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Research Article

On the Solutions of Linear Fractional Differential Equations of Order $2q$ Including Small Delay Where $0 < q < 1$

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Abstract. The main goal of this study is to find the solutions of linear fractional differential equations of order $2q$, including small delay, where $0 < q < 1$ which has various applications. The fractional derivatives are taken in the sense of Caputo which is more suitable than Riemann-Liouville sense. We assume that the order q satisfy the condition $nq = 1$ for some natural number n which determines the number of the linearly independent solutions. Since the delay term is small, the linear fractional differential equation is expanded in powers series of which reduce the problem to regular or singular perturbation problem for which it is easier to find the solution. The solution is obtained in the form of a series expansion of E. To demonstrate the accuracy and the effectiveness of the proposed approach, some illustrative examples are presented.

Keywords. Fractional differential; Small delay

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1. Introduction

Since the fractional differential equations play an important role in modelling for the wide range of problems which model systems in nature, including past memories, in various scientific research areas such as bioengineering, thermo-dynamics, viscoelasticity, control theory, aerodynamics, electromagnetics, signal processing, chemistry, finance, it is growing considerable interest in recent years [2, 6]. It is well known that fractional derivative is global in nature whereas the integer derivative is local in nature which makes the ODEs of fractional order intriguing topic for many scientists. This property makes fractional ODEs the best possible choice in modelling physical problems involving past memory. Since the derivatives in the Caputo sense is closer to integer order derivatives, the analysis of the ODEs involving Caputo derivatives, gives more useful results.

By making use of Mittag-Leffler function, characteristic equations of fractional ODEs are solved and solutions of them are constructed efficiently. In this sense, the Mittag-Leffler function takes the role of the exponential function which is used in the determination of solutions for ODEs with integer derivatives.

Delay differential equations (DDE's) can be defined as the generalization of the ordinary differential equations which are appropriate for modelling physical systems with memory. The delay is an unavoidable fact of the physical systems such as transportation of energy and materials, feedback control systems kinetics, damping behavior of viscoelastic materials, so on.

The proposed approach is based on applying Taylor series expansion to simplify linear fractional differential equations with small delay. The analytical solution which is in the form of a series expansion for the linear fractional differential equations with small delay is obtained by solving the resulting regular or singular perturbation problem. In regular perturbation problems a small change in the problem causes a small change solution whereas a small change in the problem causes a large change in the solution in singular perturbation problem. Perturbation which is a local method, shows how to find the solutions approximately for the problem under the influence of the perturbation by using the solutions to the unperturbed problem [3, 10, 11].

2. Linear Fractional Differential Equations of Order $2q$, Including Small Delay Where $0 < q < 1$

Let us consider the following linear fractional differential equations with small delay

$$D^{2q}u(t)+BD^{q}u(t)+Cu(t)+Eu(t-\varepsilon)=0, \tag{1}$$

where ε is a small delay term. By taking advantage of this fact we expand the function including delay in the following series

$$u(t-\varepsilon)\approx u(t)-\varepsilon Du(t)+O(\varepsilon^2).$$

Substituting the expansion into (1), the following linear fractional differential equations is obtained:

$$D^{2q}u(t)+BD^{q}u(t)+Cu(t)+E(u(t)-\varepsilon Du(t)+O(\varepsilon^2))=0 \tag{2}$$

which does not contain the function including delay. Now we assume that $qn=1$ for some natural number n. Hence, we have $D_t=D_t^{qn}$. By making use of this fact we can rewrite the equation (2) in the following form:

$$D^{2q}u(t)+BD^{q}u(t)+(C+E)u(t)-E\varepsilon D_t^{qn}u(t)=0. \tag{3}$$

Let $u(t)=E_{q,1}u(rt^q)$ be the solution of (3), then the characteristic equation for (3) is obtained as follows [10]

$$r^2+Br+C+E(1-\varepsilon r^n)=0$$

after some arrangement the characteristic equation can be rewritten in the following form

$$\varepsilon r^n - \frac{1}{E}r^2 - \frac{B}{E}r - \left(\frac{C}{E}+1\right) = 0, \tag{4}$$

where we use perturbation theory to obtain the solution of the characteristic equation. If we have a regular perturbation problem, the solution can be represented in the following series form:

$$r(\varepsilon) = r_0 + \varepsilon r_1 + O(\varepsilon^2) \tag{5}$$

At this stage, we consider the various cases by using the assumption:

