lambda_{n+1,N}M_3\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\ &\leq M_4\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|, \end{aligned} \tag{34}$$

where M_5 is an appropriate constant such that $M_4 \geq \max\{M_1, M_3\}$. Similarly, we have

$$\|W_{n+1}x_n - W_n x_n\| \leq M_5\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|. \tag{35}$$

Substituting (29), (34) and (35) into (30)

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq [1-\alpha_{n+1}(\bar{\gamma}-\alpha\gamma)]\|x_{n+1}-x_n\| \\ &\quad + M_5\left(\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}| + |s_n - s_{n+1}| + |\alpha_n - \alpha_{n+1}|\right),\end{aligned} \tag{36}$$

where M_5 is an appropriate constant such that

$$M_5 \geq \max\left\{M_4, \|BW_n\rho_n\|, \gamma\sup_{n\geq 1}\{\|f(W_n x_n)\|\}, \|Dy_n\|\right\}.$$

An application of Lemma 3 to (36) implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{37}$$

Observe that (28), (37) and condition (C2), we have

$$\lim_{n\to\infty}\|\rho_{n+1}-\rho_n\| = \lim_{n\to\infty}\|y_{n+1}-y_n\| = 0. \tag{38}$$

Step 3. We will prove that $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Since $x_n = \alpha_{n-1}\gamma f(W_{n-1}x_{n-1}) + (I-\alpha_{n-1}B)W_{n-1}\rho_{n-1}$, we have

$$\begin{aligned}\|x_n - W_n\rho_n\| &\leq \|x_n - W_{n-1}\rho_{n-1}\| + \|W_{n-1}\rho_{n-1} - W_n\rho_n\| \\ &\leq \alpha_{n-1}\|\gamma f(W_{n-1}x_{n-1}) - BW_{n-1}\rho_{n-1}\| + \|\rho_{n-1}-\rho_n\| + M_4\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|,\end{aligned} \tag{39}$$

which on combining with conditions (C1), (C4) and (38) gives

$$\lim_{n\to\infty}\|x_n - W_n\rho_n\| = 0. \tag{40}$$

For $p \in \Theta$, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}p\|^2 \\ &\leq \|(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \xi^2\|A^*(I-T_{r_n}^{F_2})Ax_n\|^2 + 2\xi\langle p - x_n, A^*(I-T_{r_n}^{F_2})Ax_n\rangle \\ &\leq \|x_n - p\|^2 + \xi^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I-T_{r_n}^{F_2})Ax_n\rangle + 2\xi\langle A(p-x_n), Ax_n - T_{r_n}^{F_2}Ax_n\rangle \\ &\leq \|x_n - p\|^2 + L\xi^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\rangle + 2\xi\langle A(p-x_n), Ax_n - T_{r_n}^{F_2}Ax_n\rangle \\ &\quad -(Ax_n - T_{r_n}^{F_2}Ax_n), Ax_n - T_{r_n}^{F_2}Ax_n\rangle\end{aligned}$$

$$
\begin{aligned}
&\le \|x_n-p\|^2+L\xi^2\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\\
&\quad +2\xi(\langle Ap-T_{r_n}^{F_2}Ax_n, Ax_n-T_{r_n}^{F_2}Ax_n\rangle-\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2)\\
&\le \|x_n-p\|^2+L\xi^2\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2+2\xi\left(\frac{1}{2}\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2-\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\right)\\
&= \|x_n-p\|^2+\xi(L\xi-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2. \qquad (41)
\end{aligned}
$$

From (24), (25) and (41), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(W_nx_n)-Bq)+(I-\alpha_nB)(W_n\rho_n-p)\|^2\\
&\le (\alpha_n\|\gamma f(W_nx_n)-Bp\|+(1-\alpha_n\bar{\gamma})\|\rho_n-p\|)^2\\
&\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(1-\alpha_n\bar{\gamma})\|\rho_n-p\|^2+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\
&\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(1-\alpha_n\bar{\gamma})\|y_n-p\|^2+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\
&\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(1-\alpha_n\bar{\gamma})[\|x_n-p\|^2+\xi(L\xi-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2]\\
&\quad +2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\
&\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2+(1-\alpha_n\bar{\gamma})\xi(L\xi-1)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\\
&\quad +2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\
&= \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-(1-\alpha_n\bar{\gamma})\xi(1-L\xi)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\\
&\quad +2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \qquad (42)
\end{aligned}
$$