Case 1: $n = 1$

Let us consider the case $n = 1$ which makes the characteristic equation (4)

$$\varepsilon r - \frac{1}{E}r^2 - \frac{B}{E}r - \left(\frac{C}{E}+1\right) = 0.$$

Arranging this equation, we get

$$r^2 + (B - E\varepsilon)r + (C+E), \tag{6}$$

where we have regular perturbation problem which means that we seek the solution in the form of (5). Substituting (5) into (6), we get

$$r_0^2 + 2\varepsilon r_0 r_1 + \varepsilon^2 r_1^2 + Br_0 + B\varepsilon r_1 - E\varepsilon r_0 - E\varepsilon^2 r_1 + C + E + O(\varepsilon^3) = 0. \tag{7}$$

Equating to zero the successive terms of the series on the left-hand side of (7):

$$\varepsilon^0 : r_0^2 + Br_0 + C + E = 0$$

$$\varepsilon^1 : 2r_0 r_1 + Br_1 - Er_0 = 0 \Rightarrow r_1 = \frac{Er_0}{2r_0 + B}$$

Solving the first equation above we have

$$r_0^{+,-} = \frac{-B \pm \sqrt{\Delta}}{2},$$

where $\Delta = B^2 - 4(C+E)$.

Similarly, for the second equation we obtain

$$r_1^{+,-} = \frac{E\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)}{2\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)+B} = \frac{E\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)}{\pm\sqrt{\Delta}}.$$

Hence the approximate solution of characteristic equation becomes

$$r^+(\varepsilon) = r_0^+ + \varepsilon r_1^+ + O(\varepsilon^2) = \frac{-B+\sqrt{\Delta}}{2} + \varepsilon\frac{E\left(\frac{-B+\sqrt{\Delta}}{2}\right)}{\sqrt{\Delta}} + O(\varepsilon^2),$$

$$r^-(\varepsilon) = r_0^- + \varepsilon r_1^- + O\left(\varepsilon^2\right) = \frac{-B-\sqrt{\Delta}}{2} + \varepsilon\frac{E\left(\frac{B+\sqrt{\Delta}}{2}\right)}{\sqrt{\Delta}} + O(\varepsilon^2).$$

As a result, the approximate solution of the equations of linear fractional differential equations with small delay can be obtained in the series form of ε as follows:

$$u(t,\varepsilon) = \alpha_1 E_{1,1}\left(r^+(\varepsilon)t\right) + \alpha_2 E_{1,1}(r^-(\varepsilon)t),$$

where

$$E_{q,1}(r(\varepsilon)t) = \sum_{k=0}^{\infty}\frac{(r(\varepsilon)t)^k}{\Gamma(k+1)}.$$

Case 2: $n = 2$

Let us consider the case $n = 2$ which makes the characteristic equation (4)

$$\varepsilon r^2 - \frac{1}{E}r^2 - \frac{B}{E}r - \left(\frac{C}{E}+1\right) = 0.$$

Arranging this equation, we have

$$\left(\varepsilon - \frac{1}{E}\right)r^2 - \frac{B}{E}r - \left(\frac{C}{E}+1\right) = 0, \tag{8}$$

where we have a regular perturbation problem which means that we seek the solution in the form of (5). Substituting (5) into (8), we get

$$[r_0^2 + 2\varepsilon r_0 r_1 + \varepsilon^2 r_1^2 + O(\varepsilon^3)]\left(\varepsilon - \frac{1}{E}\right) - [r_0 + \varepsilon r_1 + O(\varepsilon^2)]\frac{B}{E} - \left(\frac{C}{E}+1\right) = 0. \tag{9}$$

Equating to zero, the successive terms of the series on the left-hand side of (9):

$$\varepsilon^0 : r_0^2 + Br_0 + C + E = 0$$

$$\varepsilon^1 : r_0^2 - \frac{1}{E}2r_0 r_1 - \frac{B}{E}r_1 = 0 \ \Rightarrow\ r_1 = \frac{Er_0^2}{2r_0 + B}$$

Solving the first equation above

$$r_0^{+,-} = \frac{-B \pm \sqrt{\Delta}}{2},$$

where $\Delta = B^2 - 4(C+E)$.

Similarly, for the second equation we obtain

$$r_1^{+,-} = \frac{E\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)^2}{2\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)+B} = \frac{E\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)^2}{\pm\sqrt{\Delta}}.$$

Hence the approximate solution of characteristic equation becomes

$$r^+(\varepsilon) = r_0^+ + \varepsilon r_1^+ + O(\varepsilon^2) = \frac{-B+\sqrt{\Delta}}{2} + \varepsilon\frac{E\left(\frac{-B+\sqrt{\Delta}}{2}\right)^2}{\sqrt{\Delta}} + O(\varepsilon^2),$$

$$r^-(\varepsilon) = r_0^- + \varepsilon r_1^- + O(\varepsilon^2) = \frac{-B-\sqrt{\Delta}}{2} - \varepsilon\frac{E\left(\frac{B+\sqrt{\Delta}}{2}\right)^2}{\sqrt{\Delta}} + O(\varepsilon^2.$$

As a result, the approximate solution of the equations of linear fractional differential equations with small delay can be obtained in the series form of ε as follows

$$u(t,\varepsilon) = \alpha_1 E_{\frac{1}{2},1}\left(r^+(\varepsilon)t^{\frac{1}{2}}\right) + \alpha_2 E_{\frac{1}{2},1}(r^-(\varepsilon)t^{\frac{1}{2}}),$$

where

$$E_{\frac{1}{2},1}\left(r(\varepsilon)t^{\frac{1}{2}}\right) = \sum_{k=0}^{\infty}\frac{\left(r(\varepsilon)t^{\frac{1}{2}}\right)^k}{\Gamma(\frac{1}{2}k+1)}.$$

Case 3: $n = 3$

Let us consider the case $n = 3$ which makes the characteristic equation (4)

$$\varepsilon r^3 - \frac{1}{E}r^2 - \frac{B}{E}r - \left(\frac{C}{E}+1\right) = 0 \tag{10}$$

which is a singular perturbation problem. Both by seeking the roots in the form of (5) and by scaling $r = \varepsilon^b x$ and balancing the suitable terms to recover the solutions of equation (10)

Substituting (5) into (10), we obtain

$$\varepsilon[r_0^3+3\varepsilon r_0^2 r_1+O(\varepsilon^2)]-\frac{1}{E}[r_0^2+2\varepsilon r_0 r_1+O(\varepsilon^2)]-\frac{B}{E}[r_0+\varepsilon r_1+O(\varepsilon^2)]-\left(\frac{C}{E}+1\right)=0.$$

Equating to zero, the successive terms of the series:

$$\varepsilon^0 : r_0^2+Br_0+C+E=0$$

$$\varepsilon^1 : Er_0^3-2r_0 r_1-Br_1=0 \Rightarrow r_1=\frac{Er_0^3}{2r_0+B}$$

Solving the first equation above

$$r_0^{+,-}=\frac{-B\pm\sqrt{\Delta}}{2},$$

where $\Delta=B^2-4(C+E)$

Similarly, for the second equation we obtain

$$r_1^{+,-}=\frac{E\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)^3}{2\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)+B}=\frac{E\left(\frac{-B\pm\sqrt{\Delta}}{2}\right)^3}{\pm\sqrt{\Delta}}.$$

Hence the approximate roots of characteristic equation becomes

$$r_1(\varepsilon)=r_0^+ +\varepsilon r_1^+ +O(\varepsilon^2)=\frac{-B+\sqrt{\Delta}}{2}+\varepsilon\frac{E\left(\frac{-B+\sqrt{\Delta}}{2}\right)^3}{\sqrt{\Delta}}+O(\varepsilon^2),$$

$$r_2(\varepsilon)=r_0^- +\varepsilon r_1^- +O(\varepsilon^2)=\frac{-B-\sqrt{\Delta}}{2}+\varepsilon\frac{E\left(\frac{B+\sqrt{\Delta}}{2}\right)^3}{\sqrt{\Delta}}+O(\varepsilon^2).$$