That is,

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma})\xi(1-L\xi)\|Ax_n-T_{r_n}^{F_2}Ax_n\|^2\\
&\quad \le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\
&\quad \le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \qquad (43)
\end{aligned}
$$

Since $\xi(1-L\xi)>0$, it follows by conditions (C1), (37) and the existence of $\lim_{n\to\infty}\|x_n-p\|$ that

$$
\lim_{n\to\infty}\|Ax_n-T_{r_n}^{F_2}Ax_n\|=0. \qquad (44)
$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I-\xi A^*(I-T_{r_n}^{F_2})A$ is nonexpansive, we have

$$
\begin{aligned}
\|y_n-p\|^2 &= \|T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)x_{n+1}-T_{r_n}^{F_1}p\|^2\\
&\le \langle T_{r_n}^{F_1}(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n-T_{r_n}^{F_1}p,(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n-p\rangle\\
&= \langle y_n-p,(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n-p\rangle\\
&= \frac{1}{2}\left(\|y_n-p\|^2+\|(I-\xi A^*(I-T_{r_n}^{F_2})A)x_n-p\|^2-\|y_n-x_n-\xi A^*(I-T_{r_n}^{F_2})A)x_n-p\|^2\right)\\
&\le \frac{1}{2}\Big(\|y_n-p\|^2+\|x_n-p\|^2-\Big(\|y_n-x_n\|^2+\xi^2\|A^*(I-T_{r_n}^{F_2})Ax_n\|^2\\
&\qquad -2\xi\langle y_n-x_n,A^*(I-T_{r_n}^{F_2})Ax_n\rangle\Big)\Big), \qquad (45)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|y_n-p\|^2 &\le \|x_n-p\|^2-\|y_n-x_n\|^2+2\xi\langle y_n-x_n,A^*(I-T_{r_n}^{F_2})Ax_n\rangle\\
&\le \|x_n-p\|^2-\|y_n-x_n\|^2+2\xi\|y_n-x_n\|\|A^*(I-T_{r_n}^{F_2})Ax_n\|. \qquad (46)
\end{aligned}
$$

From (24), (25) and (46), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(W_nx_n)-Bp)+(I-\alpha_nB)(W_n\rho_n-p)\|^2\\ &\le (\alpha_n\|\gamma f(W_nx_n)-Bp\|+(1-\alpha_n\bar{\gamma}\|\rho_n-p\|)^2\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(1-\alpha_n\bar{\gamma})\|\rho_n-p\|^2+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(1-\alpha_n\bar{\gamma})\|y_n-p\|^2+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(1-\alpha_n\bar{\gamma})[\|x_n-p\|^2-\|y_n-x_n\|^2\\ &\quad +2\xi\|y_n-x_n\|\|A^*(I-T^{F_2}_{r_n})Ax_n\|]+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \end{aligned} \tag{47}$$

Therefore,

$$\begin{aligned}(1-\alpha_n\bar{\gamma})\|y_n-x_n\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2\xi(1-\alpha_n\bar{\gamma})\|y_n-x_n\|\|A^*(I-T^{F_2}_{r_n})Ax_n\|\\ &\quad +2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad +2\xi(1-\alpha_n\bar{\gamma})\|y_n-x_n\|\|A^*(I-T^{F_2}_{r_n})Ax_n\|\\ &\quad +2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \end{aligned} \tag{48}$$

It follows from condition (C1), (37), (44) and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{49}$$

Step 4. We will prove that $\lim_{n\to\infty}\|x_n-W_n\rho_n\|=0$.
For $p\in\Theta$, we have

$$\begin{aligned}\|\rho_n-p\|^2 &= \|P_C(I-s_nD)y_n-P_C(I-s_nD)p\|^2\\ &\le \|(y_n-p)-s_n(Dy_n-Dp)\|^2\\ &= \|y_n-p\|^2-2s_n\langle y_n-p,Dy_n-Dp\rangle+s_n^2\|Dy_n-Dp\|^2\\ &\le \|x_n-p\|^2-2s_n[-u\|Dy_n-Dp\|^2+v\|y_n-p\|^2]+s_n^2\|Dy_n-Dp\|^2\\ &\le \|x_n-p\|^2+2s_nu\|Dy_n-Dp\|^2-2s_nv\|y_n-p\|^2+s_n^2\|Dy_n-Dp\|^2\\ &\le \|x_n-p\|^2+\left(2s_nu+s_n^2-\frac{2s_nv}{u^2}\right)\|Dy_n-Dp\|^2. \end{aligned} \tag{50}$$