Let us find the root in the following form

$$r_3(\varepsilon)=\varepsilon^b x. \tag{11}$$

Substituting (11) into (10), we have

$$E\varepsilon^{3b+1}x^3-\varepsilon^{2b}x^2-B\varepsilon^b x-C-E=0.$$

In order to find b, examine the three possible pairs of terms. If we balance $\varepsilon^{2b}=\varepsilon^b$, we find $b=0$ recover the first root above. If we balance $\varepsilon^{3b+1}=\varepsilon^b$, we find $b=-\frac{1}{2}$, but the term ε^{2b} is the dominant term cannot be balanced by either of the other two terms. Finally, if we balance $\varepsilon^{3b+1}=\varepsilon^b$, we find $b=-1$.

Substituting $b=-1$

$$E\varepsilon^{-2}x^3-\varepsilon^{-2}x^2-B\varepsilon^{-1}x-C-E=0.$$

Let us times with ε^2 both of equation

$$Ex^3-x^2-B\varepsilon x-\varepsilon^2(C+E)=0, \tag{12}$$

$$x(\varepsilon)=x_0+\varepsilon x_1+O(\varepsilon^2). \tag{13}$$

Substituting (13) into (12), we have

$$Ex_0^3+3Ex_0^2x_1-x_0^2-2\varepsilon x_0x_1-B\varepsilon x_0+O(\varepsilon^2)=0. \tag{14}$$

Equating to zero, the successive terms of series on the left-hand side of (14)

$$\varepsilon^0 : Ex_0^3-x_0^2=0$$

$$\varepsilon^1: -3Ex_0^2x_1 - 2\varepsilon x_0x_1 - B\varepsilon x_0 = 0$$

Solving the first equation above, we have

$$x_0 = \frac{1}{E}, \quad x_0 = 0.$$

Similarly, solving the second equation above for $x_0 \neq 0$, we obtain

$$x_1 = \frac{B}{3Ex_0 - 2}$$

and it is clear that

$$x_1 = B$$

for $x_0 = \frac{1}{E}$. Therefore, an approximate solution of characteristic equation becomes

$$x(\varepsilon) = \frac{1}{E} + \varepsilon B + O(\varepsilon^2). \tag{15}$$

Substituting (15) into (11), we have

$$r_3(\varepsilon) = \frac{1}{\varepsilon}\frac{1}{E} + B + O(\varepsilon^2).$$

As a result, the approximate solution of the equations of linear fractional differential equations with small delay can be obtained in the series form of delay term ε as follows

Let $r_1(\varepsilon)$, $r_2(\varepsilon)$, $r_3(\varepsilon)$ denote the solutions of equation (10). Hence, the approximate solution of the equations of linear fractional differential equations with small delay can be obtained as a linear combination of corresponding Mittag-Leffler functions as follows:

$$u(t,\varepsilon) = \alpha_1 E_{\frac{1}{3},1}\left(r_1(\varepsilon)t^{\frac{1}{3}}\right) + \alpha_2 E_{\frac{1}{3},1}\left(r_2(\varepsilon)t^{\frac{1}{3}}\right) + \alpha_3 E_{\frac{1}{3},1}\left(r_3(\varepsilon)t^{\frac{1}{3}}\right),$$

where

$$E_{\frac{1}{3},1}\left(r(\varepsilon)t^{\frac{1}{3}}\right) = \sum_{k=0}^{\infty} \frac{\left(r(\varepsilon)t^{\frac{1}{3}}\right)^k}{\Gamma\left(\frac{1}{3}k+1\right)}.$$

3. Illustrative Examples

In this section, various examples are illustrated for the three cases above to show that how this method is effective and accurate.

Example 1. Let us consider the following linear fractional differential equation with small delay

$$D^1u(t) + 3D^{\frac{1}{2}}u(t) + u(t) + u(t-\varepsilon) = 0,$$

where $u(0) = 0$ and $D^{\frac{1}{2}}u(0) = 1$.

After using Taylor Series expansion and some arrangement we have

$$(1-\varepsilon)D^1u(t) + 3D^{\frac{1}{2}}u(t) + 2u(t) = 0.$$

Hence, the solution can be represented in the following form:

$$u(t,\varepsilon) = \left(\frac{1}{1+5\varepsilon}\right)E_{\frac{1}{2},1}\left((-1+\varepsilon)t^{\frac{1}{2}}\right) - \left(\frac{1}{1+5\varepsilon}\right)E_{\frac{1}{2},1}((-2-4\varepsilon)t^{\frac{1}{2}}). \tag{16}$$

The graph of the solution $u(t,\varepsilon)$ is given in Figure 1 for $\varepsilon = 0.5, 0.4, 0.3, 0.2, 0.1, 0.01$.

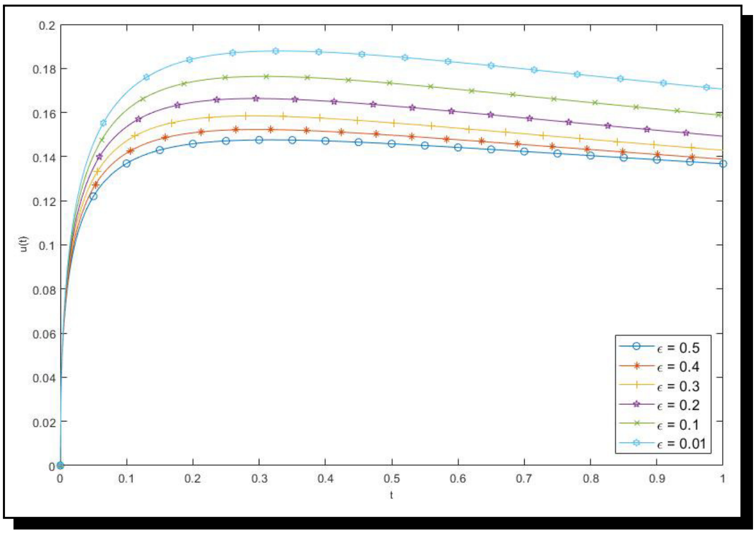

Figure 1. Graph for the solution (16) when $\varepsilon = 0.5, 0.4, 0.3, 0.2, 0.1, 0.01$

Example 2. Let us consider the following linear fractional differential equation with small delay

$$D^1 u(t) + 2D^{\frac{1}{2}} u(t) + u(t) + u(t-\varepsilon) = 0,$$

where $u(0) = 0$ and $D^{\frac{1}{2}} u(0) = -1$.

After using Taylor Series expansion and some arrangement, we have

$$(1-\varepsilon) D^1 u(t) + 2D^{\frac{1}{2}} u(t) + 2u(t) = 0.$$

Hence, the solution can be represented in the following form:

$$u(t,\varepsilon) = \frac{i}{2} E_{\frac{1}{2},1}\left((-1-\varepsilon+i)t^{\frac{1}{2}}\right) - \frac{i}{2} E_{\frac{1}{2},1}((-1-\varepsilon-i)t^{\frac{1}{2}}), \tag{17}$$

where $i^2 = -1$.

The graph of the solution $u(t,\varepsilon)$ is given in Figure 2 for $\varepsilon = 0.5, 0.4, 0.3, 0.2, 0.1, 0.01$.