Observe that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(\gamma f(W_nx_n)-Bp)+(I-\alpha_nB)(W_n\rho_n-p)\|^2\\ &\le (\alpha_n\|\gamma f(W_nx_n)-Bp\|+\|I-\alpha_nB\|\|W_n\rho_n-p\|)^2\\ &\le (\alpha_n\|\gamma f(W_nx_n)-Bp\|+(1-\alpha_n\bar{\gamma})\|\rho_n-p\|)^2\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|\rho_n-p\|^2+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \end{aligned} \tag{51}$$

Substituting (50) into (51), we have

$$\begin{aligned}\|x_n-p\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2+\left(2s_nu+s_n^2-\frac{2s_nv}{u^2}\right)\|Dy_n-Dp\|^2\\ &\quad +2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \end{aligned} \tag{52}$$

It follows from the condition (C3) that

$$\begin{aligned}\Big(\frac{2av}{u^2}-2bu-b^2\Big)\|Dy_n-Dp\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\ &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|. \end{aligned} \tag{53}$$

From conditions (C1) and (37) that

$$\lim_{n\to\infty}\|Dy_n-Dp\|=0. \tag{54}$$

On the other hand, we have

$$\begin{aligned}\|\rho_n-p\|^2&=\|P_C(I-s_nD)y_n-P_C(I-s_nD)p\|^2\\ &\le\langle(I-s_nD)y_n-(I-s_nD)p,\rho_n-p\rangle\\ &=\frac{1}{2}\{\|(I-s_nD)y_n-(I-s_nD)p\|^2+\|\rho_n-p\|^2-\|(I-s_nD)y_n-(I-s_nD)p-(\rho_n-p)\|^2\}\\ &\le\frac{1}{2}\{\|y_n-p\|^2+\|\rho_n-p\|^2-\|(y_n-\rho_n)-s_n(Dy_n-Dp)\|^2\}\\ &=\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|y_n-\rho_n\|^2-s_n^2\|Dy_n-Dp\|^2+2s_n\langle y_n-\rho_n,Dy_n-Dp\rangle\},\end{aligned} \tag{55}$$

which yields that

$$\|\rho_n-p\|^2\le\|x_n-p\|^2-\|y_n-\rho_n\|^2+2s_n\|y_n-\rho_n\|\|Dy_n-Dp\|. \tag{56}$$

Substituting (56) into (51) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|y_n-\rho_n\|^2\\ &\quad+2s_n\|y_n-\rho_n\|\|Dy_n-Dp\|+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|.\end{aligned} \tag{57}$$

It follows that

$$\begin{aligned}\|y_n-\rho_n\|^2&\le\alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2s_n\|y_n-\rho_n\|\|Dy_n-Dp\|+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|\\ &\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad+2s_n\|y_n-\rho_n\|\|Dy_n-Dp\|+2\alpha_n\|\gamma f(W_nx_n)-Bp\|\|\rho_n-p\|.\end{aligned} \tag{58}$$

From conditions (C1), (37) and (54), we have

$$\lim_{n\to\infty}\|y_n-\rho_n\|=0. \tag{59}$$

Observe that

$$\begin{aligned}\|y_n-W_ny_n\|&\le\|W_ny_n-W_n\rho_n\|+\|W_n\rho_n-x_n\|+\|x_n-y_n\|+\|y_n-\rho_n\|\\ &\le2\|y_n-\rho_n\|+\|W_n\rho_n-x_n\|+\|x_n-y_n\|.\end{aligned} \tag{60}$$

From conditions (40), (49) and (59), we have

$$\lim_{n\to\infty}\|y_n-W_ny_n\|=0. \tag{61}$$

Observe that $P_\Theta(\gamma f+(I-B))$ is a contraction. Indeed, for $\forall\ x,y\in H_1$, we have

$$\begin{aligned}\|P_\Theta(\gamma f+(I-B))(x)-P_\Theta(\gamma f+(I-B))(y)\| &\le \|(\gamma f+(I-B))(x)-(\gamma f+(I-B))(y)\|\\ &\le \gamma\|f(x)-f(y)\|+\|I-B\|\|x-y\|\\ &\le \gamma\alpha\|x-y\|+(1-\bar{\gamma})\|x-y\|\\ &=(\gamma\alpha+1-\bar{\gamma})\|x-y\|. \end{aligned}\tag{62}$$