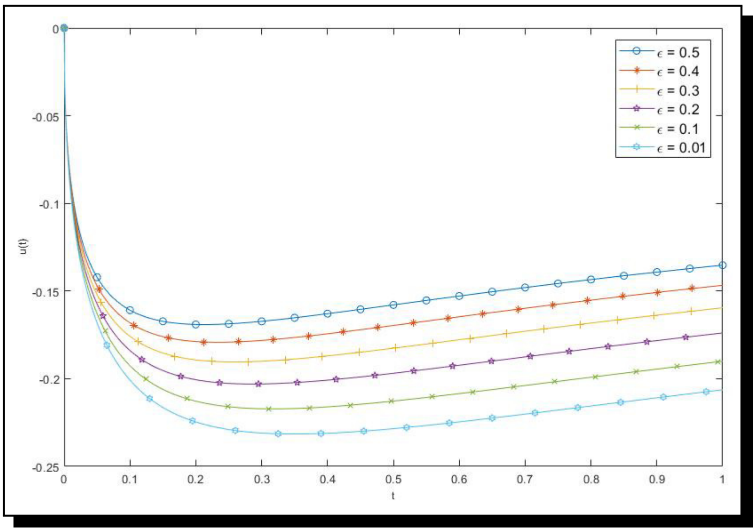

Figure 2. Graph for the solution (17) when $\varepsilon = 0.5, 0.4, 0.3, 0.2, 0.1, 0.01$

Example 3. Let us consider the following linear fractional differential equation with small delay

$$D^{\frac{2}{3}}u(t)+3D^{\frac{1}{3}}u(t)+u(t)+u(t-\varepsilon)=0,$$

where $u(0)=0$, $D^{\frac{1}{3}}u(0)=-1$ and $D^{\frac{2}{3}}u(0)=1$.

After using Taylor Series expansion and some arrangement we have

$$D^{\frac{2}{3}}u(t)+3D^{\frac{1}{3}}u(t)+2u(t)-\varepsilon D^{1}u(t)=0.$$

Hence, the solution can be represented in the following form:

$$u(t,\varepsilon)=\alpha_1E_{\frac{1}{3},1}\left((-1-\varepsilon)t^{\frac{1}{3}}\right)+\alpha_2E_{\frac{1}{3},1}\left((-2-8\varepsilon)t^{\frac{1}{3}}\right)+\alpha_3E_{\frac{1}{3},1}\left(\left(\frac{1}{\varepsilon}+3\right)t^{\frac{1}{3}}\right), \tag{18}$$

where $\alpha_1=\dfrac{8\varepsilon^2-2\varepsilon-1}{7\varepsilon^3+29\varepsilon^2+11\varepsilon+1}$, $\alpha_2=\dfrac{-\varepsilon^2+3\varepsilon+1}{(7\varepsilon+1)(8\varepsilon^2+5\varepsilon+1)}$ and $\alpha_3=\dfrac{-9\varepsilon^3-2\varepsilon^2}{(\varepsilon^2+4\varepsilon+1)(8\varepsilon^2+5\varepsilon+1)}$.

The graph of the solution $u(t,\varepsilon)$ is given in Figure 3 for $\varepsilon=0.5,0.4,0.3,0.28,0.25,0.2,0.15$.

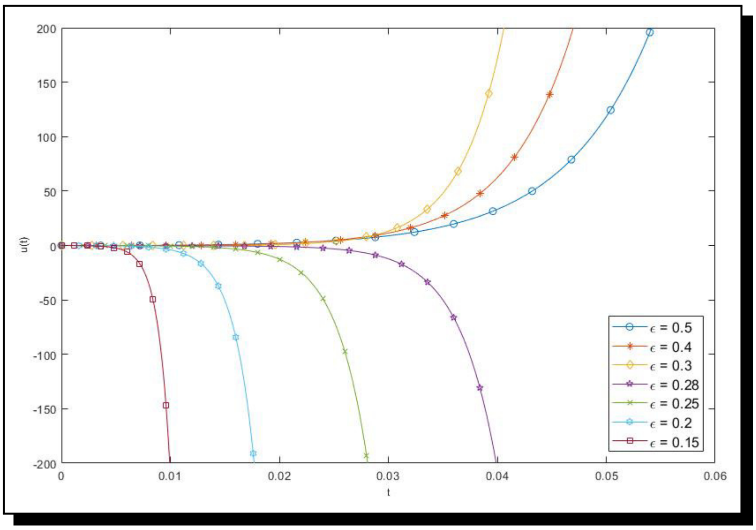

Figure 3. Graph for the solution (18) when $\varepsilon=0.5,0.4,0.3,0.28,0.25,0.2,0.15$

Example 4. Let us consider the following linear fractional differential equation with small delay

$$D^{\frac{2}{3}}u(t)-D^{\frac{1}{3}}u(t)-u(t)+u(t-\varepsilon)=0,$$

where $u(0)=0$, $D^{\frac{1}{3}}u(0)=-1$ and $D^{\frac{2}{3}}u(0)=1$.