The Banach's Contraction Mapping Principle guarantees that $P_\Theta(\gamma f+(I-B))$ has a unique fixed point, say $q\in H_1$. That is, $q=P_\Theta(\gamma f+(I-B))(q)$. Next, we show that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle\le 0. \tag{63}$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle=\lim_{i\to\infty}\langle\gamma f(q)-Bq,x_{n_i}-q\rangle. \tag{64}$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. Without loss generality, we can assume that $\{y_{n_i}\}\rightharpoonup w$

Step 5. We will prove that $w\in\Theta$.
Since Hilbert spaces are Opial's space, from (61), we have

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i}-w\| &< \liminf_{i\to\infty}\|y_{n_i}-W_nw\|\\ &=\liminf_{i\to\infty}\|y_{n_i}-W_ny_{n_i}+W_ny_{n_i}-W_nw\|\\ &\le\liminf_{i\to\infty}\|W_ny_{n_i}-W_nw\|\\ &\le\liminf_{i\to\infty}\|y_{n_i}-w\|,\end{aligned}\tag{65}$$

which derives a contraction. Thus, we have $w\in Fix(W_n)$. It follows from $Fix(W_n)=\cap_{i=1}^N Fix(K_i)$. Next, let us show that $w\in VI(C,B)$. Put

$$Mw_1=\begin{cases}Dw_1+N_Cw_1, & w_1\in C;\\ \emptyset, & w_1\notin C.\end{cases}$$

Since D is relaxed (u,v)-cocoercive and condition (C3), we have

$$\langle Dx-Dy,x-y\rangle\ge(-u)\|Dx-Dy\|^2+v\|x-y\|^2\ge(v-u\mu^2)\|x-y\|^2\ge 0,$$

which yields that D is monotone. Thus M is maximal monotone. Let $(w_1,w_2)\in G(M)$. Since $w_2-w_1\in N_Cw_1$ and $\rho_n\in C$, we have

$$\langle w_1-\rho_n,w_2-Dw_1\rangle\ge 0.$$

On the other hand, from $\rho_n=P_C(I-s_nD)y_n$, we have

$$\langle w_1-\rho_n,\rho_n-(I-s_nDy_n)\rangle\ge 0, \tag{66}$$

and hence

$$\begin{aligned}\langle w_1-\rho_n,w_2\rangle &\ge \langle w_1-\rho_{n_i},Dw_1\rangle\\ &\ge\langle w_1-\rho_{n_i},Dw_1\rangle-\left\langle w_1-\rho_{n_i},\frac{\rho_{n_i}-y_{n_i}}{s_{n_i}}+Dy_{n_i}\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \left\langle w_1 - \rho_{n_i}, Dw_1 - \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} - Dy_{n_i} \right\rangle \\
&= \langle w_1 - \rho_{n_i}, Dw_1 - D\rho_{n_i} \rangle + \langle w_1 - \rho_{n_i}, D\rho_{n_i} - Dy_{n_i} \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \right\rangle \\
&\geq \langle w_1 - \rho_{n_i}, D\rho_{n_i} - Dy_{n_i} \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} \right\rangle,
\end{aligned} \tag{67}
$$

which implies that $\langle w_1 - w, w_2 \rangle \geq 0$. We have $w \in M^{-1}0$ and hence $w \in VI(C,D)$.

Next, we show that $w \in MEP(F_1, \varphi)$. Since $y_n = T_{r_n}^{F_1}(I - \xi A^*(I - T_{r_n}^{F_2})A)x_n$, we have

$$F_1(y_n, y) + \varphi(y) - \varphi(y_n) + \frac{1}{r_n}\langle y - y_n, y_n - x_n - \xi A^*(I - T_{r_n}^{F_2})Ax_n \rangle \geq 0, \quad \forall\ y \in C,$$

which implies that

$$F_1(y_n, y) + \varphi(y) - \varphi(y_n) + \frac{1}{r_n}\langle y - y_n, y_n - x_n \rangle - \frac{1}{r_n}\langle y - y_n, \xi A^*(I - T_{r_n}^{F_2})Ax_n \rangle \geq 0, \ \ \forall\ y \in C.$$

From Assumption 1 (A2), we have

$$
\begin{aligned}
&\varphi(y) - \varphi(y_n) + \frac{1}{r_n}\langle y - y_n, y_n - x_n \rangle - \frac{1}{r_n}\langle y - y_n, \xi A^*(I - T_{r_n}^{F_2})Ax_n \rangle \\
&\quad \geq -F_1(y_n, y) \geq F_1(y, y_n), \quad \forall\ y \in C,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\varphi(y) - \varphi(y_{n_i}) + \frac{1}{r_{n_i}}\langle y - y_{n_i}, y_{n_i} - x_{n_i} \rangle - \frac{1}{r_{n_i}}\langle y - y_{n_i}, \xi A^*(I - T_{r_{n_i}}^{F_2})Ax_{n_i} \rangle \\
&\quad \geq F_1(y, y_{n_i}), \quad \forall\ y \in C,
\end{aligned}
$$

This implies by $y_{n_i} \rightharpoonup w$, condition (C5), (44), (49), Assumption 1 (A2), and the proper lower semicontinuity of φ that

$$F_1(y, w) + \varphi(w) - \varphi(y) \leq 0, \quad \forall\ y \in C.$$

Put $y_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Consequently, we get $y_t \in C$ and hence $F_1(y_t, w) + \varphi(w) - \varphi(y) \leq 0$. So, by Assumption 1 (A1)-(A4), we have

$$
\begin{aligned}
0 &= F_1(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\
&\leq t(F_1(y_t, y) + \varphi(y) - \varphi(y_t)).
\end{aligned}
$$

Hence, we have

$$F_1(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0, \quad \forall\ y \in C.$$

Letting $t \to 0$, by Assumption 1 (A3) and the proper lower semicontinuity of φ, we have

$$F_1(w, y) + \varphi(y) - \varphi(w) \geq 0, \quad \forall\ y \in C.$$

This implies that $w \in MEP(F_1, \varphi)$.

Since A is a bounded linear operator, we have $Ax_{n_i} \rightharpoonup Aw$. The it follows from (44) that

$$T_{r_{n_i}}^{F_2} Ax_{n_i} \rightharpoonup Aw, \ \text{ as } \ i \to \infty. \tag{68}$$

By the definition of $T^{F_2}_{r_{n_i}}Ax_{n_i}$, we have

$$F_2(T^{F_2}_{r_{n_i}}Ax_{n_i},y)+\phi(y)-\phi(T^{F_2}_{r_{n_i}}Ax_{n_i})+\frac{1}{r_{n_i}}\langle y-T^{F_2}_{r_{n_i}}Ax_{n_i},T^{F_2}_{r_{n_i}}Ax_{n_i}-Ax_{n_i}\rangle\geq 0,\quad\forall\ y\in Q.$$

Since F_2 is upper semicontinuous in the first argument, it implies by (68) that

$$F_2(Aw,y)+\phi(y)-\phi(Aw)\geq 0,\quad\forall\ y\in Q.$$

This shows that $Aw\in MEP(F_2,\phi)$. Therefore, $w\in SMEP(F_1,\varphi,F_2,\phi)$ and hence $w\in\Theta$.

Since $q=P_\Theta(\gamma f+(I-B))(q)$, we have

$$\begin{aligned}\limsup_{i\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle&=\lim_{n\to\infty}\langle\gamma f(q)-Bq,x_{n_i}-q\rangle\\&=\langle\gamma f(q)-Bq,w-q\rangle\leq 0.\end{aligned}$$

That is (63) holds.