After using Taylor Series expansion and some arrangement we have

$$D^{\frac{2}{3}}u(t)-D^{\frac{1}{3}}u(t)-\varepsilon D^{1}u(t)=0.$$

Hence, the solution can be represented in the following form:

$$u(t,\varepsilon)=\alpha_1+\alpha_2E_{\frac{1}{3},1}\left((1+\varepsilon)t^{\frac{1}{3}}\right)+\alpha_3E_{\frac{1}{3},1}\left(\left(\frac{1}{\varepsilon}-1\right)t^{\frac{1}{3}}\right), \tag{19}$$

where $\alpha_1=\dfrac{-\varepsilon^2-\varepsilon-1}{\varepsilon^2-1}$, $\alpha_2=\dfrac{1}{(\varepsilon+1)(\varepsilon^2+2\varepsilon-1)}$ and $\alpha_3=\dfrac{\varepsilon^3-2\varepsilon^2}{(\varepsilon+1)(\varepsilon^2+2\varepsilon-1)}$ for $\varepsilon^2+2\varepsilon-1\neq 0$.

The graph of the solution $u(t,\varepsilon)$ is given in Figure 4 for $\varepsilon = 0.5, 0.4, 0.3, 0.2, 0.1, 0.15$.

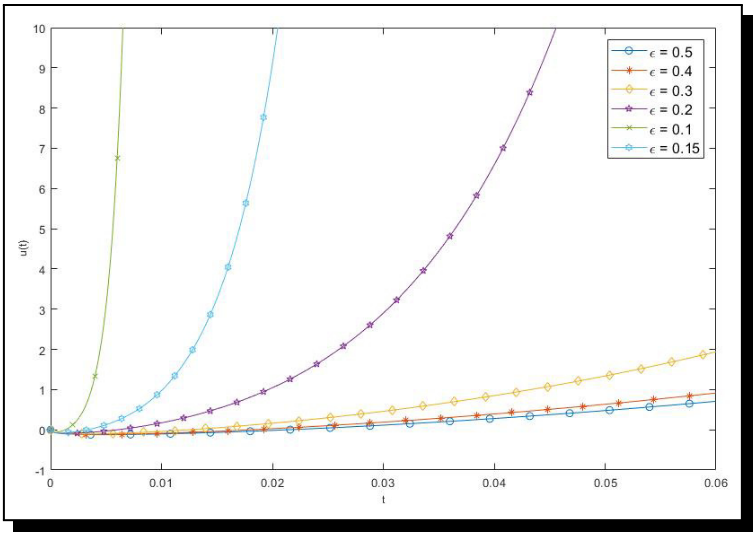

Figure 4. Graph for the solution (19) when $\varepsilon = 0.5, 0.4, 0.3, 0.2, 0.1, 0.15$

4. Conclusion

In this research, analytic or approximate solutions of linear fractional differential equations of order $2q$ with small delay is obtained in terms of Mittag-Leffler function, where $0 < q < 1$. The fractional derivatives are taken in the sense of Caputo which is more suitable than Riemann-Liouville sense. Since delay is small, taking power series expansion of delayed term into account, the problem is reduced into a problem of regular or singular perturbation problem for which it is easier to obtain under the assumption $n \cdot q = 1$ for some natural number n. Therefore the solution is obtained in the form of a series expansion of small delay. It is observed that in regular perturbation problems a small change in the problem causes a small change solution whereas a small change in the problem causes a large change in the solution in singular perturbation problem. The illustrative examples demonstrate the accuracy and the effectiveness of the proposed approach. The obtained results imply that analytic or approximate solutions of much more complicated fractional delay differential equations could be obtained by improving the method applied in this research.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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