Step 6. We will prove that $x_n\to q$ as $n\to\infty$.
We consider

$$\begin{aligned}\|x_{n+1}-q\|^2&=\|(I-\alpha_nB)(W_n\rho_n-q)+\alpha_n(\gamma f(W_nx_n)-Bq)\|^2\\&\leq\|(I-\alpha_nB)(W_n\rho_n-q)\|^2+2\alpha_n\langle\gamma f(W_nx_n)-Bq,x_{n+1}-q\rangle\\&\leq(1-\alpha_n\bar{\gamma})^2\|\rho_n-q\|^2+2\alpha_n\langle\gamma f(W_nx_n)-Bq,x_{n+1}-q\rangle\\&\leq(1-\alpha_n\bar{\gamma})^2\|y_n-q\|^2+2\alpha_n\gamma\langle\gamma f(W_nx_n)-f(q),x_{n+1}-q\rangle+2\alpha_n\langle\gamma f(q)-Bq,x_{n+1}-q\rangle\\&\leq(1-\alpha_n\bar{\gamma})^2\|x_n-q\|^2+2\alpha_n\gamma\alpha\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\langle\gamma f(q)-Bq,x_{n+1}-q\rangle\\&\leq(1-\alpha_n\bar{\gamma})^2\|x_n-q\|^2+\alpha_n\gamma\alpha(\|x_n-q\|^2+\|x_{n+1}-q\|^2)+2\alpha_n\langle\gamma f(q)-Bq,x_{n+1}-q\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-q\|^2&\leq\frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Bq,x_{n+1}-q\rangle\\&=\frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\alpha\gamma)}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Bq,x_{n+1}-q\rangle\\&\leq\left[1-\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\right]\|x_n-q\|^2+\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\left[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Bq,x_{n+1}-q\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_6\right],\end{aligned}$$

where M_6 is an appropriate constant such that $M_6\geq\sup_{n\geq 1}\|x_n-q\|^2$. Put $l_n=\frac{2\alpha_n(\bar{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}$ and $t_n=\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Bq,x_{n+1}-q\rangle+\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_6$. That is,

$$\|x_{n+1}-q\|^2\leq(1-l_n)\|x_n-q\|^2+l_nt_n. \tag{69}$$

It follows from condition (C1) and (63) that

$$\lim_{n\to\infty}l_n=0,\ \sum_{n=1}^{\infty}l_n=\infty\ \text{ and }\ \limsup_{n\to\infty}t_n\leq 0. \tag{70}$$

Apply Lemma 3 to (70) to conclude that $x_n\to q$ as $n\to\infty$. This complete the proof. □

4. Corollary

Corollary 1. *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, let $K_1, K_2, K_3, \ldots, K_N$ be a finite family of nonexpansive mapping of C into H_1 and let D be a μ-Lipschitzian, relaxed (μ, v)-cocoercive map of C into H_1. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 1, let $\varphi : C \times \mathbb{R} \cup \{+\infty\}$ and $\phi : Q \times \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex functions such that $C \cap dom\, \varphi \neq \emptyset$ and $Q \cap dom\, \varphi \neq \emptyset$, respectively, and F_2 is upper semicontinuous in the first argument. Let f be a contraction of H_1 into itself with coefficient α $(0 < \alpha < 1)$ and let B be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ such that $\|B\| \leq 1$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\Theta = \cap_{i=1}^{N} Fix(K_i) \cap VI(C, B) \cap SMEP(F_1, \varphi, F_2, \phi) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} y_n = T_{r_n}^{F_1}(I - \xi A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} = \alpha_n \gamma_n f(W_n x_n) + (I - \alpha_n B)W_n y_n, \quad n \geq 1, \end{cases} \tag{71}$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0, \infty)$ and $\xi \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of the operator A^*A, A^* is the adjoint of A. Assume that the following conditions are satisfied:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$, *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(C3) $\{s_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq \frac{2(v - u\mu^2)}{\mu^2}$, $v \geq u\mu^2$;

(C4) $\sum\limits_{n=1}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$, *for all* $i = 1, 2, \ldots, N$;

(C5) $\liminf\limits_{n\to\infty} r_n > 0$.

Then, the both sequence $\{x_n\}$ and $\{y_n\}$ generated by (71) converges strongly to $q \in \Theta$, which solves the following variational inequality

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall\, p \in \Theta. \tag{72}$$

Proof. Taking $\{s_n\} = 0$ for all n, in Theorem 1, we get the desired conclusion easily. □

5. Conclusion

In this paper, we first propose a modified iterative scheme (19) in Theorem 1 and then we prove some strong convergence of the sequence $\{x_n\}$ generated by (19) to a common solution of finite family of nonexpansive mappings and split mixed equilibrium problem. We divide the proof into 6 steps and our theorem is extend and improve the corresponding results of Kazmi and Rizvi [23] and Suantai et al. [30].

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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